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# GENERALIZED MONOTONE MAPPING AND RESOLVENT EQUATION TECHNIQUE WITH AN APPLICATION

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**Abstract.** The objective of this paper is to study generalized monotone mapping, which is the addition of cocoercive mapping and monotone mapping. First resolvent operator is obtained and discussion of its few properties. Then we give the resolvent equation associated with the resolvent operator and find a solution to a variational-like inclusion problem.

**Keywords**: generalized monotone mapping; resolvent operator; algorithms; variational-like inclusions; semi-inner product space.

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## 1. Introduction

Variational inclusion is a natural generalization of variational inequalities. Since monotonicity is a key factor in the study of variational inclusions. Therefore, mathematicians introduced and studied many types of monotonicity e.g. maximal monotone mapping, relaxed monotone mapping, $H$-monotone mapping, $A$-monotone mapping etc., and discussed the solvability of

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different variational inclusion problems with the help of underlying different monotone mappings, see [4, 5],[7]-[10],[15],[22, 23],[25]-[27],[28, 29]. The resolvent operator technique which is the generalized form of projection technique, is very efficient tool to solve variational inclusions and their generalizations. Many heuristics generalized the monotonicity such as $(H,\eta)$-monotone, $(A,\eta)$-monotone, $(A,\eta)$-maximal relaxed monotone etc. They introduced and studied different variational inclusions problems involving these monotone mapping in Hilbert spaces (Benach spaces), see [8, 9, 22, 25, 28].

Recently, Sahu et al. [26] proved the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces, which is more general than the results studied in [27]. Very recently Luo and Huang [23], introduced and studied $(H,\varphi)$-$\eta$-monotone mapping in Banch spaces. Bhat and Zahoor [4, 5] introduced and studied $(H,\phi)$-$\eta$-monotone mapping in semi-inner product space. For the applications point of view we refer to see [7]-[10],[17, 22, 23],[25]-[29],[31, 32]. The proposed work is impelled by the noble research works mentioned above. First we study the generalized monotone mapping which is the addition of cocoercive mapping and monotone mapping and call it $H(.,.,.)$-$\varphi$-$\eta$-cocoercive mapping in semi-inner product spaces. Then, resolvent operator and its resolvent equation are obtained and discuss its few properties. In last existence and convergence results are obtained for a variational inclusion problem in 2-uniformly smooth Banach spaces. Our work is extension and refinement of some result. For details, see [7]-[10],[12]-[18],[22, 23],[25]-[29],[31, 32].

**Definition 1.1.** [24, 26] Let us consider the vector space $Y$ over the field $F$ of real or complex numbers. A functional $[.,.]: Y\times Y\to F$ is called a semi inner product if

(i) $[u^1+u^2,v^1]=[u^1,v^1]+[u^2,v^1],\ \forall u^1,u^2,v^1\in Y$

(ii) $[\alpha u^1,v^1]=\alpha[u^1,v^1],\ \forall \alpha\in F,\ u^1,v^1\in Y$

(iii) $[u^1,u^1]\geq 0,\ for\ u^1\neq 0$

(iv) $|[u^1,v^1]|^2\leq [u^1,u^1][v^1,v^1],\ \forall u^1,v^1\in Y$

The pair $(Y,[.,.])$ is called a semi-inner product space.

"We observed that $\|u^1\|=[u^1,u^1]^{1/2}$ is a norm and we can say a semi-inner product space is a normed linear space with the norm. Every normed linear space can be made into a semi-inner product space in infinitely many different ways. Giles [11] had shown that if the underlying space $Y$ is a uniformly convex smooth Banach space then it is possible to define a semi-inner product uniquely. For a detailed study and fundamental results on semi-inner product spaces, one may refer to Lumer [24], Giles [11] and Koehler [21]," [4].

**Definition 1.2.** [26, 30] The $Y$ be a Banach space, then

(i) modulus of smoothness of $Y$ defined as

$$\rho_Y(s) \;=\; \sup\left\{\frac{\|u^1+v^1\|+\|u^1-v^1\|}{2}-1:\; \|u^1\|\leq 1,\; \|v^1\|\leq s\right\}.$$

(ii) $Y$ be uniformly smooth if $\lim_{s\to 0}\rho_Y(s)/s \;=\; 0$

(iii) $Y$ be $p$-uniformly smooth for $p>1$, if there exists $c>0$ such that $\rho_Y(s)\leq cs^p$.

