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DECOMPOSITION OF GENERALIZED PETERSEN GRAPHS INTO CLAWS, CYCLES AND PATHS

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Abstract. Let $G = (V,E)$ be a finite graph with n vertices. Let n and k be positive integers where $n \geq 3$ and $1 \leq k < \frac{n}{2}$. The *Generalized Petersen Graph GP(n,k)* is a graph with vertex set $\{u_0, u_1, u_2, ..., u_{n-1}, v_0, v_1, v_2, ..., v_{n-1}\}$ and edge-set consisting of all edges of the form $u_i u_{i+1}, u_i v_i$ and $v_i v_{i+k}$ where $0 \leq i \leq n-1$, the subscripts being reduced modulo n. Obviously $GP(n,k)$ is always a cubic graph and $GP(5,2)$ is the well-known Petersen graph. In this paper, we show that $GP(n,1), n \geq 3$ can be decomposed into n copies of S_3 if n is even and P_4 and $(n-1)$ copies of S_3 if n is odd. Also, we show that $GP(n,2), n \geq 5$ can be decomposed into $\frac{n}{2}$ copies of S_3, 2 copies of $C_{\frac{n}{2}}$ and $\frac{n}{2}$ copies of P_2 if n is even and C_n, P_4 , $\lfloor \frac{n}{2} \rfloor$ copies of S_3 and $\left(\lfloor \frac{n}{2} \rfloor - 1 \right)$ copies of P_2 if n is odd. $GP(n,2), n \geq 5$ and $n = 3d, d = 2,3,...$ can be decomposed into $2d$ copies of S_3 and d copies of P_4. $GP(n,2), n \geq 5$ and $n = 4d, d = 2,3,...$ can be decomposed into $3d$ copies of S_3 and d copies of P_4. $GP(n,3), n \geq 8$ can be decomposed into n copies of S_3 if n is even and P_6, P_2 and $(n-2)$ copies of S_3 if n is odd.

Keywords: Generalized Petersen Graph; Decomposition; claw.

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1. Introduction

The origin of graph decomposition is from the combinatorial problems most of which emerged in the 19th century. The first one was the prize question for the year in the *Lady's* and *Gentlemen's Diary* of 1844 stated by *W.S.B.Woolhouse*: Determine the number of combinations that can be made out of n symbols, p symbols in each such that no combination of q symbols which may appear in any one of them may appear in any other. When every q symbols appear in exactly one of the p subsets, such a configuration is known as a *Steiner system*. When $q = 2$ and $p = 3$, the configuration is known as *Steiner Triple system*. In 1847, *T.P.Kirkman* settled the existence question for the *Steiner Triple system*. It is equivalent to the decomposition of K_n into triangles.

The other best problems are *Kirkman's problem* of 15 strolling school girls, *Dudeney problem* of 9 handcuffed prisoners, *Euler's problem* of 36 army officers, *Kirkman's problem of knights, Lucas' dance around problem* and *the four color problem*. However, the earliest works in the direction are not explicitly related to *graph decompositions*. The paper dealing directly with *graph decomposition* appeared after the turn of the 19th century. Since then, the interest in graph decomposition has been on increase and a real up surge is witnessed after 1950. Now a days, *graph decomposition* rank among the most prominent areas of graph theory and combinatorics. Many types of decomposition have been well studied in the literature. There are lot of applications of *decomposition of graph* which include group testings, DNA library screening, scheduling problems, sharing scheme and synchronous optical networks.

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [8] and to study about the decomposition of graphs into paths, stars and cycles is referred to [1], [2], [3] and [4].

As usual C_n denotes the cycle of length n, P_{n+1} denotes the path of length n and S_3 denotes the claw.

2. Basic Definitions and Examples of Generalized Petersen Graphs

In this section, we see some basic definitions of graph decomposition, Generalized Petersen Graph and examples of Generalized Petersen Graph.

