FRACTIONAL ORDER EXPLICIT FINITE DIFFERENCE SCHEME FOR TIME FRACTIONAL RADON DIFFUSION EQUATION IN CHARCOAL MEDIUM

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Abstract: In this paper, we introduce the fractional order radon diffusion equation and develop explicit finite difference scheme for time fractional radon diffusion equation (TFRDE). Also, we discuss the stability and convergence of the scheme, as an application of this scheme, we obtain the numerical solutions of the test problem and it represented graphically.

Keywords: fractional calculus; finite difference; Caputo formula; Mathematica; convergence.

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1. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis which deals with the investigation and applications of differentiation and integration to arbitrary non-integer order. At present fractional calculus has been simulated by many applications in physics, engineering, bioscience,

applied mathematics etc. [1, 2, 6, 17, 18, 19, 20, 21, 22, 23, 24]. The analytical solution of fractional diffusion equation is very difficult to find hence researchers developed the finite difference schemes to find numerical solution [3, 4, 7, 8, 9, 10, 11]. Radon is a colorless, odorless, radioactive gas. It forms naturally from the decay of radioactive elements, such as uranium, which are found in different amounts in soil and rock throughout the world. Radon gas in the soil and rock can move into the air and into underground water and surface water. Radon is present outdoors and indoors. Due to dangerous nature of radon many researchers study the radon transport through soil, air, concrete, activated charcoal, etc., [5, 12, 13, 14, 15, 16]. The diffusion theory came from the famous physiologist Adolf Fick. He stated that the flux density J is proportional to the gradient of concentration. This gives,

$$J = -D\frac{\partial R}{\partial t} \tag{1.1}$$

where J is the radon flux density is diffusion coefficient, $\dfrac{\partial R}{\partial t}$ is gradient of radon concentration and D is diffusivity coefficient of radon. Now the change in concentration to change in time and position is stated by the Fick's second law which is the extension of Fick's first law, that gives,

$$\frac{\partial R(z,t)}{\partial t} = D\frac{\partial^2 R(z,t)}{\partial z^2} - \lambda R(z,t) \tag{1.2}$$

where $\lambda = 2.1 \times 10^{-6}\text{s}^{-1}$ is the decay constant.

In this paper, we develop the time fractional explicit finite difference method for fractional order radon diffusion equation. We consider the following time fractional radon diffusion equation [TFRDE].

$$\frac{\partial^\alpha R(z,t)}{\partial t^\alpha} = D\frac{\partial^2 R(z,t)}{\partial z^2} - \lambda R(z,t), 0 < z < L, 0 \le \alpha \le 1, t \ge 0 \tag{1.3}$$

initial conditions: $R(z, 0) = 0, 0 < z < L$ \hfill (1.4)

boundary conditions: $R(0,t) = R_0$ and $\dfrac{\partial R(z,t)}{\partial t} = 0, t \ge 0$ \hfill (1.5)

Definition 1.1   *The Caputo time-fractional derivative of order α, (0 ≤ α ≤ 1) is defined by,*

$$\frac{\partial^\alpha R(z,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial R(z,\eta)}{\partial \eta} \frac{d\eta}{(t-\eta)^\alpha}; 0 < \alpha < 1$$

$$= \frac{\partial R(z,t)}{\partial t}; \quad \alpha = 1$$

We organize the paper as follows: In section 2, we develop explicit finite difference scheme for time fractional radon diffusion equation (TFRDE). The section 3, is devoted for stability of the solution of the scheme and the convergence of the approximated finite difference scheme is proved in section 4. In the last section we solved some text problems and their solutions are represented graphically by mathematical software Mathematica.

2. FINITE DIFFERENCE SCHEME

We consider the following time fractional radon diffusion equation [TFRDE],

$$\frac{\partial^\alpha R(z,t)}{\partial t^\alpha} = D\frac{\partial^2 R(z,t)}{\partial z^2} - \lambda R(z,t), 0 < z < L, 0 \le \alpha \le 1, t \ge 0 \tag{2.1}$$

Initial conditions: $R(z,0) = 0, \ 0 < z < L$ \hfill (2.2)

Boundary conditions: $R(0, t) = R_0, \ t \ge 0$

$$R(L,t) = 0 \ \text{ or } \ \frac{\partial R(L,t)}{\partial t} = 0, t \ge 0 \tag{2.3}$$

We introduce the finite difference approximation to discretize the time fractional derivative.

We define,

$\quad t_k = kT; \ k = 0, 1, 2, ..., N$ and

$\quad z_i = ih \ ; \ i = 0, 1, 2, ..., M$

where $\ \tau = \dfrac{T}{N}$ and $h = \dfrac{L}{M}$

Let (z_i, t_k); i= 0, 1, 2, ..., M and k = 0, 1, 2, ..., N be the exact solution of the TFRDE (2.1)-(2.3) at the mesh point (z_i, t_k). Let $\ R_i^k\ $ be the numerical approximation of the point R(ih, kτ).

