



Available online at <http://scik.org>

J. Math. Comput. Sci. 11 (2021), No. 2, 1176-1192

<https://doi.org/10.28919/jmcs/5308>

ISSN: 1927-5307

COUPLED FIXED POINT THEOREMS FOR TWISTED $(\alpha, \beta) - \psi$ -EXPANSIVE MAPPINGS

MANOJ KUMAR*, PREETI BHARDWAJ

Department of Mathematics, Baba Mastnath University, Asthal Bohar 124021, Rohtak, India

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, we introduce a new concept of cyclic and cyclic ordered $\alpha - \psi$ expansive mappings and investigate the existence of a fixed point for the mappings in this class. Further, we shall derive coupled fixed point theorems in complete metric spaces. The presented theorems generalize and improve many existing results in the literature. Moreover, some examples are given to illustrate our results.

Keywords: expansive mapping; fixed point; complete metric space; coupled fixed point.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis, since it provides a simple proof for the existence and uniqueness of the solutions to various mathematical models. Its core subject is concerned with the conditions for the existence of one or more fixed points of a mapping f from a topological space X into itself. The Banach contraction principle [1] is one of the most versatile elementary results in fixed point theory. Moreover, being based on iteration process, it can be implemented on a computer to find the fixed point of a contractive mapping. This principle has many applications and was extended by several

*Corresponding author

E-mail address: manojantil18@gmail.com

Received December 08, 2020

authors. Among them, we mention the $\alpha - \psi$ -contractive mapping, which was introduced by Samet et al. [2] via α -admissible mappings.

In 1984, Wang et al. [3] presented some interesting work on expansive mappings in metric spaces which correspond to some contractive mapping in [4]. Khan et al. [5] generalized the result of [3] by using functions. Also, Rhoades [6] and Taniguchi [7] generalized the results of Wang [3] for a pair of mappings. Kang [8] generalized the results of Khan et al. [5], Rhoades [6] and Taniguchi [7] for expansive mappings. Shahi et al. [9] introduced the concept of (ξ, α) -expansive mappings in complete metric spaces. Salimi et al. [10] introduced the new notion of twisted $(\alpha, \beta) - \psi$ -contractive mappings in metric spaces. Recently, Kang et al. [11] introduced the notion of twisted $(\alpha, \beta) - \psi$ -expansive mappings in metric spaces.

In this paper, we introduce a new concept of cyclic and cyclic ordered $\alpha - \psi$ expansive mappings and establish various fixed point theorems for such mappings in complete metric spaces. The presented theorems extend, generalize and improve many existing results in the literature. Some examples are given to illustrate our results.

For the sake of completeness, we recall some basic definitions and fundamental results.

Wang et al. [3] defined expansion mappings in the form of following theorem.

Theorem 1.1. Let (X, d) be a complete metric space. If $f : X \rightarrow X$ is an onto mapping and there exists a constant $k > 1$ such that $d(fx, fy) \geq kd(x, y)$, for each $x, y \in X$. Then f has a unique fixed point in X .

Recently, Samet et al. [2] introduced the following concepts.

Definition 1.2. Let Ψ denote the family of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfy the following :

- (i) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n th iterate of ψ ;
- (ii) ψ is non-decreasing.

Definition 1.3. Let (X, d) be a metric space and $f : X \rightarrow X$ be a given self mapping. f is said to be an $\alpha - \psi$ -contractive mapping if there exists two functions $\alpha : X \times X \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that $\alpha(x, y)d(fx, fy) \leq (\psi(d(x, y)))$ for all $x, y \in X$.

Definition 1.4. Let $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. f is said to be α -admissible if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$.

In 2013, P. Salimi et al. [10] introduced the concept of twisted (α, β) -admissible mappings in the following form:

Definition 1.5. Let $f : X \rightarrow X$ and $\alpha, \beta : X \times X \rightarrow [0, \infty)$. f is said to be a twisted (α, β) -admissible mapping if $x, y \in X$,
$$\begin{cases} \alpha(x, y) \geq 1, \\ \beta(x, y) \geq 1, \end{cases} \Rightarrow \begin{cases} \alpha(fy, fx) \geq 1, \\ \beta(fy, fx) \geq 1. \end{cases}$$

Definition 1.6. Let (X, d) be a metric space and $f : X \rightarrow X$ be a twisted (α, β) -admissible mapping. Then f is said to be a

(i) twisted $(\alpha, \beta) - \psi$ -contractive mapping of type I, if

$$\alpha(x, y)\beta(x, y)d(fx, fy) \leq \psi(d(x, y))$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

(ii) twisted $(\alpha, \beta) - \psi$ -contractive mapping of type II, if there is $0 < p \leq 1$ such that

$$(\alpha(x, y)\beta(x, y) + p)^{d((fx, fy))} \leq (1 + p)^{\psi(d(x, y))}$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