Let $L = \{H_1, H_2, ..., H_r\}$ be a family of subgraphs of G. An $L-decomposition$ of G is an edge- disjoint decomposition of G into positive integer α_i copies of H_i, where $i \in \{1,2,...,r\}$. Furthermore, if each H_i $(i \in \{1,2,...,r\})$ is isomorphic to a graph H, we say that G has an $H-decomposition$. It is easily seen that $\sum_{i=1}^{r} \alpha_i \; e(H_i) = e(G)$ is one of the obvious necessary conditions for the existence of a $\{H_1, H_2, ..., H_r\}-$ *decomposition* of G. For convenience, we call the equation, $\sum_{i=1}^{r} \alpha_i \; e(H_i) = e(G)$, a *necessary sum condition*.

Let n and k be positive integers, $n \geq 3$ and $1 \leq k < \frac{n}{2}$. The *Generalized Petersen Graph* $GP(n,k)$ is a graph with vertex set $\{u_0, u_1, u_2, ..., u_{n-1}, v_0, v_1, v_2, ..., v_{n-1}\}$ and edge-set consisting of all egdes of the form $u_i u_{i+1}, u_i v_i$ and $v_i v_{i+k}$ where $0 \leq i \leq n-1$, the subscripts being reduced modulo n.

Obviously $GP(n,k)$ is always a cubic graph and $GP(5,2)$ is the well-known Petersen graph. Thus $GP(5,2)$ is the Petersen graph and is represented in Figure 1(d).
Also $GP(3,1)$, $GP(4,1)$, $GP(5,1)$ and $G(10,4)$ are represented in Figure 1(a), Figure 1(b), Figure 1(c) and Figure 1(e).

3. Decomposition of Generalized Petersen Graph $GP(n,1)$ into Claws and Path

In this section, we characterize the theorem of decomposition of Generalized Petersen Graph $GP(n,1)$ into claws and paths.

Theorem 3.1. *$GP(n,1), n \geq 3$ can be decomposed into n copies of S_3 if n is even and P_4 and $(n-1)$ copies of S_3 if n is odd.*