(iv) $Y$ be 2-uniformly smooth if there exists $c>0$ such that $\rho_Y(s)\leq cs^2$.

**Lemma 1.3.** [26, 30] Let $p>1$ be a real number and $Y$ be a smooth Banach space. Then the following statements are equivalent:

(i) $Y$ is 2-uniformly smooth.

(ii) There is a constant $k>0$ such that for every $v^1, w^1\in Y$, the following inequality holds

$$\|v^1+w^1\|^2\leq\|v^1\|^2+2\langle w^1, f_{v^1}\rangle+k\|w^1\|^2, \tag{1.1}$$

where $f_{v^1}\in J(v^1)$ and $J(v^1)=\{v^{1*}\in Y^*:\langle v^1, v^{1*}\rangle=\|v^1\|^2 \text{ and } \|v^{1*}\|=\|v^1\|\}$ is the normalized duality mapping.

"Every normed linear space $Y$ is a semi-inner product space (see [24]). Infact, by Hahn-Banach theorem, for each $v^1\in Y$, there exists at least one functional $f_{v^1}\in Y^*$ such that $\langle v^1, f_{v^1}\rangle=\|v^1\|^2$. Given any such mapping $f: Y\to Y^*$, we can verify that $[w^1, v^1]=\langle w^1, f_{v^1}\rangle$ defines a semi-inner product. Hence we can write the inequality (2.1) as

$$\|v^1+w^1\|^2\leq\|v^1\|^2+2[w^1, f_{v^1}]+s\|w^1\|^2. \tag{1.2}$$

The constant $s$ is known as constant of smoothness of $Y$, is chosen with best possible minimum value", [26].

## 2. PRELIMINARIES

Let $Y$ be a 2-uniformly smooth Banach space. Its norm and topological dual space is given by $\|.\|$ and $Y^*$, respectively. The semi-inner product $[.,.]$ signify the dual pair among $Y$ and $Y^*$. In order to proceed the next, we recall some basic concepts, which will be needed in the subsequent sections.

**Definition 2.1.** [23, 26] Let $Y$ be real 2-uniformly smooth Banach space. Let single-valued mapping $Q: Y \to Y$ and mapping $\eta: Y \times Y \to Y$, then
(i) $Q$ is $(r,\eta)$-strongly monotone if there $\exists$ constant $r > 0$ such that

$$[Q(u) - Q(u'), \eta(u,u')] \ \geq \ r \, \|u - u'\|^2, \ \forall u, \ u' \in Y;$$

(ii) $Q$ is $(s,\eta)$-cocoercive if there $\exists$ constant $s > 0$ such that

$$[Q(u) - Q(u'), \eta(u,u')] \ \geq \ s \, \|Q(u) - Q(u')\|^2, \ \forall u, \ u' \in Y;$$

(iii) $Q$ is $(s',\eta)$-relaxed cocoercive if there $\exists$ constant $s > 0$ such that

$$[Q(u) - Q(u'), \eta(u,u')] \ \geq \ -s' \, \|Q(u) - Q(u')\|^2, \ \forall u, \ u' \in Y;$$

(iv) $Q$ is $\alpha$-expansive if there $\exists$ constant $\alpha > 0$

$$\|Q(u) - Q(u')\| \ \geq \ \alpha \, \|u - u'\|, \ \forall u, \ u' \in Y;$$

(v) $\eta$ is be $\tau$-Lipschitz continuous if there $\exists$ constant $\tau > 0$ such that

$$\|\eta(u,u')\| \ \leq \ \tau \, \|u - u'\|, \ \forall u, \ u' \in Y.$$

**Definition 2.2.** [17] Let us consider the single-valued mappings $Q,R,S: Y \to Y$, mapping $\eta$ : $Y \times Y \to Y$, $H: Y \times Y \times Y \to Y$, then
(i) $H(Q,.,.)$ is $(\mu,\eta)$-cocoercive in regards $R$ if there $\exists$ constant $\mu > 0$ such that

$$[H(Qu,x,x) - H(Qu',x,x), \ \eta(u,u')] \ \geq \ \mu \, \|Qu - Qu'\|^2, \ \forall x, \ u, \ u' \in Y;$$

