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CERTAIN EQUIVALENCE RELATIONS ON AN EPIGROUP

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Abstract. In this paper, certain equivalences on an epigroup are introduced and the congruences generated by these equivalences are investigated. In addition, the join between Green's relations $\mathcal{H}$ (or $\mathcal{D}$) and anyone of these given equivalences is considered and the sublattices generated by them are depicted.

Keywords: epigroups; equivalences; lattces.

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1. Introduction and preliminaries

An epigroup is a semigroup in which some power of any element lies in a subgroup of the given semigroup. We will denote by $\mathcal{K}$ the equivalence relation on an epigroup S corresponding to the partition of the given epigroup S into its unipotency classes and $\mathcal{H}$, $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$, $\mathcal{J}$ are the well known Green's relations. Sedlock [7] determined necessary and sufficient conditions on a periodic semigroup S in order that $\mathcal{K}$ coincide with any one of these Green's relations. A characterization of periodic semigroups for which Green's relations $\mathcal{J}$ is included in $\mathcal{K}$ was given by Miller [3]. Results on periodic semigroups are generalized to epigroups by Madison, Mukherjee and Sen [2]. In this paper we will define

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a new equivalence relation and illustrate some properties of certain given equivalences. The join between Green's relations $\mathcal{D}$ (or $\mathcal{H}$) and anyone of them is considered and the sublattices generated by them are depicted.

Now we give precise definitions of the notions mentioned above and the ones that will be used in the further text. For undefined notions and notations we refer to [1], [5] and [8].

The set of all natural numbers is denoted by $\mathbb{N}$. For an arbitrary subset M of a semigroup S we put

$$\sqrt{M} = \{x \in S | x^n \in M \text{ for some } n \in \mathbb{N}\}.$$

The set of all idempotents of a semigroup S is denoted by $E(S)$. If $e \in E(S)$, then

$$G_e = \{x \in S | x \in eS \cap Se,\ e \in xS \cap Sx\}$$

is the largest subgroup of S having e as its identity, called the maximal subgroup of S determined by e, and the set K_e is defined by $K_e = \sqrt{G_e}$. An element a of S is group-bound if at least one of its powers lies in some subgroup of S. There is exactly one such maximal subgroup. A semigroup S is called an epigroup if every element a of S is group-bound. By definition, for any element a of an epigroup S there exists $e \in E(S)$ such that $a \in K_e$. That the idempotent e is defined here uniquely is a consequence of the following property already mentioned in [4].

Lemma 1.1. *If $a^m \in G_e$, then $a^n \in G_e$ for any $n \geq m$.*

Any epigroup S is partitioned into the subsets K_e called unipotency classes. The idempotent of the unipotency class to which an element a belongs will be denoted by a^ω. The element $\overline{a} = (aa^\omega)^{-1}$ is the inverse of aa^ω in the group G_{a^ω}, where $aa^\omega = a^\omega a \in G_{a^\omega}$ was also mentioned in [4]. This element is called the pseudo-inverse of a. The smallest positive integer n such that $a^n \in G_{a^\omega}$ is called the index of an element a and will be denoted by $\mathrm{ind}(a)$. The following equalities hold in any epigroup S:

$$\overline{a}a = a\overline{a} = a^\omega,\ a^\omega\overline{a} = \overline{a},\ a^m a^\omega = a^m \text{ for some } m \in \mathbb{N}.$$

Clearly for an epigroup S, S is completely regular if and only if for each $a \in S$, $a = \overline{\overline{a}}$; S is a band if and only if for each $a \in S$, $a = a^{\omega}$.

Let S be an epigroup. For any $n \in \mathbb{N}$ we easily get to the following properties.

Lemma 1.2. a) $a^{\omega} = (a^n)^{\omega}$; b) $\overline{a}^n = \overline{a^n}$; c) $a^n = (\overline{\overline{a}})^n$, $n \geq \text{ind}(a)$.

Lemma 1.2.a) shows that each unipotency class is closed under raising an element to a power; if $a \in K_e$, then $\langle a \rangle \subseteq K_e$.

A semigroup is called stable if for any of its $\mathcal{J}$-classes the sets of $\mathcal{L}$-classes and $\mathcal{R}$-classes contained in it is minimal with respect to the partial order ($L_a \leq L_b$ means $L(a) \subseteq L(b)$, $R_a \leq R_b$ means $R(a) \subseteq R(b)$). It is easy to check that any epigroup is a stable semigroup and in a stable semigroup Green's relation $\mathcal{J}$ and $\mathcal{D}$ coincide, so do in an epigroup.