(iii) twisted $(\alpha, \beta) - \psi$ -contractive mapping of type III, if there is $p \geq 1$ such that

$$(d(fx, fy) + p)^{\alpha(x, y)\beta(x, y)} \leq \psi(d(x, y)) + p$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

Recently, Kang et al. [11] introduced the concept of twisted $(\alpha, \beta) - \psi$ -expansive mappings in metric spaces as follows:

Definition 1.7. Let (X, d) be a metric space and $f : X \rightarrow X$ be a twisted (α, β) -admissible mapping. Then f is said to be a

(i) twisted $(\alpha, \beta) - \psi$ -expansive mapping of type I, if

$$(1) \quad \psi(d(fx, fy)) \geq \alpha(x, y)\beta(x, y)d(x, y),$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

(ii) twisted $(\alpha, \beta) - \psi$ -expansive mapping of type II, if there is $0 < p \leq 1$ such that

$$(2) \quad (1 + p)^{\psi(d(fx, fy))} \geq (\alpha(x, y)\beta(x, y) + p)^{d(x, y)},$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

(iii) twisted $(\alpha, \beta) - \psi$ -expansive mapping of type III, if there is $p \geq 1$ such that

$$(3) \quad \psi(d(fx, fy)) + p \geq (d(x, y) + p)^{\alpha(x,y)\beta(x,y)},$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

In what follows, we present the main results of Kang et al. [11].

Theorem 1.8. Let (X, d) be a complete metric space and $f : X \rightarrow X$ be a bijective, twisted $(\alpha, \beta) - \psi$ -expansive mapping of type I or type II or type III satisfying the following conditions:

- (i) f^{-1} is twisted (α, β) -admissible;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, f^{-1}x_0) \geq 1, \beta(x_0, f^{-1}x_0) \geq 1$;
- (iii) f is continuous.

Then f has a fixed point, that is, there exists $z \in X$ such that $fz = z$.

In what follows, Kang et al. [11] proved that Theorem 1.8 still holds for f not necessarily continuous, assuming the following condition:

(M): If $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(f^{-1}x, f^{-1}x_{2n}) \geq 1$ and $\beta(f^{-1}x, f^{-1}x_{2n}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Theorem 1.9. If in Theorem 1.8, continuity of f is replaced by the condition (M), then the result holds true.

Here, we give some suitable examples to illustrate the results given by Kang et al. [11].

Example 1.10. Let $X = \mathbb{R}$ be endowed with the usual metric

$d(x, y) = |x - y|$, for all $x, y \in X$. Define $f : X \rightarrow X$ by

$$fx = \begin{cases} -3x, & \text{if } x \in [-1, 1] \\ 2x + \frac{11}{6}, & \text{if } x \in \mathbb{R} \setminus [-1, 1]. \end{cases}$$

Define also $\alpha, \beta : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 1, & \text{if } x \in [-1, 0] \text{ and } y \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

and $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(a) = 2a$ for all $a \geq 0$.

We prove that Theorem 1.9 can be applied to f .

Proof. Let $\alpha(x, y) \geq 1$ for $x, y \in X$. Then $x \in [-1, 0]$ and $y \in [0, 1]$, and so $f^{-1}y \in [-1, 0]$ and

$f^{-1}x \in [0, 1]$, that is, $\alpha(f^{-1}y, f^{-1}x) \geq 1$. Also assume $\beta(x, y) \geq 1$ for $x, y \in X$. Therefore $x \in [-1, 0]$ and $y \in [0, 1]$, and so $f^{-1}y \in [-1, 0]$ and $f^{-1}x \in [0, 1]$, that is, $\beta(f^{-1}y, f^{-1}x) \geq 1$. Also $\alpha(0, f^{-1}0) \geq 1$ and $\beta(0, f^{-1}0) \geq 1$. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$. This implies that $\{x_{2n+1}\} \subset [0, 1]$ and $\{x_{2n}\} \subset [-1, 0]$. Thus, $x = 0$ and so $\alpha(f^{-1}x_{2n}, f^{-1}x) \geq 1$ and $\beta(f^{-1}x_{2n}, f^{-1}x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Moreover, for $x \in [-1, 0]$ and $y \in [0, 1]$, we have

$$2(d(fx, fy)) \geq \alpha(x, y)\beta(x, y)d(x, y).$$

Otherwise $\alpha(x, y)\beta(x, y) = 0$ and (1) trivially holds. Then f is a twisted $(\alpha, \beta) - \psi$ -expansive mapping of type I and, by Theorem 1.9, f has a fixed point. Clearly, 0 and $\frac{-11}{6}$ are two fixed points of f .

Example 1.11. Let X, d, α and β be as in Example 1.9 and $f : X \rightarrow X$ be defined by $fx = \begin{cases} -4e^2x, & \text{if } x \in [-1, 1] \\ 2x + \frac{5}{2}, & \text{if } x \in \mathbb{R} \setminus [-1, 1]. \end{cases}$

Define also $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(a) = 2a$ for all $a \geq 0$.

We prove that Theorem 1.9 can be applied to f .