In the TFRDE (2.1) – (2.3) time fractional derivative is approximated in the caputo sense by following scheme,

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$$\frac{\partial^\alpha R(z_i,t_k)}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)} \int\limits_0^{t_{k+1}} \frac{1}{(t_{k+1}-\eta)^\alpha} \cdot \frac{\partial R(z_i,\eta)}{\partial \eta} d\eta$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{R(z_i,t_{j+1})-R(z_i,t_j)}{\tau} \int\limits_{j\tau}^{(j+1)\tau} \frac{d\eta}{(t_{k+1}-\eta)^\alpha} + o(\tau)$$

Put $t_{k+1}-\eta=\xi \Rightarrow -d\eta=d\xi$

When $\eta = j\tau \Rightarrow \xi = t_{k+1} - j\tau = (k+1)\tau - j\tau = (k+1-j)\tau$

Again, when $\eta = (j+1)\tau \Rightarrow \xi = (k-j)\tau$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{R(z_i,t_{j+1})-R(z_i,t_j)}{\tau} \int\limits_{(k-j)\tau}^{(k+1-j)\tau} \frac{d\xi}{\xi^\alpha} + o(\tau)$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{R(z_i,t_{k+1-j})-R(z_i,t_{k-j})}{\tau} \int\limits_{j\tau}^{(j+1)\tau} \frac{d\xi}{\xi^\alpha} + o(\tau)$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{R(z_i,t_{k+1-j})-R(z_i,t_{k-j})}{\tau} \left[\frac{(j+1)^{1-\alpha} - j^{1-\alpha}}{1-\alpha} \right] \tau^{1-\alpha} + o(\tau)$$

$$= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[R(z_i,t_{k+1})-R(z_i,t_k) \right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^k \left[R(z_i,t_{k+1-j})-R(z_i,t_{k-j}) \right] \times \left[(j+1)^{1-\alpha} - j^{1-\alpha} \right] + o(\tau)$$

$$= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[R_i^{k+1} - R_i^k \right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^k b_j \left[R_i^{k-j+1} - R_i^{k-j} \right] + o(\tau)$$

Where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}; j = 1,2,...,N$

Now, for approximating the second order space derivative, we adopt a symmetric second order difference quotient in space at time level $t = t_k$

$$\frac{\partial^2 R(z_i,t_k)}{\partial z^2} = \frac{R(z_{i-1},t_k)-2R(z_i,t_k)+R(z_{i+1},t_k)}{h^2}$$

$$\frac{\partial^2 R(z_i,t_k)}{\partial z^2} = \frac{R_{i-1}^k - 2R_i^k + R_{i+1}^k}{h^2}$$

Therefore, substituting in equation (2.1), we get

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[R_i^{k+1} - R_i^k\right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k} b_j\left[R_i^{k-j+1} - R_i^{k-j}\right]$$

$$= D\left[\frac{R_{i-1}^k - 2R_i^k + R_{i+1}^k}{h^2}\right] - \lambda R(z_i, t_k)$$

Where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}; j = 1, 2, ..., N$

$$\left[R_i^{k+1} - R_i^k\right] + \sum_{j=1}^{k} b_j\left[R_i^{k-j+1} - R_i^{k-j}\right]$$

$$= \frac{D\Gamma(2-\alpha)\cdot\tau^\alpha}{h^2}\left[R_{i-1}^k - 2R_i^k + R_{i+1}^k\right] - \lambda\Gamma(2-\alpha)\tau^\alpha R(z_i, t_k)$$

Put $r = \frac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}$ and $\mu = \lambda\Gamma(2-\alpha)\tau^\alpha$

We have,

$$\left[R_i^{k+1} - R_i^k\right] + \sum_{j=1}^{k} b_j\left[R_i^{k-j+1} - R_i^{k-j}\right] = r\left[R_{i-1}^k - 2R_i^k + R_{i+1}^k\right] - \mu R_i^k \qquad (2.4)$$

After Simplification,

$$R_i^{k+1} = rR_{i-1}^k + (1 - 2r - \mu)R_i^k + rR_{i+1}^k - \sum_{j=1}^{k} b_j\left[R_i^{k-j+1} - R_i^{k-j}\right]$$

$$= rR_{i-1}^k + (1 - 2r - \mu)R_i^k + rR_{i+1}^k + b_1\left[R_i^{k-1} - R_i^k\right] + b_2\left[R_i^{k-2} - R_i^{k-1}\right]$$

$$+ b_3\left[R_i^{k-3} - R_i^{k-2}\right] + ... + b_{k-1}\left[R_i^1 - R_i^2\right] + b_k\left[R_i^0 - R_i^1\right]$$

$$R_i^{k+1} = rR_{i-1}^k + (1 - 2r - \mu - b_1)R_i^k + rR_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})R_i^{k-j} + b_k R_i^0 \qquad (2.5)$$