The division relation $|$ and the relation $\longrightarrow$ on a semigroup S are defined by

$$a|b \iff (\exists x, y \in S^1)\ b = xay, \quad a \longrightarrow b \iff (\exists k \in \mathbb{N})\ a|b^k.$$

Let $\longleftrightarrow$ be the symmetric openings of $\longrightarrow$, that is, $\longleftrightarrow\ =\longrightarrow \cap \longrightarrow^{-1}$. The transitive closure of $\longleftrightarrow$ is denoted by $\longleftrightarrow^{\infty}$. Putcha [5] proved that $\longleftrightarrow^{\infty}$ is the least semilattice congruence on S.

If ρ is a binary relation on a semigroup S, we denote by $\sqrt{\rho}$ the binary relation on S defined by the following condition:

$$x\sqrt{\rho}y \iff \text{there exist } m, n \in \mathbb{N} \text{ such that } x^m \rho y^n.$$

The relation ρ^* is the congruence on S generated by ρ. Explicitly, we have

$$a\rho^* b \iff a = b \text{ or } a = x_1u_1y_1, x_1v_1y_1 = x_2u_2y_2, \ldots, x_nv_ny_n = b$$

for some $x_i, y_i \in S^1, u_i, v_i \in S$ such that either $u_i\rho v_i$ or $v_i\rho u_i$ $i = 1, 2, \ldots, n$.

2. Certain equivalence relations

For an epigroup S, besides the equivalence $\mathcal{K}$, we define two equivalences as follows:

$$a \sim b \quad \text{if} \quad a^m = b^n \text{ for some } m, n \in \mathbb{N},$$

$$a\mathcal{P}b \quad \text{if} \quad \overline{a} = \overline{b},$$

where $\sim$ is sometimes known as Schwartz's equivalence. Clearly for an epigroup S, we have

$$a\mathcal{K}b \iff a^\omega = b^\omega,\ a\mathcal{P}b \iff \overline{\overline{a}} = \overline{\overline{b}}\ \text{ (or } aa^\omega = bb^\omega).$$

Note that for any $a \in S$, $\overline{a} = \overline{\overline{\overline{a}}}$, so $a\mathcal{P}\overline{a}$ holds.

Lemma 2.1. *For an epigroup S, we have $\sim = \sqrt{\mathcal{P}}$.*

Proof. Let $a \sim b$, for $a, b \in S$. Then $a^m = b^n$, for some $m, n \in \mathbb{N}$, so $\overline{a^m} = \overline{b^n}$. It follows $a^m\mathcal{P}b^n$, hence $a\sqrt{\mathcal{P}}b$. Conversely, if $a\sqrt{\mathcal{P}}b$, then $a^m\mathcal{P}b^n$, for some $m, n \in \mathbb{N}$. It folows $\overline{\overline{a^m}} = \overline{\overline{b^n}}$, which means $a^m(a^m)^\omega = b^n(b^n)^\omega$. As $a^m(a^m)^\omega = (a^m)^\omega a^m$, $b^n(b^n)^\omega = (b^n)^\omega b^n$, we have $a^{km} = b^{kn}$ for $k = \max\{\mathrm{ind}(a^m), \mathrm{ind}(b^n)\}$. This shows $a \sim b$. □

Corollary 2.2. *For an epigroup S, $\mathcal{P} \subseteq \sim \subseteq \mathcal{K}$.*

Proof. Together with Lemma 2.1 and Lemma 1.2.a) it is easy to verify $\mathcal{P} \subseteq \sim \subseteq \mathcal{K}$. □

Recall that a retract of a semigroup S is a subsemigroup T for which there exists a homomorphism $\varphi : S \to T$ which is the identity mapping on T; such a homomorphism φ is called a retraction. A retract ideal is an ideal that is a retract. As is known that in an epigroup S, the mapping $x \mapsto \overline{\overline{x}}$ is an endomorphism if and only if the set GrS is a retract (see [8] 3.42 Lemma). In this case, the indicated mapping is a unique retraction of S onto GrS.