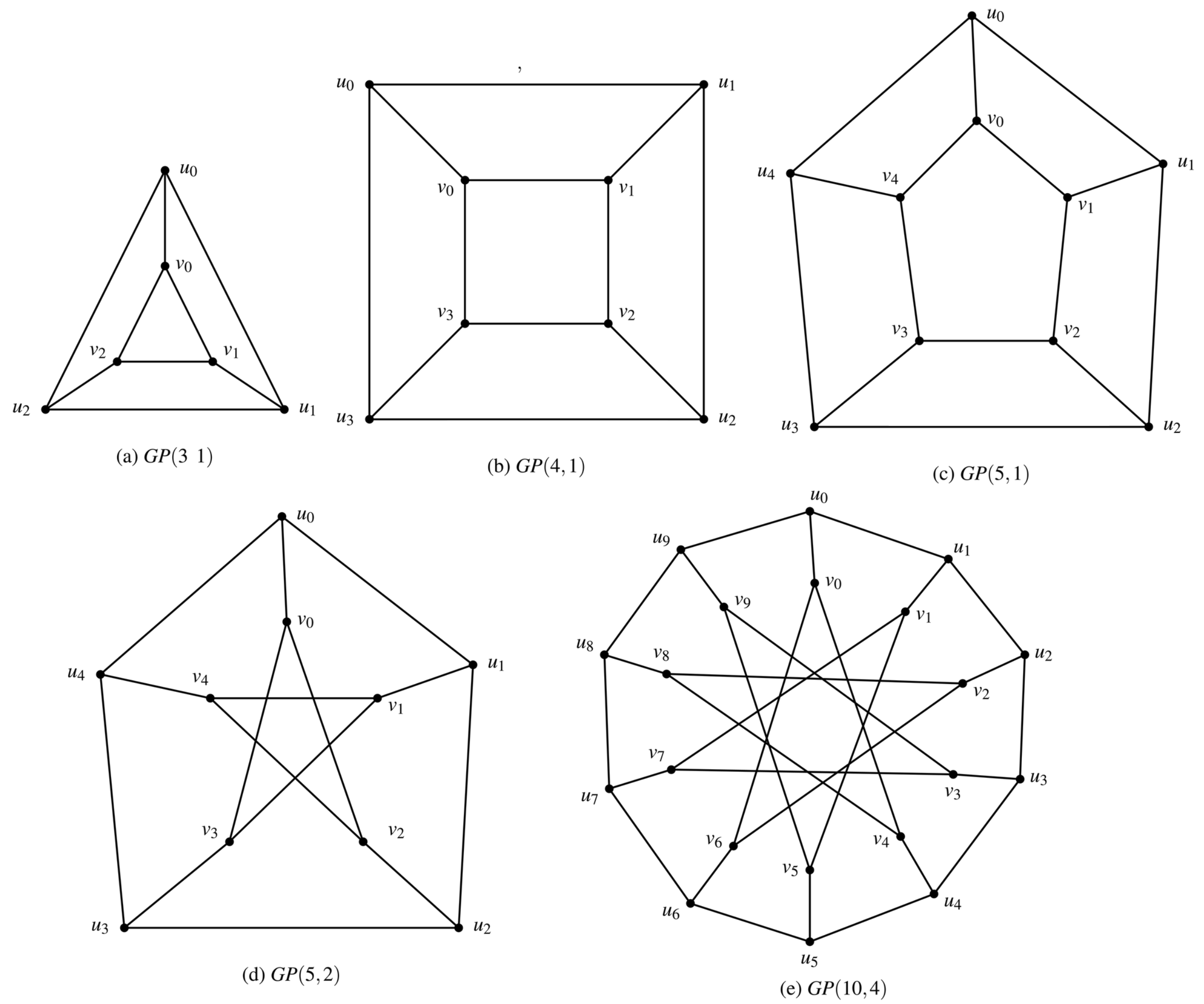

FIGURE 1. Examples of Generalized Peterson Graphs

Proof. Let $V(GP(n,1)) = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and $E(GP(n,1)) = \left\{e_0, e_1, ..., e_{n-1}, e_0^{'}, e_1^{'}, ..., e_{n-1}^{'}, e_0^{''}, e_1^{''}, ..., e_{n-1}^{''}\right\}$ where e_i and $e_i^{'}$, $0 \leq i \leq n-1$ are edges in outer and inner cycles and $e_i^{''}$, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.

Case 1. If n is even. That is $n = 2d, d = 2, 3, ...$

Let $E_i = \left\{e_i, e_{i+1}^{''}, e_{i+1}\right\}$ where $i = 0, 2, 4, ..., 2d-2$ and $E_j = \left\{e_j^{'}, e_{j+1}^{''}, e_{j+1}^{'}\right\}$ where $j = 1, 3.5, ..., 2d-1$.

Then each edge induced subgraph $< E_i >$ and $< E_j >$ forms d disjoint claws. Therefore $GP(n,1)$ can be decomposed into $d+d=2d=n$ disjoint claws. Hence $GP(n,1)$ can be decomposed into n distinct claws.

Case 2. If n is odd. That is $n=2d-1, d=2,3,4,....$

Let $E_i = \left\{ e_i, e''_{i+1}, e_{i+1} \right\}$ where $i = 0,2,4,...,2d-4$ and $E_j = \left\{ e'_j, e''_{j+1}, e'_{j+1} \right\}$ where $j = 1,3.5,...,2d-3$ and $E_k = \left\{ e_{2d-2}, e''_0, e'_0 \right\}$.

Then each edge induced subgraph $< E_i >$ and $< E_j >$ forms d disjoint claws and the edge induced subgraph $< E_k >$ forms a path of length 4 (i.e, P_4). Therefore $GP(n,1)$ can be decomposed into $d+d=2d=n-1$ disjoint claws and P_4. Hence $GP(n,1)$ can be decomposed into $n-1$ disjoint claws and P_4. □

Illustration: The above theorem can be explained through the following Figure 2.