Proof. Proceeding as in the proof of Example 1.10, we deduce that f^{-1} is a twisted (α, β) -admissible mapping and that the conditions (i) and (ii) of Theorem 1.9 hold.

Moreover, if $x \in [-1, 0], y \in [0, 1]$ and $0 < p \leq 1$, we have

$$(1 + p)^{2(d(fx, fy))} \geq (\alpha(x, y)\beta(x, y) + p)^{m(x, y)}.$$

Otherwise $\alpha(x, y)\beta(x, y) = 0$ and (2) trivially holds. Then f is a twisted $(\alpha, \beta) - \psi$ -expansive mapping of type II and, by Theorem 1.9, f has a fixed point.

Clearly 0 and $\frac{-5}{2}$ are two fixed points of f .

Example 1.12. Let $X = [0, \infty)$ be endowed with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$ and $f : X \rightarrow X$ be defined by

$$fx = \begin{cases} 2x, & \text{if } x \in [0, 1] \\ 3x - \frac{3}{2}, & \text{if } x \in (1, \infty). \end{cases}$$

Define also $\alpha, \beta : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 1, & \text{if } x \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

and $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(a) = 2a$ for all $a \geq 0$.

We prove that Theorem 1.9 can be applied to f .

Proof. Proceeding as in the proof of Example 1.10, we deduce that f^{-1} is a twisted (α, β) -admissible mapping and that the conditions (i) and (ii) of Theorem 1.9 hold. Moreover, if $x, y \in [0, 1]$ and $p \geq 1$, we have $2(d(fx, fy)) + p \geq (m(x, y) + p)^{\alpha(x,y)\beta(x,y)}$. Otherwise $\alpha(x, y)\beta(x, y) = 0$ and (3) trivially holds. Then f is a twisted $(\alpha, \beta) - \psi$ -expansive mapping of type III and, by Theorem 1.9, f has a fixed point. Clearly, 0 and $\frac{3}{4}$ are two fixed points of f .

To ensure the uniqueness of the fixed point in Theorems 1.8 and 1.9 Kang et al. [11] consider the following condition:

(P): For all $u, v \in X$ with $u \neq v$, there exists $w \in X$ such that $\alpha(u, w) \geq 1$, $\alpha(v, w) \geq 1$ and $\beta(u, w) \geq 1$, $\beta(v, w) \geq 1$.

2. CYCLIC RESULTS

In this section, we show how is possible to apply the results of Kang et al. [11] for proving, in a natural way, some analogous fixed point results involving a cyclic mapping.

Definition 2.1. Let (X, d) be a metric space and A, B be two non-empty and closed subsets of X . Let $f : A \cup B \rightarrow A \cup B$ be a bijective mapping with $A \subseteq f(B)$ and $B \subseteq f(A)$ such that $\alpha(f^{-1}y, f^{-1}x) \geq 1$ if $\alpha(x, y) \geq 1$, where $x \in A$ and $y \in B$. Then f is said to be a

(i) cyclic $\alpha - \psi$ -expansive mapping of type I, if

$$\psi(d(fx, fy)) \geq \alpha(x, y)d(x, y),$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

(ii) cyclic $\alpha - \psi$ -expansive mapping of type II, if there is $0 < p \leq 1$ such that

$$(1 + p)^{\psi(d(fx, fy))} \geq (\alpha(x, y) + p)^{d(x, y)}$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

(iii) cyclic $\alpha - \psi$ -expansive mapping of type III, if there is $p \geq 1$ such that

$$\psi(d(fx, fy)) + p = (d(x, y) + p)^{\alpha(x, y)}$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

Now, we prove the following result for a continuous cyclic mapping.

Theorem 2.2. Let (X, d) be a complete metric space and A, B be two non-empty and closed subsets of X such that $\alpha : X \times X \rightarrow [0, \infty)$ and $f : A \cup B \rightarrow A \cup B$ be a bijective, continuous and generalized cyclic $\alpha - \psi$ -expansive mapping of type I or type II or type III. If there exists $x_0 \in A$ such that $\alpha(x_0, f^{-1}x_0) \geq 1$, then f has a fixed point in $A \cap B$.

Proof. Let $Y = A \cup B$ and $\beta : Y \times Y \rightarrow [0, \infty)$ be the function defined by

$$\beta(x, y) = \begin{cases} 1, & \text{if } x \in A \quad \text{and} \quad y \in B \\ 0, & \text{otherwise.} \end{cases}$$

Then (Y, d) is a complete metric space and f^{-1} is a twisted (α, β) -admissible mapping. Now, if $x_0 \in A$ is such that $\alpha(x_0, f^{-1}x_0) \geq 1$, then also $\beta(x_0, f^{-1}x_0) \geq 1$ and hence all the hypotheses of Theorem 1.8 hold with $X = Y$. Consequently, f has a fixed point in $A \cup B$, say z . Since $z \in A$ implies $z = f^{-1}z \in B$ and $z \in B$ implies $z = f^{-1}z \in A$, then $z \in A \cap B$.