Lemma 2.3. *For an epigroup S, if GrS is a retract, then $\mathcal{P}$ is a congruence.*

Proof. Suppose $a\mathcal{P}b$, then $\overline{\overline{a}} = \overline{\overline{b}}$. For any $c \in S$, $\overline{\overline{a}}\,\overline{\overline{c}} = \overline{\overline{b}}\overline{\overline{c}}$, hence $\overline{\overline{ac}} = \overline{\overline{bc}}$, since Gr$S$ is a retract. Thus $ac\mathcal{P}bc$, it follows that $\mathcal{P}$ is a right congruence. Dually, $\mathcal{P}$ is a left congruence, so the corollary is proved. □

In a regular semigroup S, $\mathcal{J}^*(= \mathcal{D}^*)$ is the least semilattce congruence (see [1] Theorem 1.4.17), that is, $\mathcal{J}^* = \text{—}^\infty$ $(\mathcal{D}^* = \text{—}^\infty)$. In the following we say about $\mathcal{P}^*, \mathcal{K}^*$ for an epigroup S.

Theorem 2.4. *For an epigroup S,*

(i). $\mathcal{P}^*$ is the least completely regular congruence on S;

(ii). $\mathcal{K}^*$ is the least band congruence on S.

Proof. (i) Since a homomorphic image of an epigroup is an epigroup, then $S/\mathcal{P}^*$ is an epigroup. To show that $\mathcal{P}^*$ is a completely regular congruence on S, for $a \in S$, we only need prove that $\overline{\overline{a\mathcal{P}^*}} = a\mathcal{P}^*$, which means $a\mathcal{P}^*$ is a group element in $S/\mathcal{P}^*$, that is, $\overline{\overline{a}}\mathcal{P}^*a$. Since $\overline{\overline{\overline{a}}} = \overline{a}$, we have $\overline{\overline{a}}\mathcal{P}a$, obviously $\overline{\overline{a}}\mathcal{P}^*a$. Hence $S/\mathcal{P}^*$ is a completely regular semigroup.

Conversely let $a\mathcal{P}^*b$, $a, b \in S$ and ρ be a completely regular congruence. We have

$$a = x_1u_1y_1,\ x_1v_1y_1 = x_2u_2y_2,\ \ldots,\ x_nv_ny_n = b,$$

for some $x_i, y_i \in S^1, u_i, v_i \in S, u_i\mathcal{P}v_i$, $i = 1, 2, \ldots, n$. As ρ is a completely regular congruence, then $u_i\rho\overline{\overline{u_i}} = \overline{\overline{v_i}}\rho v_i$, thus $u_i\rho v_i$. So

$$a = x_1u_1y_1\, \rho\, x_1v_1y_1\, \rho\, x_2u_2y_2\, \ldots\, \rho\, x_nv_ny_n = b,$$

hence $a\rho b$. Consequently $\mathcal{P}^* \subseteq \rho$ and the minimality of $\mathcal{P}^*$ is proved.

(ii) Obviously $a\mathcal{K}a^\omega$, then $a\mathcal{K}^*a^\omega$ and thus $\mathcal{K}^*$ is a band congruence. Conversely let ρ be a band congruence and $a\mathcal{K}^*b$, $a, b \in S$. Then we have

$$a = x_1u_1y_1,\ x_1v_1y_1 = x_2u_2y_2,\ \ldots,\ x_nv_ny_n = b, \tag{1}$$

where $x_i, y_i \in S^1, u_i, v_i \in S, u_i\mathcal{K}v_i$, $i = 1, 2, \ldots, n$. As ρ is a band congruence, then $u_i\rho {u_i}^\omega = {v_i}^\omega\rho v_i$, thus $u_i\rho v_i$. So by (1), $a\rho b$. Consequently we have $\mathcal{K}^* \subseteq \rho$ and the minimality of $\mathcal{K}^*$ is proved. □

For an epigroup S, in general, $\mathcal{K} \subseteq \mathcal{D}$ doesn't prevail (see section 3 Table 3.1). But for a regular epigroup S from Theorem 2.4 we can see $\mathcal{K}^*, \mathcal{D}^*$ are comparable with respect to the partial order ordered by inclusion, since for a regular epigroup, $\mathcal{D}^*$ is the the least semilattice congruence on S.

Corollary 2.5. *For a regular epigroup S, $\mathcal{P}^* \subseteq \mathcal{K}^* \subseteq \mathcal{D}^*$.*

3. Sublattice generated by some equivalences

For a semigroup S, the lattice of equivalences of S is denoted by $(\mathcal{E}(S), \subseteq, \cap, \vee)$, which are partially ordered by inclusion. In this section the join between $\mathcal{H}$ (or $\mathcal{D}$) and anyone of the equivalences considered in the preceding section will be given. Furthermore the sublattices generated by some equivalences of them are depicted.