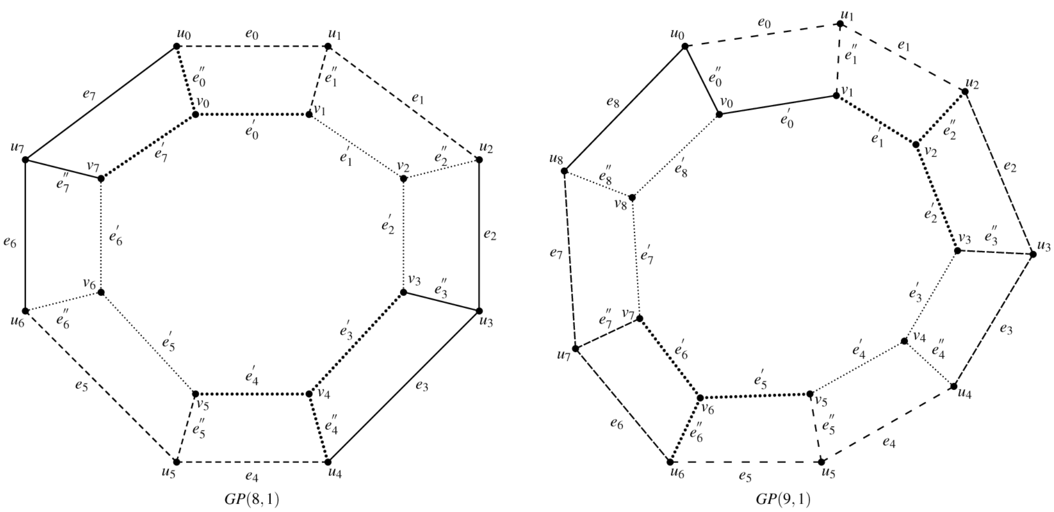

FIGURE 2

The above two figures represents decomposition of $GP(8,1)$ into 8 claws and decomposition of $GP(9,1)$ into 8 claws and a path P_4 respectively.

All edges of the claws and path differentiated in the above Figure 2.

4. Decomposition of Generalized Petersen Graph $GP(n,2)$ into Claws and Paths

In this section, we characterize the theorem of decomposition of Generalized Petersen Graph $GP(n,2)$ into claws, cycles and paths when n is a multiple of $2,3$ and 4 respectively.

Theorem 4.1. *$GP(n,2), n \geq 5$ can be decomposed into $\frac{n}{2}$ copies of S_3, 2 copies of $C_{\frac{n}{2}}$ and $\frac{n}{2}$ copies of P_2 if n is even and C_n, P_4 , $\lfloor \frac{n}{2} \rfloor$ copies of S_3 and $\left(\lfloor \frac{n}{2} \rfloor - 1\right)$ copies of P_2 if n is odd.*

Proof. Let $V(GP(n,1)) = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and
$E(GP(n,2)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+2} \;/\; 0 \leq i \leq n-1\}$ with indices taken modulo n.
Now, $E(GP(n,2)) = \left\{e_0, e_1, ..., e_{n-1}, e_0^{'}, e_1^{'}, ..., e_{n-1}^{'}, e_0^{''}, e_1^{''}, ..., e_{n-1}^{''}\right\}$ where e_i and $e_i^{'}$, $0 \leq i \leq n-1$ are edges in outer and inner cycles and $e_i^{''}$, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.
Case 1. If n is even. That is $n = 2d,\ d = 3, 4, ...$
Let $E_i = \left\{e_i, e_{i+1}^{''}, e_{i+1}\right\}$ where $i = 0, 2, 4, ..., 2d-2$, $E_j = \left\{e_0^{'}, e_2^{'}, e_4^{'}, ..., e_{2d-2}^{'}\right\}$,
$E_k = \left\{e_1^{'}, e_3^{'}, e_5^{'}, ..., e_{2d-1}^{'}\right\}$ and $E_l = \left\{e_0^{''}, e_2^{''}, e_4^{''}, ..., e_{2d-2}^{''}\right\}$.
Then, the edge induced subgraph $< E_i >$ forms $d = \frac{n}{2}$ disjoint claws, each edge induced subgraph $< E_j >$ and $< E_k >$ forms two disjoint cycles of length $d = \frac{n}{2}$ and the edge induced subgraph $< E_l >$ forms $d = \frac{n}{2}$ disjoint path of length one (i.e, P_2). Hence $GP(n,2)$ can be decomposed into $\frac{n}{2}S_3, 2C_{\frac{n}{2}}$ and $\frac{n}{2}P_2$.
Case 2. If n is odd. That is $n = 2d-1,\ d = 3, 4, ...$
Let $E_i = \left\{e_i, e_{i+1}^{''}, e_{i+1}\right\}$ where $i = 0, 2, 4, ..., 2d-4$,
$E_j = \left\{e_0^{'}, e_2^{'}, e_4^{'}, ..., e_{2d-2}^{'}, e_1^{'}, e_3^{'}, e_5^{'}, ..., e_{2d-3}^{'}\right\}$, $E_k = \left\{e_{2d-2}^{''}, e_{2d-2}, e_0^{''}\right\}$ and
$E_l = \left\{e_2^{''}, e_4^{''}, e_6^{''}, ..., e_{2d-4}^{''}\right\}$.
Then, the edge induced subgraph $< E_i >$ forms $\lfloor \frac{n}{2} \rfloor$ disjoint claws, the edge induced subgraph $< E_j >$ forms a cycle of length n, the edge induced subgraph $< E_k >$ forms a path of length three (i.e, P_4) and the edge induced subgraph $< E_l >$ forms $(\lfloor \frac{n}{2} \rfloor - 1)$ disjoint path of length one (i.e, P_2). Hence $GP(n,2)$ can be decomposed into $\lfloor \frac{n}{2} \rfloor$ claws, C_n, P_4 and $(\lfloor \frac{n}{2} \rfloor - 1)P_2$. ☐