Where, $r = \frac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}, \mu = \lambda\Gamma(2-\alpha)\tau^\alpha$

$b_j = (j+1)^{1-\alpha} - j^{1-\alpha}; j = 1, 2, ...N$, $i = 0, 1, ..., M$ and $k = 0, 1, ..., N$

The initial condition is approximated as

$R_i^0 = 0, i = 0, 1, ..., M$

For the two boundary points Z_0 and Z_M, the corresponding discretization schemes are

$$R_0^k = 0 \text{ and } \frac{\partial R(L,t)}{\partial z} = 0 \text{ implies}$$

$$\frac{R_{M+1}^k - R_{M-1}^k}{2h} = 0 \Rightarrow R_{M+1}^k = R_{M-1}^k$$

Therefore, from equation (2.4) for $k = 0$ the fractional approximated initial boundary value problem is

$$R_i^1 - R_i^0 = r\left[R_{i-1}^0 - 2R_i^0 + R_{i+1}^0\right] - \mu R_i^0$$

$$R_i^1 = rR_{i-1}^0 + (1 - 2r - \mu)R_i^0 + rR_{i+1}^0$$

Therefore, the complete fractional approximated initial boundary value problem is,

$$R_i^1 = rR_{i-1}^0 + (1 - 2r - \mu)R_i^0 + rR_{i+1}^0 \text{ for } k = 0 \tag{2.6}$$

$$R_i^{k+1} = rR_{i-1}^k + (1 - 2r - \mu - b_1)R_i^k + rR_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})R_i^{k-j} + b_k R_i^0, \text{ for } k \geq 1 \tag{2.7}$$

With initial conditions: $R_i^0 = 0; i = 0, 1, 2, \ldots M$ \hfill (2.8)

Boundary conditions: $R_0^k = 0$ and $\dfrac{\partial R(L,t)}{\partial z} = 0$

$$R_{M+1}^k = R_{M-1}^k; k = 0, 1, 2, \ldots \tag{2.9}$$

Where, $r = \dfrac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}, \mu = \lambda\Gamma(2-\alpha)\tau^\alpha$

$$b_j = (j+1)^{1-\alpha} - j^{1-\alpha}; j = 1, 2, 3, \ldots, k, \ i = 0, 1, 2, \ldots, M \text{ and } k = 0, 1, 2, \ldots$$

The problem $(2.6) - (2.9)$ is a complete discretization of the problem $(2.1) - (2.3)$

The fractional approximated initial boundary value problem $(2.6) - (2.9)$ can be written in the following matrix equation form as follows,

Further more,

for, $k = 0$, $R_i^1 = rR_{i-1}^0 + (1 - 2r - \mu)R_i^0 + rR_{i+1}^0$

for, $i = 1$, $\quad R_1^1 = rR_0^0 + (1 - 2r - \mu) R_1^0 + rR_2^0$

for, $i = 2$, $\quad R_2^1 = rR_1^0 + (1 - 2r - \mu) R_2^0 + rR_3^0$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

for, $i = M$, $\quad R_M^1 = rR_{M-1}^0 + (1 - 2r - \mu) R_M^0 + rR_{M+1}^0$

$$= 2rR_{M-1}^0 + (1 - 2r - \mu) R_M^0 \qquad\qquad \left(\because R_{M+1}^k = R_{M-1}^k \right)$$

$$
\begin{bmatrix} R_1^1 \\ R_2^1 \\ R_3^1 \\ | \\ | \\ R_M^1 \end{bmatrix}
=
\begin{bmatrix}
1-2r-\mu & r & - & \cdots & - & - \\
r & 1-2r-\mu & r & - & - & - \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
- & - & - & \cdots & 2r & 1-2r-\mu
\end{bmatrix}
\begin{bmatrix} R_1^0 \\ R_2^0 \\ R_3^0 \\ | \\ | \\ R_M^0 \end{bmatrix}
+
\begin{bmatrix} rR_0^0 \\ 0 \\ | \\ | \\ | \\ 0 \end{bmatrix}
$$

$$R^1 = AR^0 + R' \qquad \text{for } k = 0$$

$$
A =
\begin{bmatrix}
1-2r-\mu & r & - & \cdots & - & - \\
r & 1-2r-\mu & r & - & - & - \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
| & | & | & | & | & | \\
- & - & - & \cdots & 2r & 1-2r-\mu
\end{bmatrix}
$$

$$
R^1 = \begin{bmatrix} R_1^1 \\ R_2^1 \\ | \\ | \\ | \\ R_M^0 \end{bmatrix}
\qquad
R^0 = \begin{bmatrix} R_1^0 \\ R_2^0 \\ | \\ | \\ | \\ R_M^0 \end{bmatrix}
\qquad
R' = \begin{bmatrix} rR_0^0 \\ 0 \\ | \\ | \\ | \\ 0 \end{bmatrix} = 0
$$