Also for cyclic $\alpha - \psi$ -expansive mappings, we can omit the continuity condition as is shown in the following theorem:

Theorem 2.3. Let (X, d) be a complete metric space and A, B be two non-empty and closed subsets of X such that $\alpha : X \times X \rightarrow [0, \infty)$ and $f : A \cup B \rightarrow A \cup B$ be a bijective and cyclic $\alpha - \psi$ -expansive mapping of type I or type II or type III. Also suppose that the following conditions hold:

- (i) there exists $x_0 \in A$ such that $\alpha(x_0, f^{-1}x_0) \geq 1$;
- (ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(f^{-1}x, f^{-1}x_{2n}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then f has a fixed point in $A \cap B$.

Proof. Let $Y = A \cup B$ and define the function $\beta : Y \times Y \rightarrow [0, \infty)$ as in the proof of Theorem 2.2. Let $\{x_n\}$ be a sequence in Y such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x_{2n} \in A$ and $x_{2n+1} \in B$. Now, since B is closed, then $x \in B$ and hence $\alpha(f^{-1}x, f^{-1}x_{2n}) \geq 1$ and $\beta(f^{-1}x, f^{-1}x_{2n}) \geq 1$. We deduce that all the hypotheses of Theorem 1.9 are satisfied with $X = Y$ and hence f has a fixed point.

3. CYCLIC ORDERED RESULTS

By using the similar arguments to those presented in the previous section, we are able to obtain results in the setting of ordered complete metric spaces.

Definition 3.1. Let (X, d, \preceq) be an ordered metric space and A, B be two non-empty and closed subsets of X . Let $\alpha : X \times X \rightarrow [0, \infty)$ and $f : A \cup B \rightarrow A \cup B$ be a bijective mapping with $A \subseteq f(B)$ and $B \subseteq f(A)$ such that $\alpha(f^{-1}y, f^{-1}x) \geq 1$ if $\alpha(x, y) \geq 1$, where $x \in A$ and $y \in B$. Then f is said to be a

(i) cyclic ordered $\alpha - \psi$ -expansive mapping of type I, if

$$\psi(d(fx, fy)) \geq \alpha(x, y)d(x, y),$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

(ii) cyclic ordered $\alpha - \psi$ -expansive mapping of type II, if there is $0 < p \leq 1$ such that

$$(1 + p)\psi(d(fx, fy)) \geq (\alpha(x, y) + p)d(x, y)$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

(iii) cyclic ordered $\alpha - \psi$ -expansive mapping of type III, if there is $p \geq 1$ such that

$$\psi(d(fx, fy)) + p = (d(x, y) + p)\alpha(x, y)$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

Theorem 3.2. Let (X, d, \preceq) be an ordered complete metric space and A, B be two non-empty and closed subsets of X . Let $\alpha : X \times X \rightarrow [0, \infty)$ and $f : A \cup B \rightarrow A \cup B$ be a bijective, continuous and cyclic ordered $\alpha - \psi$ -expansive mapping of type I or type II or type III. If there exists $x_0 \in A$ such that $\alpha(x_0, f^{-1}x_0) \geq 1$ and $x_0 \preceq f^{-1}x_0$, then f has a fixed point in $A \cap B$.

Proof. Let $Y = A \cup B$ and $\beta : Y \times Y \rightarrow [0, \infty)$ be the function defined by

$$\beta(x, y) = \begin{cases} 0, & \text{if } x \in A, \ y \in B \text{ with } x \preceq y \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, (1)(respectively (2) or (3)) holds for all $x, y \in Y$. Let $\beta(x, y) \geq 1$ for $x, y \in X$, then $x \in A$ and $y \in B$ with $x \preceq y$. It follows that $f^{-1}x \in B$ and $f^{-1}y \in A$ with $f^{-1}y \preceq f^{-1}x$, since f is decreasing. Therefore $\beta(f^{-1}y, f^{-1}x) \geq 1$, implies that, f^{-1} is a twisted (α, β) -admissible mapping. Now, let $\alpha(x_0, f^{-1}x_0) \geq 1$ with $x_0 \in A$ and $x_0 \preceq f^{-1}x_0$. From $x_0 \in A$, we have $f^{-1}x_0 \in B$

with $x_0 \preceq f^{-1}x_0$, that is, $\beta(x_0, f^{-1}x_0) \geq 1$. Then all the conditions of Theorem 1.8 are satisfied with $X = Y$ and f has a fixed point in $A \cup B$, say z . Since $z \in A$ implies $z = f^{-1}z \in B$ and $z \in B$ implies $z = f^{-1}z \in A$, then $z \in A \cap B$.