Theorem 4.2. *$GP(n,2), n \geq 5$ and $n = 3d, d = 2, 3, ...$ can be decomposed into $2d$ copies of S_3 and d copies of P_4.*

Proof. Let $V(GP(n,1)) = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and
$E(GP(n,2)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+2} \ / \ 0 \leq i \leq n-1\}$ with indices taken modulo n.
Now, $E(GP(n,2)) = \left\{e_0, e_1, ..., e_{n-1}, e'_0, e'_1, ..., e'_{n-1}, e''_0, e''_1, ..., e''_{n-1}\right\}$ where e_i and e'_i, $0 \leq i \leq n-1$ are edges in outer and inner cycles and e''_i, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.

Given $n \geq 5$ and $n = 3d$, $d = 2, 3, ...$.

Let $E_i = \left\{e_i, e''_{i+1}, e_{i+1}\right\}$ where $i = 0, 3, 6, ..., 3d-3$, $E_j = \left\{e'_j, e''_{j+2}, e'_{j+2}\right\}$ where $j = 1, 4, 7, ..., 3d-2$, and $E_k = \left\{e_k, e''_k, e'_k\right\}$ where $k = 2, 5, 8, ..., 3d-1$.

Then each edge induced subgraph $< E_i >$ and $< E_j >$ forms d disjoint claws and the edge induced subgraph $< E_k >$ forms d disjoint paths of length 3 (i.e, P_4). Hence $GP(n,2), (n \geq 5, n = 3d)$ can be decomposed into $2d$ claws and dP_4. $\square$

Theorem 4.3. *$GP(n,2), n \geq 5$ and $n = 4d, d = 2, 3, ...$ can be decomposed into $3d$ copies of S_3 and d copies of P_4.*

Proof. Let $V(GP(n,1)) = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and
$E(GP(n,2)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+2} \ / \ 0 \leq i \leq n-1\}$ with indices taken modulo n.
Now, $E(GP(n,2)) = \left\{e_0, e_1, ..., e_{n-1}, e'_0, e'_1, ..., e'_{n-1}, e''_0, e''_1, ..., e''_{n-1}\right\}$ where e_i and e'_i, $0 \leq i \leq n-1$ are edges in outer and inner cycles and e''_i, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.

Given $n \geq 5$ and $n = 4d$, $d = 2, 3, ...$.

Let $E_i = \left\{e_i, e''_{i+1}, e_{i+1}\right\}$ where $i = 0, 4, 8, ..., 4d-4$, $E_j = \left\{e'_j, e''_{j+2}, e'_{j+2}\right\}$ where $j = 1, 4, 7, ..., 3d-2$, and $E_k = \left\{e''_{k-1}, e_{k-1}, e_k\right\}$ where $k = 3, 7, ..., 4d-1$.

Then, the edge induced subgraph $< E_i >$ forms d disjoint claws, the edge induced subgraph $< E_j >$ forms $2d$ disjoint claws and the edge induced subgraph $< E_k >$ forms d disjoint path of length three (i.e, P_4). Hence $GP(n,2), (n \geq 5, n = 4d)$ can be decomposed into $3d$ claws and dP_4. $\square$

5. Decomposition of Generalized Petersen Graph $GP(n,3)$ into Claws and Paths

In this section, we characterize the theorem of decomposition of Generalized Petersen Graph $GP(n,3)$ into claws and paths.