Further more, from equation (2.7)

for, i = 1 $R_1^{k+1} = rR_0^k + \left(1 - 2r - \mu - b_1\right)R_1^k + rR_2^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)R_1^{k-j} + b_k R_1^0$

for, i = 2 $R_2^{k+1} = rR_1^k + \left(1 - 2r - \mu - b_1\right)R_2^k + rR_3^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)R_2^{k-j} + b_k R_2^0$

for, i = 3 $R_3^{k+1} = rR_2^k + \left(1 - 2r - \mu - b_1\right)R_3^k + rR_4^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)R_3^{k-j} + b_k R_3^0$

$\vdots$ $\qquad$ $\vdots$ $\qquad$ $\vdots$ $\qquad$ $\vdots$

$\vdots$ $\qquad$ $\vdots$ $\qquad$ $\vdots$

for, i = M $\qquad R_M^{k+1} = rR_{M-1}^k + \left(1 - 2r - \mu - b_1\right)R_M^k + rR_{M+1}^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)R_M^{k-j} + b_k R_M^0$

$$= 2rR_{M-1}^k + \left(1 - 2r - \mu - b_1\right)R_M^k + rR_{M+1}^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)R_M^{k-j} + b_k R_M^0$$

$$\begin{bmatrix} R_1^{k+1} \\ R_2^{k+1} \\ | \\ | \\ | \\ | \\ R_M^{k+1} \end{bmatrix} = \begin{bmatrix} 1-2r-\mu-b_1 & r & - & \cdots & - & & - \\ r & 1-2r-\mu-b_1 & r & - & - & & - \\ | & | & | & | & | & & | \\ | & | & | & | & | & & | \\ | & | & | & | & | & & | \\ | & | & | & | & | & & | \\ - & - & & - & \cdots & 2r & 1-2r-\mu-b_1 \end{bmatrix} \begin{bmatrix} R_1^k \\ R_2^k \\ | \\ | \\ | \\ | \\ R_M^k \end{bmatrix}$$

$$+ \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)\begin{bmatrix} R_1^{k-j} \\ R_2^{k-j} \\ | \\ | \\ | \\ R_M^{k-j} \end{bmatrix} + b_k \begin{bmatrix} R_1^0 \\ R_2^0 \\ | \\ | \\ | \\ R_M^0 \end{bmatrix} + \begin{bmatrix} rR_0^0 \\ 0 \\ 0 \\ | \\ | \\ 0 \end{bmatrix}$$

$$R^{k+1} = BR^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)R^{k-j} + b_k R^0 + R'$$

Where, $R^k = \left[R_1^k, R_2^k, ..., R_M^k\right]^T$

Therefore using the above equations the fractional approximate initial boundary value problem (2.6) – (2.9) can be written in the matrix form as follows:

$$R^1 = AR^0 + R'; \text{ for } k = 0 \tag{2.10}$$

$$R^{k+1} = BR^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})R^{k-j} + b_k R^0 + R'; \text{ for } k \geq 1 \tag{2.11}$$

Where $R^k = \left[R_1^k, R_2^k, ..., R_M^k\right]^T; k = 0, 1, 2, ..., N$

$$r = \frac{D\Gamma(2-\alpha)\tau^\alpha}{h^2}, \mu = \lambda\Gamma(2-\alpha)\tau^\alpha, b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$$

$j = 1, 2, ..., k$

$$A = \begin{bmatrix} 1-2r-\mu & r & - & \cdots & - & - \\ r & 1-2r-\mu & r & - & - & - \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ - & - & - & \cdots & 2r & 1-2r-\mu \end{bmatrix}$$

$$B = \begin{bmatrix} 1-2r-\mu-b_1 & r & - & \cdots & - & - \\ r & 1-2r-\mu-b_1 & r & - & - & - \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ - & - & - & \cdots & 2r & 1-2r-\mu-b_1 \end{bmatrix}$$

$$R' = 0 = \begin{bmatrix} rR_0^0 \\ | \\ | \\ | \\ | \\ | \\ 0 \end{bmatrix}$$

$$R^1 = \left[R_1^1, R_2^1, R_3^1, ..., R_M^1\right]^T; R^0 = \left[R_1^0, R_2^0, R_3^0, ..., R_M^0\right]^T$$