Theorem 3.3. Let (X, d, \preceq) be an ordered complete metric space and A, B be two non-empty and closed subsets of X . Let $\alpha : X \times X \rightarrow [0, \infty)$ and $f : A \cup B \rightarrow A \cup B$ be a bijective, cyclic ordered $\alpha - \psi$ -expansive mapping of type I or type II or type III. Also suppose that the following conditions hold:

- (i) there exists $x_0 \in A$ such that $\alpha(x_0, f^{-1}x_0) \geq 1$ and $x_0 \preceq f^{-1}x_0$.
- (ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(f^{-1}x, f^{-1}x_{2n}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
- (iii) if $\{x_n\}$ is a sequence in X such that $x_{2n} \preceq x_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x_{2n} \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point in $A \cap B$.

Proof. Consider the complete metric space (Y, d) , where $Y = A \cup B$ and define the function $\beta : Y \times Y \rightarrow [0, \infty)$ as in the proof of Theorem 3.2. Let $\{x_n\}$ be a sequence in Y such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x_{2n} \in A$ and $x_{2n+1} \in B$ with $x_{2n} \preceq x_{2n+1}$. Since B is closed and by (iii), we deduce that $x \in B$ and $x_{2n} \preceq x$, that is, $\beta(f^{-1}x, f^{-1}x_{2n}) \geq 1$. Also $\alpha(f^{-1}x, f^{-1}x_{2n}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. We deduce that all the hypotheses of Theorem 1.9 are satisfied with $X = Y$ and hence f has a fixed point.

4. COUPLED FIXED POINT

Now, we shall show that the coupled fixed point theorems in complete metric spaces can also be derived from these results. Before proving the result, we recall the following definition due to Bhaskar and Lakshmikantham [12].

Definition 4.1. Let $f : X \times X \rightarrow X$ be a given mapping. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

Samet et al. [2] proved the following Lemma for the existence of coupled fixed point, which we

shall use in the proof of our main results.

Lemma 4.2. Let $F : X \times X \rightarrow X$ be a given mapping. Define the mapping $f : X \times X \rightarrow X \times X$ by

$$(4) \quad f(x, y) = (F(x, y), F(y, x)),$$

for all $(x, y) \in X \times X$. Then (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of f .

Now, we prove our main results.

Theorem 4.3. Let (X, d) be a complete metric space and $F : X \times X \rightarrow X$ be a given bijective mapping. Suppose that there exists $\psi \in \Psi$ and functions $\alpha, \beta : X^2 \times X^2 \rightarrow [0, \infty)$ such that

$$(5) \quad \psi(d(F(x, y), F(u, v))) \geq \frac{1}{2} \alpha((x, y), (u, v)) \beta((x, y), (u, v)) [d(x, u) + d(y, v)],$$

for all $(x, y), (u, v) \in X \times X$. Suppose also that

(i) for all $(x, y), (u, v) \in X \times X$, we have

$$\alpha((x, y), (u, v)) \Rightarrow \alpha(F^{-1}u, F^{-1}x) \geq 1,$$

$$\beta((x, y), (u, v)) \Rightarrow \beta(F^{-1}u, F^{-1}x) \geq 1;$$

(ii) there exists $(x_0, y_0) \in X \times X$ such that

$$\alpha((x_0, y_0), (a, b)) \geq 1 \text{ and } \alpha((b, a), (y_0, x_0)) \geq 1,$$

$$\beta((x_0, y_0), (a, b)) \geq 1 \text{ and } \beta((b, a), (y_0, x_0)) \geq 1,$$

$$\text{where } F^{-1}(x_0) = (a, b);$$

(iii) F is continuous.

Then F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Proof. For the proof of our result, we consider the mapping f given by (4) as a bijective mapping such that

$$f^{-1}(x, y) = F^{-1}x.$$

Also, consider the complete metric space (Y, ρ) , where $Y = X \times X$ and

$$\rho((x, y), (u, v)) = d(x, u) + d(y, v),$$

for all $(x, y), (u, v) \in Y$.

From (4), we have

$$(6) \quad \psi(d(F(x,y), F(u,v))) \geq \frac{1}{2} \alpha((x,y), (u,v)) \beta((x,y), (u,v)) [d(x,u) + d(y,v)]$$

and

$$(7) \quad \psi(d(F(v,u), F(y,x))) \geq \frac{1}{2} \alpha((v,u), (y,x)) \beta((v,u), (y,x)) [d(v,y) + d(u,x)].$$

Define the functions

$\eta_1, \eta_2 : Y \times Y \rightarrow [0, \infty)$ by

$$(8) \quad \eta_1((\mu_1, \mu_2), (v_1, v_2)) = \min\{\alpha((\mu_1, \mu_2), (v_1, v_2)), \alpha((v_2, v_1), (\mu_2, \mu_1))\}$$

and

$$(9) \quad \eta_2((\mu_1, \mu_2), (v_1, v_2)) = \min\{\beta((\mu_1, \mu_2), (v_1, v_2)), \beta((v_2, v_1), (\mu_2, \mu_1))\},$$

for all $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$.

Summing up the inequalities (6-7) and using (8-9), we get

$$(10) \quad \psi(d(F(\mu_1, \mu_2), F(v_1, v_2))) + \psi(d(F(v_2, v_1), F(\mu_2, \mu_1))) \geq \eta_1(\mu, v) \eta_2(\mu, v) \rho(\mu, v),$$

for all $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$.