Let $\sqrt{\mathcal{D}}^{\infty}$ and $\sqrt{\mathcal{H}}^{\infty}$ be transitive closure of $\sqrt{\mathcal{D}}$ and $\sqrt{\mathcal{H}}$ respectively. it is to be noted that for any epigroup S, $\sqrt{\mathcal{D}}^{\infty} = —^{\infty}$. The result has been obtained by Putcha [6], who proved that in an epigroup, the transitive closure of $\sqrt{\mathcal{J}}$ (for an epigroup $\mathcal{J} = \mathcal{D}$) is the least semilattice congruence.

Theorem 3.1. *For an epigroup S, we have*

(i). $\mathcal{P} \vee \mathcal{D} = \sim \vee \mathcal{D} = \mathcal{K} \vee \mathcal{D} = —^{\infty}$;

(ii) $\mathcal{P} \vee \mathcal{H} = \sim \vee \mathcal{H} = \mathcal{K} \vee \mathcal{H} = \sqrt{\mathcal{H}}^{\infty}$.

Proof. (i) Since $\mathcal{P} \subseteq \sim \subseteq \mathcal{K}$, we have $\mathcal{P}\vee\mathcal{D} \subseteq \sim \vee\mathcal{D} \subseteq \mathcal{K}\vee\mathcal{D}$. Note that $\mathcal{P}\vee\mathcal{D}, \sim \vee\mathcal{D}, \mathcal{K}\vee\mathcal{D}$ are all equivalences on S. It is sufficient to show $\mathcal{K} \vee \mathcal{D} = —^{\infty}$ and $\mathcal{K} \vee \mathcal{D} \subseteq \mathcal{P} \vee \mathcal{D}$.

It is easy to verify that $\mathcal{K} \subseteq —$, $\mathcal{D} \subseteq —$. Since the join $\mathcal{K} \vee \mathcal{D}$ is the least equivalence containing $\mathcal{K}$ and $\mathcal{D}$ and $—^{\infty}$ is an equivalence, it follows that $\mathcal{K} \vee \mathcal{D} \subseteq —^{\infty}$. Conversely, by virtue of Corollary 3 in [6], $—^{\infty}$ is the transitive closure of $\sim \circ\mathcal{D}$ (or $\mathcal{D}\circ \sim$) on S. But then as $\sim \subseteq \mathcal{K}$, we have $—^{\infty} \subseteq \mathcal{K} \vee \mathcal{D}$.

For any $a, b \in S$, Let $a(\mathcal{K}\vee\mathcal{D})b$, then for some $n \in \mathbb{N}$, there exist elements $x_1, x_2, \ldots, x_{2n-1}$ in S such that

$$a\mathcal{K}x_1\mathcal{D}x_2\mathcal{K}x_3\mathcal{D}x_4 \ldots \mathcal{K}x_{2n-1}\mathcal{D}b.$$

Thus $a\mathcal{P}\overline{\overline{a}}\mathcal{D}a^{\omega} = x_1^{\omega}\mathcal{D}\overline{\overline{x_1}}\mathcal{P}x_1\mathcal{D}x_2 \ldots x_{2n-1}\mathcal{D}b$, which implies that $a(\mathcal{P} \vee \mathcal{D})b$, so $\mathcal{K} \vee \mathcal{D} \subseteq \mathcal{P} \vee \mathcal{D}$.

(ii) An argument similar to the part proof of (i) establishes the first three equalities and we only need show $\mathcal{P}\vee\mathcal{H} = \sqrt{\mathcal{H}}^{\infty}$. Now $\mathcal{K}, \mathcal{H} \subseteq \sqrt{\mathcal{H}}, \mathcal{P} \subseteq \mathcal{K}$, we have $\mathcal{P}\vee\mathcal{H} \subseteq \sqrt{\mathcal{H}}^{\infty}$, since the join $\mathcal{P} \vee \mathcal{H}$ is the least equivalence containing $\mathcal{P}$ and $\mathcal{H}$ and $\sqrt{\mathcal{H}}^{\infty}$ is an equivalence containing $\mathcal{P}$ and $\mathcal{H}$.