$$R^k = \left[R_1^k, R_2^k, R_3^k, ..., R_M^k\right]^T; R' = \left[rR_0^0, 0, 0, ..., 0\right]^T$$

$$r = \frac{D\Gamma(2-\alpha)\tau^{\alpha}}{h^2}; \mu = \lambda\Gamma(2-\alpha)\tau^{\alpha};$$

$$b_j = (j+1)^{1-\alpha} - j^{1-\alpha}; j = 1,2,...,k, \quad i = 0,1,2,...,M \text{ and } k = 0,1,2,...,N$$

The above system of equation is solved by using mathematica software.

3. STABILITY

Theorem 3.1

The solution of the implicit finite difference scheme (2.6) – (2.9) for time fractional Radon diffusion equation (2.1) – (2.3), is stable, when

$$r \le \min\left\{\frac{2-\mu}{3}, \frac{2-\mu}{4} \middle| 0 \le \mu \le 1\right\}$$

Proof:

Let λ_i be an eigenvalue of matrix A to linear system of equations (2.6) – (2.9) so that,

$Az_i = \lambda z_i$ for some nonzero vector z_i choose i such that

$|z_i| = \max\{|z_j| : j = 1,2,...,k-1\}$ then

$$\sum_{j=1}^{k-1} a_{ij}z_j = \lambda z_i \text{ and therefore, } \lambda = a_{ii} + \sum_{\substack{j=1 \\ j \ne i}}^{k-1} a_{ij}\frac{z_j}{z_i} \tag{3.1}$$

Substituting the values of a_{ij} in to (3.1), we get

1) When, $i = 1$

$$\lambda = a_{11} + \sum_{\substack{j=2 \\ j \ne 1}}^{k-1} a_{ij}\frac{z_j}{z_i}$$

$$\lambda = 1 - 2r - \mu + a_{12}\frac{z_2}{z_1}$$

$$\le 1 - 2r - \mu + r\frac{z_2}{z_1} \le 1 - 2r - \mu + r \le 1 - r - \mu \le 1$$

$$\therefore \quad \lambda \le 1$$

$$\therefore \quad \lambda = 1 - 2r - \mu + r \cdot \frac{z_2}{z_1} \geq 1 - 2r - \mu - r \frac{z_2}{z_1} \geq 1 - 2r - \mu - r \geq -1$$

$$\therefore \quad \lambda \geq -1$$

When $\quad 1 - 2r - \mu - r \geq -1$

$$1 - 3r - \mu \geq -1$$

$$\therefore \quad 3r \leq 2 - \mu$$

$$r \leq \frac{2 - \mu}{3}$$

2) When $\quad 2 \leq i \leq M - 1$

$$\because \quad i = 2$$

$$\lambda = a_{22} + \sum_{\substack{j=1 \\ j \neq 2}}^{k-1} a_{ij} \frac{z_j}{z_i}$$

$$\lambda = 1 - 2r - \mu + a_{21} \frac{z_1}{z_2} + a_{23} \frac{z_3}{z_2}$$

$$\geq 1 - 2r - \mu - r - r \geq 1 - 4r - \mu \geq -1$$

When $\quad 1 - 4r - \mu \geq -1$

$$4r \leq 2 - \mu \Rightarrow r \leq \frac{2 - \mu}{4} \quad \left(\because 0 \leq \mu \leq 1 \right)$$

$$\therefore \quad -1 \leq \lambda \leq 1 \text{ for } 2 \leq i \leq M - 1 \text{ when } r \leq \frac{2 - \mu}{4}$$

3) When $\quad i = M$

$$\lambda = a_{MM} + \sum_{\substack{j=1 \\ j \neq M}}^{k-1} a_{ij} \frac{z_j}{z_i}$$

$$\lambda = a_{MM} + a_{M1} \frac{z_1}{z_M} + a_{M2} \frac{z_2}{z_M} + ... + a_{M(k-1)} \frac{z_{k-1}}{z_M}$$

$$= 1 - 2r - \mu + 0 + 0 + ... + 2r$$

$$\leq 1 - \mu \leq 1$$

$$\lambda_s \geq 1 - 2r - \mu - 2r \geq -1$$

When $\quad 1 - 4r - \mu \geq -1 \quad$ implies that

$$4r \leq 2 - \mu$$

$$r \leq \frac{2 - \mu}{4}$$

Therefore, for $\ 1 \leq i \leq M$, we get

$$-1 \leq \lambda_s \leq 1$$

When $\quad r \leq \min\left\{\dfrac{2-\mu}{3}, \dfrac{2-\mu}{4} 0 \leq \mu \leq 1\right\}$