Using the property $(\psi(a+b) = \psi(a) + \psi(b))$ of the function ψ , we obtain

$$(11) \quad \psi(\rho(f\mu, fv)) \geq \eta_1(\mu, v) \eta_2(\mu, v) \rho(\mu, v),$$

for all $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$.

Clearly, f is continuous and twisted $(\alpha, \beta) - \psi$ -expansive mapping.

Let $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$ such that $\eta_1(\mu, v) \geq 1, \eta_2(\mu, v) \geq 1$.

Using the condition (i), we obtain

$$\eta_1(f^{-1}v, f^{-1}\mu) \geq 1, \eta_2(f^{-1}v, f^{-1}\mu) \geq 1.$$

Then f^{-1} is twisted (η_1, η_2) -admissible.

Moreover, from the condition (ii) of the hypotheses of the theorem, we find that there exists $(x_0, y_0) \in Y$ such that $\eta_1((x_0, y_0), f^{-1}(x_0, y_0)) \geq 1$ and $\eta_2((x_0, y_0), f^{-1}(x_0, y_0)) \geq 1$.

So, we have transformed the problem to the complete metric space (Y, ρ) . Therefore, all the hypotheses of Theorem 1.8 are satisfied for twisted $(\alpha, \beta) - \psi$ -expansive mapping of type I, and so we deduce the existence of a fixed point of f as well as f^{-1} . Now, Lemma 4.2 gives us the existence of a coupled fixed point of F as well as F^{-1} .

Theorem 4.4. Let (X, d) be a complete metric space and $F : X \times X \rightarrow X$ be a given bijective mapping. Suppose that there exists $\psi \in \Psi$ and functions $\alpha, \beta : X^2 \times X^2 \rightarrow [0, \infty)$ such that

$$(12) \quad (1 + p)^{\psi(d(F(x,y), F(u,v)))} \geq \left(\frac{1}{2}\alpha((x,y), (u,v))\beta((x,y), (u,v)) + p\right)^{[d(x,u)+d(y,v)]},$$

for all $(x, y), (u, v) \in X \times X$ and $0 < p \leq 1$.

Suppose also that (i), (ii) and (iii) of Theorem 4.3 are satisfied.

Then F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Proof. For the proof of our result, we consider the mapping f given by (4) as a bijective mapping such that

$$f^{-1}(x, y) = F^{-1}x.$$

Also, consider the complete metric space (Y, ρ) , where $Y = X \times X$ and

$$\rho((x, y), (u, v)) = d(x, u) + d(y, v),$$

for all $(x, y), (u, v) \in Y$.

From (12), we have

$$(13) \quad (1 + p)^{\psi(d(F(x,y), F(u,v)))} \geq \left(\frac{1}{2}\alpha((x,y), (u,v))\beta((x,y), (u,v)) + p\right)^{[d(x,u)+d(y,v)]},$$

and

$$(14) \quad (1 + p)^{\psi(d(F(v,u), F(y,x)))} \geq \left(\frac{1}{2}\alpha((v,u), (y,x))\beta((v,u), (y,x)) + p\right)^{[d(v,y)+d(u,x)]}.$$

Also, we define the functions

$\eta_1, \eta_2 : Y \times Y \rightarrow [0, \infty)$ as given by (8) and (9).

From (13-14) and using (8-9), we get

$$(1+p)^{\psi(d(F(\mu_1, \mu_2), F(v_1, v_2))) + \psi(d(F(v_2, v_1), F(\mu_2, \mu_1)))} \geq (\eta_1(\mu, v)\eta_2(\mu, v) + p)^{\rho(\mu, v)},$$

for all $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$.

Using the property ($\psi(a+b) = \psi(a) + \psi(b)$) of the function ψ , we obtain

$$(15) \quad (1+p)^{\psi(\rho(f\mu, fv))} \geq (\eta_1(\mu, v)\eta_2(\mu, v) + p)^{\rho(\mu, v)},$$

for all $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$.

Clearly, f is continuous and twisted $(\alpha, \beta) - \psi$ -expansive mapping.

Let $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$ such that $\eta_1(\mu, v) \geq 1, \eta_2(\mu, v) \geq 1$.

Using the condition (i), we obtain

$$\eta_1(f^{-1}v, f^{-1}\mu) \geq 1, \eta_2(f^{-1}v, f^{-1}\mu) \geq 1.$$

Then f^{-1} is twisted (η_1, η_2) -admissible.

Moreover, from the condition (ii) of the hypotheses of the theorem, we find that there exists $(x_0, y_0) \in Y$ such that $\eta_1((x_0, y_0), f^{-1}(x_0, y_0)) \geq 1$ and $\eta_2((x_0, y_0), f^{-1}(x_0, y_0)) \geq 1$.