Theorem 5.1. *$GP(n,3), n \geq 8$ can be decomposed into n copies of S_3 if n is even and P_6, P_2 and $(n-2)$ copies of S_3 if n is odd.*

Proof. Let $V(GP(n,1)) = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and
$E(GP(n,2)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+3} \ / \ 0 \leq i \leq n-1\}$ with indices taken modulo n.
Now, $E(GP(n,2)) = \left\{e_0, e_1, ..., e_{n-1}, e_0^{'}, e_1^{'}, ..., e_{n-1}^{'}, e_0^{''}, e_1^{''}, ..., e_{n-1}^{''}\right\}$ where e_i and $e_i^{'}$, $0 \leq i \leq n-1$ are edges in outer and inner cycles and $e_i^{''}$, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.

Case 1. If n is even and $n \geq 8$.

Let $E_i = \left\{e_i, e_{i+1}^{''}, e_{i+1}\right\}$ where $i = 0, 2, 4, ..., n-2$,
and $E_j = \left\{e_j^{'}, e_{j+3}^{''}, e_{j+3}^{'}\right\}$ where $j = 1, 3, 5, ..., n-1$.

Then each edge induced subgraph $< E_i >$ and $< E_j >$ forms $\frac{n}{2}$ disjoint claws. Hence $GP(n,3), n \geq 8$ can be decomposed into nS_3 if n is even.

Case 2. If n is odd and $n \geq 8$.

Let $E_i = \left\{e_i, e_{i+1}^{''}, e_{i+1}\right\}$ where $i = 0, 2, 4, ..., n-3$, $E_j = \left\{e_j^{'}, e_{j+3}^{''}, e_{j+3}^{'}\right\} \cup \left\{e_{n-1}^{'}, e_2^{''}, e_2\right\}$ where $j = 1, 3, 5, ..., n-6$, $E_k = \left\{e_{n-4}^{'}, e_{n-1}^{''}, e_{n-1}, e_0^{''}, e_0^{'}\right\}$ and $E_l = \left\{e_{n-2}^{'}\right\}$.

Then, the edge induced subgraph $< E_i >$ forms $\left\lfloor \frac{n}{2} \right\rfloor$ disjoint claws, the edge induced subgraph $< E_j >$ forms $\left\lfloor \frac{n}{2} \right\rfloor - 1$ disjoint claws, the edge induced subgraph $< E_k >$ forms a path of length five (i.e, P_6) and the edge induced subgraph $< E_l >$ forms a path of length one (i.e, P_2). Hence $GP(n,3), n \geq 8$ can be decomposed into and $(n-2)$ disjoint claws, P_6 and P_2 if n is odd. □

Illustration: The above theorem can be explained through the following Figure 3.

All edges of the claws and path differentiated in Figure 3.

Remark 5.2. *This construction does not work with $GP(7,3)$.*

$GP(7,3)$ can be decomposed into $5S_3, C_5$ and P_2.