Implies, $\ |\lambda_s| \leq 1$ when $r \leq \min\left\{\dfrac{2-\mu}{3}, \dfrac{2-\mu}{4}\right\}$

Therefore $\ \|A\|_2 = \max|\lambda_s| \leq 1$

$$1 \leq i \leq M$$

That is $\ \|A\|_2 \leq 1$ when $r \leq \min\left\{\dfrac{2-\mu}{3}, \dfrac{2-\mu}{4}\right\}$

Therefore from equation (2.10), we have

$$\left\|R^1\right\|_2 = \left\|AR^0\right\|_2$$

$$\leq \|A\|_2 \left\|R^0\right\|_2$$

Thus $\ \left\|R^1\right\|_2 \leq \left\|R^0\right\|_2$, true for $k = 0$

We assume that $\ \left\|R^n\right\|_2 \leq \left\|R^0\right\|_2$, when $k \leq M$ is true so for $k = M + 1$ we have to prove that

$$\left\|R^{n+1}\right\|_2 \leq \left\|R^0\right\|_2$$

for $\ \|B\|_2$, we have for $1 \leq i \leq M$ the eigen values of B are given by

1) when, $\quad i = 1$

$$\lambda = b_{11} + \sum_{\substack{j=2 \\ j \neq 1}}^{k-1} b_{ij} \frac{z_j}{z_i}$$

$$\lambda = 1 - 2r - \mu - b_1 + b_{12} \frac{z_2}{z_1} \le 1 - 2r - \mu - b_1 + r \le 1 - r - \mu - b_1 \le 1$$

Also, $\quad \lambda \ge 1 - 2r - \mu - b_1 - b_{12} \dfrac{z_2}{z_1} \ge 1 - 2r - \mu - b_1 - r \ge -1$

$\therefore \quad \lambda \ge -1, \text{ when } 1 - 3r - \mu - b_1 \ge -1$

$\therefore \quad 3r \le 2 - \mu - b_1 \Rightarrow r = \dfrac{2 - \mu - b_1}{3}$

2) When $\quad 2 \le i \le M - 1$

$$\lambda = b_{11} + \sum_{\substack{j=1 \\ j \ne 2}}^{k-1} b_{ij} \frac{z_i}{z_j}$$

$$= 1 - 2r - \mu - b_1 + b_{21} \frac{z_1}{z_2} + b_{23} \frac{z_3}{z_2}$$

$$\le 1 - 2r - \mu - b_1 + r + r \le 1 - \mu - b_1 < 1$$

$\therefore \quad \lambda \le 1$

Also, $\quad \lambda \ge 1 - 2r - \mu - b_1 - r - r \ge 1 - 4r - \mu - b_1 \ge -1$

$\because \quad \lambda \ge -1 \text{ when } 1 - 4r - \mu - b_1 \ge -1$

$\qquad \therefore \quad 4r \le 2 - \mu - b_1$

$$\qquad r \le \frac{2 - \mu - b_1}{4}$$

$\therefore \quad |\lambda| \le 1 \text{ for } 2 \le i \le M \text{ where } r \le \dfrac{2 - \mu - b_1}{4}$

3) when $\quad i = M$

$$\lambda = b_{MM} + \sum_{\substack{j=1 \\ j \ne M}}^{k-1} b_{ij} \frac{z_j}{z_i}$$

$$= 1 - 2r - \mu - b_1 + 0 + \ldots + 2r \frac{z_{k-1}}{z_M}$$

$$\le 1 - 2r - \mu - b_1 + 2r \le 1 - \mu - b_1 < 1$$

$\therefore \quad \lambda \leq 1$

Also, $\quad \lambda_s \geq 1 - 2r - \mu - b_1 - 2r \geq -1$

$\therefore \quad \lambda \geq -1$ when $1 - 4r - \mu - b_1 \geq -1$

$$4r \leq 2 - \mu - b_1$$

$$r \leq \frac{2 - \mu - b_1}{4}$$

Therefore, for $1 \leq i \leq M$ we get

$$-1 \leq \lambda_s \leq 1$$

When $\quad r \leq \min\left\{\frac{2 - \mu - b_1}{3}, \frac{2 - \mu - b_1}{4} \middle| 0 \leq \mu \leq 1\right\}$