Let $a, b \in S$. Then

$$\begin{aligned} a\sqrt{\mathcal{H}}b &\implies a^m\mathcal{H}b^n \quad \text{for some } m, n \in \mathbb{N} \\ &\implies a\mathcal{P}\overline{\overline{a}}\mathcal{H}(\overline{\overline{a}})^m = \overline{\overline{a^m}}\mathcal{P}a^m\mathcal{H}b^n\mathcal{P}\overline{\overline{b^n}} = (\overline{\overline{b}})^n\mathcal{H}\overline{\overline{b}}\mathcal{P}b \text{ (by Lemma 1.2.b))} \\ &\implies a\mathcal{P} \vee \mathcal{H}b, \end{aligned}$$

so that $\sqrt{\mathcal{H}} \subseteq \mathcal{P} \vee \mathcal{H}$, which proves the inclusion $\sqrt{\mathcal{H}}^{\infty} \subseteq \mathcal{P} \vee \mathcal{H}$, since $\sqrt{\mathcal{H}}^{\infty}$ is the least equivalence relation that contains $\sqrt{\mathcal{H}}$. $\square$

Theorem 3.2. *For an epigroup S, let $\mathcal{N}$ be any of Green's relations $\mathcal{H}$, $\mathcal{D}$.*

(*i*). $(\mathcal{K} \cap \mathcal{N}) \vee \sim = \mathcal{K}$;

(*ii*). $(\sim \cap \mathcal{N}) \vee \mathcal{P} = \sim$.

Proof. (i) Clearly $(\mathcal{K} \cap \mathcal{N}) \vee \sim \subseteq \mathcal{K}$, since $(\mathcal{K} \cap \mathcal{N}) \subseteq \mathcal{K}$, $\sim \subseteq \mathcal{K}$. For the converse, we show that $\mathcal{K} \subseteq (\mathcal{K} \cap \mathcal{N}) \vee \sim$. Indeed, for $a, b \in S$,

$$\begin{aligned} a\mathcal{K}b &\implies a^{\omega} = b^{\omega} \\ &\implies a \sim a^{\mathrm{ind}(a)} (\mathcal{K} \cap \mathcal{N}) a^{\omega} = b^{\omega} (\mathcal{K} \cap \mathcal{N}) b^{\mathrm{ind}(b)} \sim b \\ &\implies a (\mathcal{K} \cap \mathcal{N}) \vee \sim b. \end{aligned}$$

(ii) Let $a, b \in S$. Then

$$\begin{aligned} a \sim b &\iff a^m = b^n \text{ for some } m, n \in \mathbb{N} \\ &\implies a^{km} = b^{kn},\ k = \max\{\mathrm{ind}(a), \mathrm{ind}(b)\} \\ &\implies a\mathcal{P}\overline{\overline{a}}(\sim \cap \mathcal{N})(\overline{a})^{km} = a^{km} = b^{kn} = (\overline{\overline{b}})^{kn}(\sim \cap \mathcal{N})\overline{\overline{b}}\mathcal{P}b \text{ (by Lemma 1.2.c))} \\ &\implies a\mathcal{P} \vee (\sim \cap \mathcal{N})b, \end{aligned}$$

which establishes the containment $\sim \subseteq \mathcal{P} \vee (\sim \cap \mathcal{N})$. Conversely, $\mathcal{P} \subseteq \sim$, $\sim \cap \mathcal{N} \subseteq \sim$ implies that $\mathcal{P} \vee (\sim \cap \mathcal{N}) \subseteq \sim$. $\square$

Example 3.3. Let $S_2 = M_2(F)$ be the all 2-by-2-matrices over the field F. This is an epigroup. Let