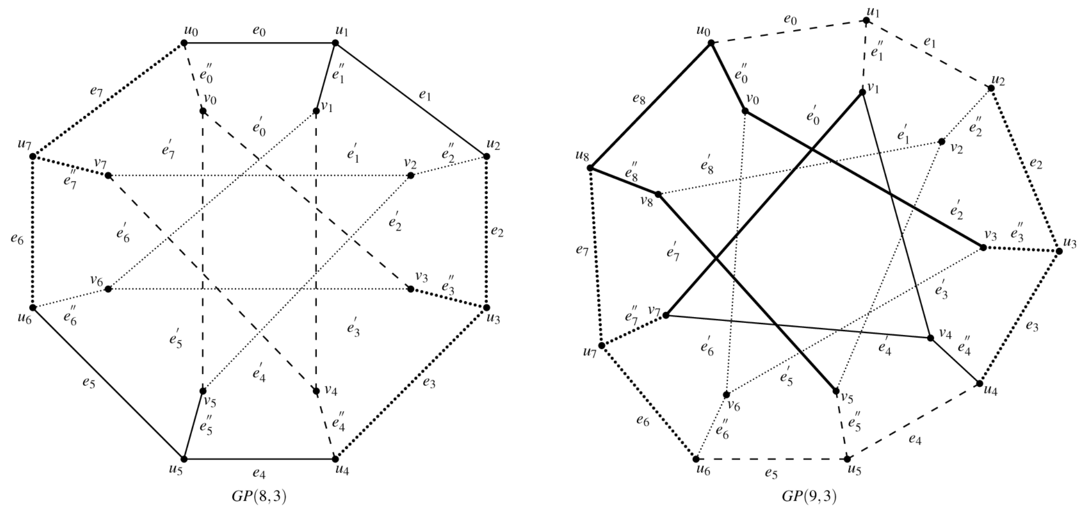

FIGURE 3

The above two figures represents decomposition of $GP(8,3)$ into 8 claws and decomposition of $GP(9,1)$ into 7 claws, a path P_6 and a path P_2 respectively.

Proof. Let $E_i = \left\{e_i, e^{''}_{i+1}, e_{i+1}\right\}$ where $i = 0,2,4$, $E_j = \left\{e^{'}_j, e^{''}_{j+3}, e^{'}_{j+3}\right\} \cup \left\{e^{'}_{n-1}, e^{''}_2, e_2\right\}$ where $j = 1$, $E_k = \left\{e^{'}_{n-4}, e^{''}_{n-1}, e_{n-1}, e^{''}_0, e^{'}_0\right\}$ and $E_l = \left\{e^{'}_{n-2}\right\}$.

Then the edge induced subgraph $< E_i >$ forms $\left\lfloor \frac{n}{2} \right\rfloor = 3$ disjoint claws, the edge induced subgraph $< E_j >$ forms $\left\lfloor \frac{n}{2} \right\rfloor - 1 = 2$ disjoint claws, the edge induced subgraph $< E_k >$ forms a cycle of length five (i.e, C_5) and the edge induced subgraph $< E_l >$ forms a path of length one (i.e, P_2). Hence $GP(7,3)$ can be decomposed into $5S_3, C_5$ and P_2.

□

Illustration: The above remark-5.2 can be explained through the following Figure 4.

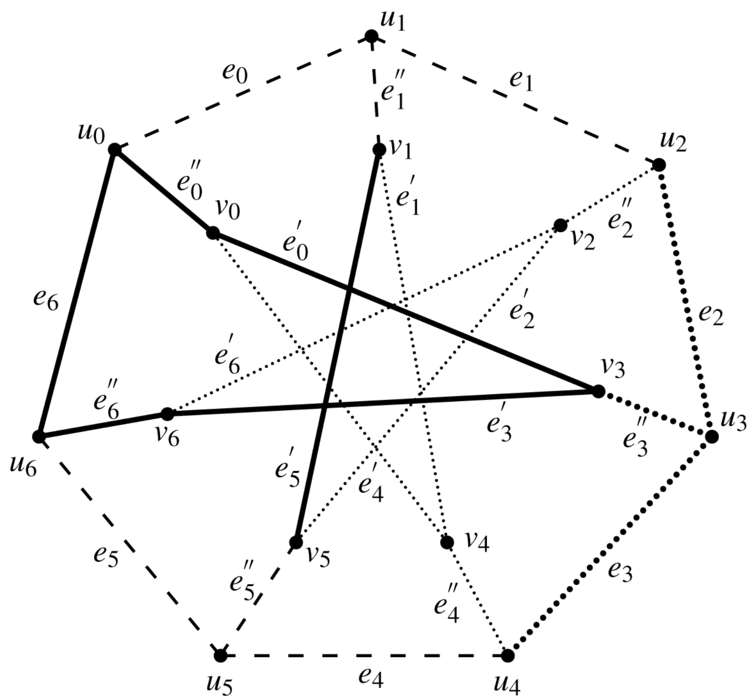

FIGURE 4. $GP(9,3)$

The above figure represents decomposition of $GP(7,3)$ into 5 claws, a cycle C_5 and a path P_2 respectively.

All edges of the claws, cycle and path differentiated in the above Figure 4.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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