So, we have transformed the problem to the complete metric space (Y, ρ) . Therefore, all the hypotheses of Theorem 1.8 are satisfied for twisted $(\alpha, \beta) - \psi$ -expansive mapping of type II, and so we deduce the existence of a fixed point of f as well as f^{-1} . Now, Lemma 4.2 gives us the existence of a coupled fixed point of F as well as F^{-1} .

Theorem 4.5. Let (X, d) be a complete metric space and $F : X \times X \rightarrow X$ be a given bijective mapping. Suppose that there exists $\psi \in \Psi$ and functions $\alpha, \beta : X^2 \times X^2 \rightarrow [0, \infty)$ such that

$$(16) \quad \psi(d(F(x, y), F(u, v))) + \frac{p}{2} \geq (d(x, u) + d(y, v) + p)^{\frac{1}{2}\alpha((x, y), (u, v))\beta((x, y), (u, v))},$$

for all $(x, y), (u, v) \in X \times X$ and $p \geq 1$.

Suppose also that (i), (ii) and (iii) of Theorem 4.3 are satisfied.

Then F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Proof. For the proof of our result, we consider the mapping f given by (4) as a bijective mapping such that

$$f^{-1}(x, y) = F^{-1}x.$$

Also, consider the complete metric space (Y, ρ) , where $Y = X \times X$ and

$$\rho((x, y), (u, v)) = d(x, u) + d(y, v),$$

for all $(x, y), (u, v) \in Y$.

From (16), we have

$$(17) \quad \psi(d(F(x, y), F(u, v))) + \frac{p}{2} \geq (d(x, u) + d(y, v) + p)^{\frac{1}{2}\alpha((x, y), (u, v))\beta((x, y), (u, v))},$$

and

$$(18) \quad \psi(d(F(v, u), F(y, x))) + \frac{p}{2} \geq (d(v, y) + d(u, x) + p)^{\frac{1}{2}\alpha((v, u), (y, x))\beta((v, u), (y, x))},$$

Also, we define the functions

$$\eta_1, \eta_2 : Y \times Y \rightarrow [0, \infty) \text{ as given by (8) and (9).}$$

Summing up the inequalities (17-18) and using (8-9), we get

$$\psi(d(F(\mu_1, \mu_2), F(v_1, v_2))) + \psi(d(F(v_2, v_1), F(\mu_2, \mu_1))) + p \geq (\rho(\mu, v) + p)^{\eta_1(\mu, v)\eta_2(\mu, v)},$$

for all $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$.

Using the property $(\psi(a + b) = \psi(a) + \psi(b))$ of the function ψ , we obtain

$$(19) \quad \psi(\rho(f\mu, fv)) + p \geq (\rho(\mu, v) + p)^{\eta_1(\mu, v)\eta_2(\mu, v)},$$

for all $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$.

Clearly, f is continuous and twisted $(\alpha, \beta) - \psi$ -expansive mapping.

Let $\mu = (\mu_1, \mu_2), v = (v_1, v_2) \in Y$ such that $\eta_1(\mu, v) \geq 1, \eta_2(\mu, v) \geq 1$.

Using the condition (i), we obtain

$$\eta_1(f^{-1}v, f^{-1}\mu) \geq 1, \eta_2(f^{-1}v, f^{-1}\mu) \geq 1.$$

Then f^{-1} is twisted (η_1, η_2) -admissible.

Moreover, from the condition (ii) of the hypotheses of the theorem, we find that there exists

$(x_0, y_0) \in Y$ such that $\eta_1((x_0, y_0), f^{-1}(x_0, y_0)) \geq 1$ and $\eta_2((x_0, y_0), f^{-1}(x_0, y_0)) \geq 1$.

So, we have transformed the problem to the complete metric space (Y, ρ) . Therefore, all the hypotheses of Theorem 1.8 are satisfied for twisted $(\alpha, \beta) - \psi$ -expansive mapping of type III, and so we deduce the existence of a fixed point of f as well as f^{-1} . Now, Lemma 4.2 gives us the existence of a coupled fixed point of F as well as F^{-1} .

In what follows, we prove that Theorems 4.3-4.5 remains valid, if we replace the continuity condition of F with the following condition:

(M_1) If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1 \text{ and } \alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \geq 1;$$

$$\beta((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1 \text{ and } \beta((y_{n+1}, x_{n+1}), (y_n, x_n)) \geq 1,$$

$x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, then

$$\alpha(f^{-1}(x, y), f^{-1}(x_n, y_n)) \geq 1 \text{ and } \alpha(f^{-1}(y_n, x_n), f^{-1}(y, x)) \geq 1;$$

$$\beta(f^{-1}(x, y), f^{-1}(x_n, y_n)) \geq 1 \text{ and } \beta(f^{-1}(y_n, x_n), f^{-1}(y, x)) \geq 1,$$

for all $n \in \mathbb{N}$.

Theorem 4.6. If we replace the continuity of F in Theorems 4.3-4.5 by the condition M_1 , then the result holds true.