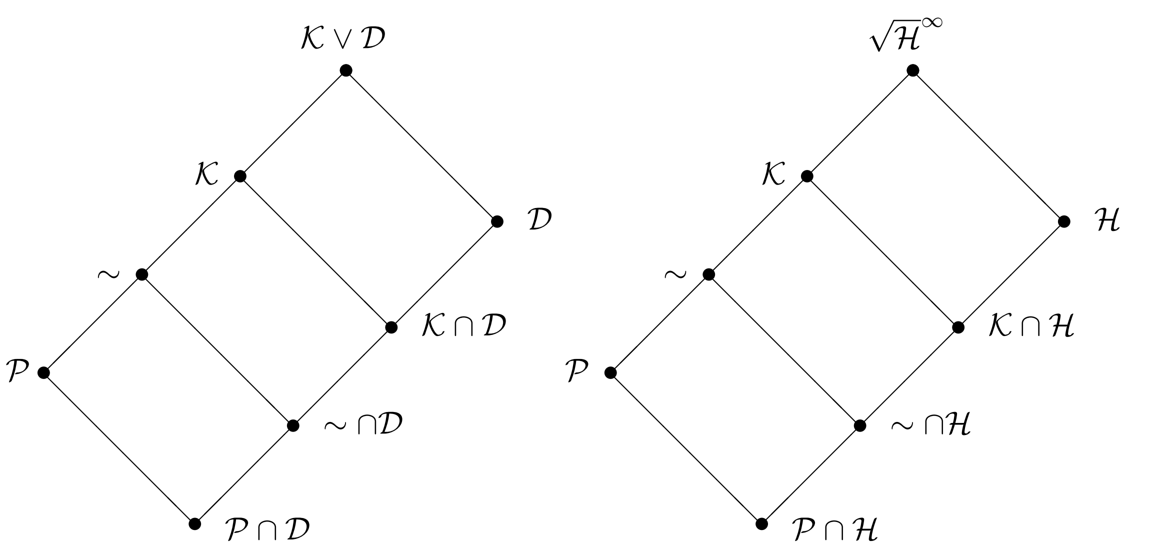
$$i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ a = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ d = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The ordered pairs in $S_2 \times S_2$ label the rows of Table 3.1, the columns of which are labelled by the members of $\{\mathcal{H}, \mathcal{D}, \mathcal{K}, \sim, \mathcal{P}\}$ and meets of any two incomparable elements among them. The entry in Table 3.1 corresponding to an ordered pair (x, y) and an equivalence relation $\mathcal{B}$ is 1 or 0 according as (x, y) belongs to or fails to belong to $\mathcal{B}$.

Table 3.1

	$\mathcal{D}$	$\mathcal{K}$	$\mathcal{K} \cap \mathcal{D}$	$\sim$	$\sim \cap \mathcal{D}$	$\mathcal{P}$	$\mathcal{P} \cap \mathcal{D}$
(f, c)	1	0	0	0	0	0	0
(o, c)	0	1	0	1	0	1	0
(a, i)	1	1	1	0	0	0	0
(b, i)	1	1	1	1	1	0	0
(c, d)	1	1	1	1	1	1	1

	$\mathcal{H}$	$\mathcal{K}$	$\mathcal{K} \cap \mathcal{H}$		$\sim \cap \mathcal{H}$		$\mathcal{P} \cap \mathcal{H}$
(c, d)	0	1	0		0		0
(a, i)	1	1	1		0		0
(b, i)	1	1	1		1		0



(a) (b)

Fig. 3.1 Lattice generated by $\{\mathcal{D}, \mathcal{K}, \sim, \mathcal{P}\}$ Lattice generated by $\{\mathcal{H}, \mathcal{K}, \sim, \mathcal{P}\}$

Remarks 3.4. From Table 3.1, for an epigroup S, $\mathcal{D}$ and $\mathcal{K}$ are incomparable with respect to the inclusion order. We may go one step further by saying that, $\mathcal{H}$ and $\mathcal{K}$ are incomparable, too. For example, in $S_3 = M_3(F)$, let

$$g = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is evident that $g\mathcal{H}h$, while $(g,h) \notin \mathcal{K}$. This together with Table 3.1, for instance, we may let

$$c' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad d' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

so $c'\mathcal{K}d'$, while $(c', d') \notin \mathcal{H}$. Thus $\mathcal{H}$ and $\mathcal{K}$ are not comparable with respect to the inclusion order.

Theorem 3.5. *Let S be an epigroup, Fig.3.1 (a) represents the sublattice of $\mathcal{E}(S)$ generated by the set $\{\mathcal{D}, \mathcal{K}, \sim, \mathcal{P}\}$. Fig.3.1 (b) represents the sublattice of $\mathcal{E}(S)$ generated by the set $\{\mathcal{H}, \mathcal{K}, \sim, \mathcal{P}\}$. Clearly, the sublattices are all distributive.*

Proof. We need check that any pair of noncomparable vertices has the meet and join as indicated in Fig.3.1 (a) or (b). It follows easily from Theorem 3.1, 3.2. Further the information contained in Table 3.1 is sufficient to verify that all equivalences indicated in Fig 3.1 (a) or (b) are distinct. Now the resulting Fig.3.1 (a) or (b) form sublattice of $\mathcal{E}(S)$ follows. $\square$

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