(ii) $H(.,R,.)$ is $(\gamma,\eta)$-relaxed cocoercive in regards $R$ if there $\exists$ constant $\gamma > 0$ such that

$$[H(x,Ru,x) - H(x,Ru',x), \eta(u,u')] \ \geq \ -\gamma \, \|Ru - Ru'\|^2, \ \forall x, \ u, \ u' \in Y;$$

(iii) $H(.,.,S)$ is $(\delta,\eta)$-strongly monotone in regards $S$ if there $\exists$ constant $\delta > 0$ such that

$$[H(x,x,Su)-H(x,x,Su'),\eta(u,u')] \ \geq \ \delta \ \|u-u'\|^2, \ \forall x, \ u, \ u' \in Y;$$

(iv) $H(Q,.,.)$ is $\kappa_1$-Lipschitz continuous in regards $Q$ if there $\exists$ constant $\kappa_1$ such that

$$\|H(Qu,x,x)-H(Qu',x,x)\| \ \leq \ \kappa_1 \ \|u-u'\|, \ \forall x, \ u,u' \in Y.$$

Similarly we can define the Lipschitz continuity for $H(.,.,.)$ in regards second and third component.

"Let $M : Y \multimap Y$ be a set-valued mapping, then graph of $M$ is given by graph$(M) = \{(v,w) : \ w \in M(v)\}$. The domain of $M$ is given by

$$\text{Dom}(M) = \{v \in Y : \exists w \in Y : (v,w) \in M\}.$$

The Range of $(M)$ is given by

$$\text{Range}(M) = \{w \in Y : \exists V \in Y : (v,w) \in M\}.$$

The inverse of $(M)$ is given by

$$M^{-1} = \{(w,v) : (v,w) \in M\}.$$

For any two set-valued mappings $N$ and $M$, and any real number $\beta$, we define

$$N + M = \{(v,w+w') : (v,w) \in N, (v,w') \in M\},$$

$$\beta M = \{(v,\beta w) : (v,w,) \in M\}.$$

For a mapping $A$ and a set-valued map $M : Y \multimap Y$, we define $A + M = \{(v,w+w') : Av = w, (v,w') \in M\}$", [4].

**Definition 2.3.** [23, 26] *A set-valued mapping $M : Y \multimap Y$ is said to be $(m,\eta)$-relaxed monotone if $\exists$ a constant $m > 0$ such that*

$$[v^* - w^*, \eta(v,w)] \ \geq \ -m \ \|v-w\|^2, \ \forall v,w \in Y, \ v^* \in M(v), \ w^* \in M(w).$$

**Definition 2.4.** Let $G, \eta : Y \times Y \to Y$ be the mappings. Then

(i) $G$ is $(\nu,\eta)$-relaxed monotone in regards first component if $\exists$ a constant $\nu>0$ such that

$$[G(v,u^*)-G(w,u^*),\eta(v,w)] \;\geq\; -\nu\|v-w\|^2,\ \forall v,w,u^*\in Y;$$

(ii) $G(.,.)$ is $\varepsilon_1$-Lipschitz continuous in regards first component if $\exists$ a constant $\varepsilon_1>0$ such that

$$\|G(v,u^*)-G(w,u^*)\| \;\leq\; \varepsilon_1\ \|v-w\|,\ \forall v,w,u^*\in Y;$$

**Definition 2.5.** [6] The Hausdorff metric $D(.,.)$ on $CB(Y)$, is defined by

$$D(A,B)=\max\left\{\sup_{u\in A}\inf_{v\in B} d(u,v),\sup_{v\in B}\inf_{u\in A} d(u,v)\right\},\ A,B\in CB(Y),$$

where $d(.,.)$ is the induced metric on $Y$ and $CB(Y)$ denotes the family of all nonempty closed and bounded subsets of $X$.