Proof. We employ the same notations of the proof of Theorem 4.3 (resp. 4.4 and 4.5). Let x_n, y_n be a sequence in Y such that $\eta((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1$ and $(x_n, y_n) \rightarrow (x, y)$ as $n \rightarrow \infty$. Using the condition M_1 , we have $\eta(f^{-1}(x, y), f^{-1}(x_n, y_n)) \geq 1$. Then all the hypotheses of Theorems 4.3-4.5 are satisfied. Thus, we deduce the existence of a fixed point of f that gives us from Lemma 4.2 the existence of a coupled fixed point of F .

To ensure the uniqueness of the coupled fixed point, we consider the following condition:

(P_1) For all $(x_1, y_1), (x_2, y_2) \in X \times X$, there exists $(x_3, y_3) \in X \times X$ such that

$$\alpha((x_1, y_1), (x_3, y_3)) \geq 1 \text{ and } \alpha((y_3, x_3), (y_1, x_1)) \geq 1;$$

$$\beta((x_1, y_1), (x_3, y_3)) \geq 1 \text{ and } \beta((y_3, x_3), (y_1, x_1)) \geq 1,$$

and

$$\alpha((x_2, y_2), (x_3, y_3)) \geq 1 \text{ and } \alpha((y_3, x_3), (y_2, x_2)) \geq 1;$$

$$\beta((x_2, y_2), (x_3, y_3)) \geq 1 \text{ and } \beta((y_3, x_3), (y_2, x_2)) \geq 1.$$

Theorem 4.7. Adding the condition (P_1) to the hypotheses of Theorems 4.3-4.5 (resp. 4.6) we

obtain the uniqueness of the coupled fixed point of F .

Proof. Clearly, under the hypothesis (P_1) , f and η satisfy the hypothesis (P) . Therefore, from Theorem 4.6 and Lemma 4.2, the result follows immediately.

Example 4.8. Let $X = \mathbb{N} \cup \{0\}$ be equipped with the usual metric for all $x, y \in X$. Then (X, d) is a complete metric space.

Define the mapping $F : X \times X \rightarrow X$ by

$$F(x, y) = 2^x y$$

Clearly, F is a continuous and bijective mapping.

Define $\alpha, \beta : X^2 \times X^2 \rightarrow [0, \infty)$ by

$$\alpha((x, y), (u, v)) = \beta((x, y), (u, v)) = \begin{cases} 1, & \text{if } x \geq u, \quad y \geq v \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show that for all $(x, y), (u, v) \in X \times X$, we have

$$\psi(d(F(x, y), F(u, v))) \geq \frac{1}{2} \alpha((x, y), (u, v)) \beta((x, y), (u, v)) [d(x, u) + d(y, v)].$$

Then (5) is satisfied with $\psi(a) = \frac{a}{2}$ for all $a \geq 0$. On the other hand, the condition (i) of Theorem 4.3 holds and the condition (ii) of the same Theorem is also satisfied with $(x_0, y_0) = (0, 0)$. All the required hypotheses of Theorem 4.3 are true and so we deduce the existence of a coupled fixed point of F . Here $(0, 0)$ is a coupled fixed point of F .

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

- [1] S. Banach, Sur les operations dans ensembles abstraits et leur application aux equations integrals, Fund. Math. 3 (1922), 133-181.
- [2] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha - \psi$ -contractive type mappings, Nonlinear Anal., Theory Meth. Appl. 75 (2012), 2154-2165.
- [3] S.Z. Wang, B.Y. Li, Z.M. Gao, K. Iseki, Some fixed point theorems on expansion mappings, Math. Japon. 29 (1984), 631-636.
- [4] B.E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257-290.
- [5] M.A. Khan, M.S. Khan, S. Sessa, Some theorems on expansion mappings and their fixed points, Demonstr. Math. 19 (1986), 673-683.

- [6] B.E. Rhoades, Some fixed point theorems for pair of mappings, *Jnanabha*, 15 (1985), 151-156.
- [7] T. Taniguchi, Common fixed point theorems on expansion type mappings on complete metric spaces, *Math. Japon.* 34 (1989), 139-142.
- [8] S.M. Kang, Fixed points for expansion mappings, *Math. Japon.* 38 (1993), 713-717.
- [9] P. Shahi, J. Kaur, S.S. Bhatia, Fixed point theorems for (ξ, α) -expansive mappings in complete metric spaces, *Fixed Point Theory Appl.* 2012 (2012), 157.
- [10] P. Salimi, C. Vetro, P. Vetro, Fixed point theorems for twisted $(\alpha, \beta) - \psi$ -contractive type mappings and applications, *Filomat*, 27(4) (2013), 605-615.
- [11] S. M. Kang, P. Nagpal, S. K. Garg, S. Kumar, Fixed point theorems for twisted $(\alpha, \beta) - \psi$ -expansive mappings in metric spaces, *Int. J. Math. Anal.* 37(9) (2015), 1805-1813.
- [12] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal., Theory Meth. Appl.* 65 (2006), 1379–1393.