Therefore $\quad \|B\|_2 = \max_{1 \leq i \leq M-1} |\lambda s| \leq 1$

$$\Rightarrow \quad \|B\|_2 \leq 1$$

Hence, $\quad \left\|R^{k+1}\right\|_2 = \left\|BR^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)R^{k-j} + b_k R^0\right\|_2$

$$\leq \|B\|_2 \left\|R^k\right\|_2 + \left(b_1 - b_2 + b_2 - b_3 + \ldots + b_{k-1} - b_k\right)\left\|R^{k-j}\right\|_2 + b_k \left\|R^0\right\|_2$$

$$\leq \left(1 - b_1 + b_1 - b_2 + b_2 - b_3 + \ldots + b_{k-1} - b_k + b_k\right)\left\|R^0\right\|_2$$

$$\leq \left\|R^0\right\|_2$$

That is, result is true for k = n + 1, Hence by induction $\left\|R^k\right\|_2 \leq \left\|R^0\right\|_2$

Therefore, this shows that the scheme is stable when

$$r \leq \min\left\{\frac{2 - \mu - b_1}{3}, \frac{2 - \mu - b_1}{4} \middle| 0 \leq \mu \leq 1\right\}.$$

4. CONVERGENCE

Theorem 4.1:

Let $\bar{R}^k$ be the exact solution of the TFRDE and R^k be the approximate solution of the TFRDE

then R^k converges to $\bar{R}^k$ as (h, τ) → (0, 0) when

$$r \leq \min\left\{\frac{2-\mu-b_1}{3}, \frac{2-\mu-b_1}{4} \middle| 0 \leq \mu \leq 1\right\}$$

Proof: Let, $R^k = \left[R_1^k, R_2^k, ..., R_M^k\right]^T$

$$\bar{R}^k = \left[\bar{R}_1^k, \bar{R}_2^k, ..., \bar{R}_M^k\right]^T$$

then $E^k = \bar{R}^k - R^k$

let us assume that,

$$\left|e_l^k\right| = \max_{1 \leq i \leq M-1} \left|e_i^k\right| = \left\|E^k\right\|_\infty \text{ for } l = 1, 2,...$$

and $T_l^k = \max_{1 \leq i \leq M-1} \left|T_i^k\right| = h^2 o(\tau + h^2)$

for $k = 1$, we have

$$\left|e_l^1\right| = \left|re_{i-1}^0 + (1-2r-\mu)e_i^0 + re_{i+1}^0\right|$$

$$\leq \left|re_{i-1}^0\right| + (1-2r-\mu)\left|e_i^0\right| + \left|re_{i+1}^0\right| + \left|T_i^1\right|$$

when, $r \leq \min\left\{\frac{2-\mu-b_1}{3}, \frac{2-\mu-b_1}{4} \middle| 0 \leq \mu \leq 1\right\}$

$$\left|e_l^1\right| \leq r\left|e_l^0\right| + (1-2r-\mu)\left|e_l^0\right| + r\left|e_l^0\right| + \left|T_l^1\right|$$

$$\leq (r+1-2r-\mu+r)\left|e_l^0\right| + \left|T_l^1\right|$$

$$\leq \left|e_l^0\right| + \left|T_l^1\right|$$

This implies,

$$\left\|E^1\right\|_\infty \leq \left\|E^0\right\|_\infty + r\left|T_l^1\right|$$

$$= \left\|E^0\right\|_\infty + rh^2 o(\tau + h^2)$$

$$\leq \left\|E^0\right\|_\infty + \tau^\alpha \Gamma(2-\alpha) o(\tau + h^2)$$

That is, the result hold for n = 1

For, n = k, we assume,

$$\left\|E^k\right\|_\infty \le \left\|E^0\right\|_\infty + kh^2 o\left(\tau + h^2\right)$$

For, n = k + 1, we have,

$$\left\|E_l^{k+1}\right\| \le \left\|re_{i-1}^k + \left(1-2r-\mu-b_1\right)e_i^k + re_{i+1}^k + \sum_{j=1}^{k-1}\left(b_j - b_{j+1}\right)e_i^{k-j} + b_k e_i^0\right\| + \left\|T_l^k\right\|$$

When r ≤ min $\left\{\dfrac{2-\mu-b_1}{3}, \dfrac{2-\mu-b_1}{4} \middle| 0 \le \mu \le 1\right\}$

$$\left|e_l^{k+1}\right| \le r\left|e_l^k\right| + \left(1-2r-\mu-b_1\right)\left|e_l^k\right| + r\left|e_l^k\right|$$

$$+ \left(b_1 - b_2 + b_2 + \ldots + b_{k-1} - b_k\right)\left|e_l^k\right| + b_k\left|e_l^k\right| + r\left|T_l^k\right|$$

$$\le \left(r + 1 - 2r - \mu - b_1 + r + b_1 - b_k + b_k\right)\left|e_l^k\right| + r\left|T_l^k\right|$$

$$\le \left|e_l^k\right| + r\left|T_l^k\right|$$

$$\left\|E^{k+1}\right\| \le \left\|E^k\right\|_\infty + r\left|T_l^k\right|$$

$$\le \left\|E^0\right\|_\infty + krh^2 o\left(\tau + h^2\right) + r\left|T_l^k\right|$$

$$\le \left\|E^0\right\|_\infty + k\tau^\alpha \Gamma(2-\alpha)o\left(\tau + h^2\right) + \tau^\alpha \Gamma(2-\alpha)$$

$$\left\|E^{k+1}\right\|_\infty \le \left\|E^0\right\|_\infty + (k+1)\tau^\alpha \Gamma(2-\alpha)o\left(\tau + h^2\right)$$