**Definition 2.6.** [6] A multi-valued mapping $S:Y\multimap CB(Y)$ is called $D$-Lipschitz continuous with constant $\lambda_S>0$, if

$$D(Sv,Sw)\leq\ \lambda_S\ \|v-w\|,\ \forall v,\ w\in Y.$$

## 3. Generalized $H(.,.,.)$-$\varphi$-$\eta$-Cocoercive Mapping

Let $Y$ be 2-uniformly smooth Banach space. Assume that $\eta,H:Y\times Y\times Y\to Y$, and $\varphi,Q,R,S:Y\to Y$ be single-valued mappings and $M:Y\multimap Y$ be a multi-valued mapping.

**Definition 3.1.** Let $H(.,.,.)$ is $(\mu,\eta)$-cocoercive in regards $Q$ with non-negative constant $\mu$, $(\gamma,\eta)$-relaxed cocoercive in regards $R$ with non-negative constant $\gamma$ and $(\delta,\eta)$-strongly monotone in regards $S$ with non-negative constant $\delta$, then $M$ is called generalized $H(.,.,.)$-$\varphi$-$\eta$-*cocoercive* in regards $Q$, $R$ and $S$ if

(i) $\varphi oM$ is $(m,\eta)$-relaxed monotone;

(ii) $(H(.,.,.)+\lambda\varphi oM)(Y) \;=\; Y,\ \lambda>0.$

Let us consider the following assumptions:

**Assumption $\mathbf{M_1}$:** Let $H$ is $(\mu,\eta)$-cocoercive in regards $Q$ with non-negative constant $\mu$, $(\gamma,\eta)$-relaxed cocoercive in regards $R$ with non-negative constant $\gamma$ and $(\delta,\eta)$-strongly monotone in regards $S$ with non-negative constant $\delta$ with $\mu>\gamma$.

**Assumption M$_2$:** Let $Q$ is $\alpha$-expansive and $R$ is $\beta$-Lipschitz continuous with $\alpha > \beta$.

**Assumption M$_3$:** Let $\eta$ is $\tau$-Lipschitz continuous.

**Assumption M$_4$:** Let $M$ is generalized $H(.,.,.)$-$\varphi$-$\eta$-cocoercive operator in regards $Q$, $R$ and $S$.

**Theorem 3.2.** *Let assumptions $M_1$, $M_2$ and $M_4$ hold good with $\ell = \mu\alpha^2 - \gamma\beta^2 + \delta > m$, then $(H(Q,R,S) + \lambda\varphi oM)^{-1}$ is single-valued.*

**Proof.** Let $y, z \in (H(Q,R,S) + \lambda\varphi oM)^{-1}(x)$ for any given $x \in Y$. It is obvious that

$$\begin{cases} -H(Qy,Ry,Sy) + x \in \ \lambda\varphi oM(y), \\ -H(Qz,Rz,Sz) + x \in \ \lambda\varphi oM(z). \end{cases}$$

Since $\varphi oM$ is $(m,\eta)$-relaxed monotone in the first argument, we have

$$\begin{aligned} -m\lambda\|y-z\|^2 \ &\leq \ [-H(Qy,Ry,Sy) + x - (-H(Qz,Rz,Sz) + x), \ \eta(y,z)] \\ &= [H(Qy,Ry,Sy) - H(Qz,Rz,Sz), \ \eta(y,z)] \end{aligned}$$

Since assumption $M_1$, $M_2$ hold, we have

$$\begin{aligned} -m\lambda\|y-z\|^2 &= -(\mu\alpha^2 - \gamma + \delta)\ \|y-z\|^2 \\ 0 &\leq -(\ell - m\lambda)\ \|y-z\|^2 \leq 0, \text{where } \ell \ = \ \mu\alpha^2 - \gamma\beta^2 + \delta. \end{aligned}$$

Since $\mu > \gamma$, $\alpha > \beta, \delta > 0$, it follows that $\|y-z\| \ \leq \ 0$. We get $y = z$, therefore $(H(Q,R,S) + \lambda\varphi oM)^{-1}$ is single-valued.