Therefore, we conclude that if we assume

$$r \le \min\left\{\frac{2-\mu-b_1}{3}, \frac{2-\mu-b_1}{4} \middle| 0 \le \mu \le 1\right\}$$

then $\left\|E^k\right\|_\infty \to 0$

as τ → 0, h → 0 which result in the convergence of R_i^k to R(z$_i$, t$_k$)

Hence the proof is complete.

5. NUMERICAL SOLUTION

In this section, we obtain the approximated solution of time fractional radon diffusion equation with initial and boundary conditions. To obtain the numerical solution of the time fractional radon

diffusion equation (TFRDE) by the finite difference scheme, it is important to use some analytical

model. Therefore, we present an example to demonstrate that TFRDE can be applied to simulate

behavior of a fractional diffusion equation by using Mathematica Software.

We consider the following, dimensionless time fractional radon diffusion equation with suitable initial and boundary conditions,

$$\frac{\partial^\alpha R(z,t)}{\partial t^\alpha} = D\frac{\partial^2 R(z,t)}{\partial z^2} - \lambda R(z,t)$$

$0 < z < L, 0 \le \alpha \le 1, t \ge 0$

Initial conditions: $R(z,0) = 0, 0 < z < L$

Boundary conditions: $R(0, t) = R_0, t \ge 0$

$$R(L,t) = 0 \text{ or } \frac{\partial R(L,t)}{\partial t} = 0, t \ge 0$$

with the radon diffusion coefficient

$D = 1.43 \times 10^6$ Bq/m^3. The numerical solutions obtained at t = 0.05.

By considering the parameters L = 1.7278 cm

$\lambda = 2.1 \times 10^{-6}$ s^{-1}, $\tau = 0.05$, k = 4 m^2/kg

$P = 0.5$ g/cm^3, $R_0 = 200$ Bq/m^3

$R(0, t) = 40 \times 10^3$, $\alpha = 0.9$, $\alpha = 0.8$

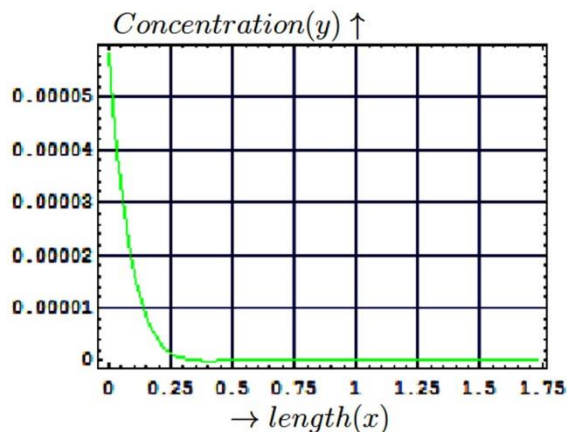is simulated in the following figure.

Figure 1: The approximate solution of
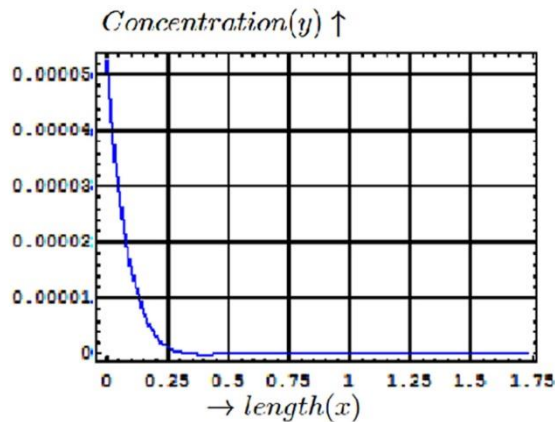radon diffusion equation $\alpha = 0.8$

Figure 2: The approximate solution of
radon diffusion equation $\alpha = 0.9$

6. CONCLUSION

We successfully developed fractional order explicit finite difference scheme for time fractional radon diffusion equation. Furthermore, we discuss its stability and convergence of the scheme. As an application of this method we obtain the numerical solution of text problems and its solutions is simulated graphically by mathematical software Mathematica.

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