**Definition 3.3.** Let assumptions $M_1$, $M_2$ and $M_4$ hold good with $\ell = \mu\alpha^2 - \gamma\beta^2 + \delta > m\lambda$ then the *resolvent operator* $R_{M,\lambda,\varphi}^{H(.,.,.)-\eta} : Y \to \ Y$ is given as

$$R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(u) \ = \ (H(Q,R,S) + \lambda\varphi oM)^{-1}(u), \ \forall\, u \in Y. \tag{3.1}$$

The next attempt is to prove the Lipschitz continuity of the resolvent operator defined by (3.1).

**Theorem 3.4.** *Let assumptions $M_1$-$M_4$ hold good with $\ell = \mu\alpha^2 - \gamma\beta^2 + \delta > m\lambda$ and $\eta$ is $\tau$-Lipschitz then $R^{H(.,.,.)-\eta}_{M,\lambda,\varphi} : Y \to Y$ is $\frac{\tau}{\ell - m\lambda}$-Lipschitz continuous, that is,*

$$\|R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(y) - R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(z)\| \leq \frac{\tau}{\ell - m\lambda}\|y - z\|, \ \forall\, y, z \in Y.$$

**Proof.** Let any given points $y, z \in Y$. From (3.3), we have

$$\begin{aligned} R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(y) &= (H(Q,R,S) + \lambda\varphi o M)^{-1}(y), \\ R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(z) &= (H(Q,R,S) + \lambda\varphi o M)^{-1}(z). \end{aligned}$$

Let $u_0 = R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(y)$ and $u_1 = R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(z)$.

$$\begin{cases} \lambda^{-1}\Big(y - H\Big(Q(u_0), R(u_0), S(u_0)\Big)\Big) \in \ \varphi o M(u_0) \\ \lambda^{-1}\Big(z - H\Big(Q(u_1), R(u_1), S(u_1)\Big)\Big) \in \ \varphi o M(u_1). \end{cases}$$

Since $\varphi o M$ is $(m, \eta)$-relaxed monotone in the first arguments, we have

$$[(y - H(Q(u_0), R(u_0), S(u_0))) - (z - H(Q(u_1), R(u_1), S(u_1))), \eta(u_0, u_1)] \ \geq \ -m\lambda \ \|u_0 - u_1\|^2,$$

which implies

$$[y - z, \eta(u_0, u_1)] \geq [H(Q(u_0), R(u_0), S(u_0)) - H(Q(u_1), R(u_1), S(u_1)), \eta(u_0, u_1)] - m\lambda \ \|u_0 - u_1\|^2.$$

Now, we have

$$\begin{aligned} \|y - z\| \ \|\eta(u_0, u_1)\| \geq \ [y - z, \ \eta(u_0, u_1)] &\geq -m\lambda \ \|u_0 - u_1\|^2 \\ &\quad + [H(Q(u_0), R(u_0), S(u_0)) - H(Q(u_1), R(u_1), S(u_1)), \eta(u_0, u_1)]. \end{aligned}$$

Since assumption $M_1$-$M_3$ hold and $\eta$ is $\tau$-Lipschitz continuous

$$\begin{aligned} \|y - z\| \ \tau \ \|u_0 - u_1\| &\geq (\mu\alpha^2 - \gamma\beta^2 + \delta) \ \|u_0 - u_1\|^2 - m\lambda \ \|u_0 - u_1\|^2 \\ &\geq \ (\ell - m\lambda) \ \|u_0 - u_1\|^2, \text{ where } \ell \ = \ (\mu\alpha^2 - \gamma\beta^2 + \delta). \end{aligned}$$

Thus

$$\|R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(y) - R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(z)\| \leq \frac{\tau}{\ell - m\lambda} \ \|y - z\|, \ \forall\, y, z \in Y.$$

Hence, we get the required result.

## 4. Formulation of the Problem and Existence of Solution

Now we make an attempt to show that generalized $H(.,.,.)$-$\varphi$-$\eta$-cocorecive operator under acceptable assumptions can be used as a powerful tool to solve variational inclusion problems.

Let $Y$ be 2-uniformly smooth Banach space. Let $V,W : Y \to CB(Y)$ be the multi-valued mappings, and let $Q,R,S,f,\varphi : Y \to Y$, $\eta, G : Y \times Y \to Y$ and $H : Y \times Y \times Y \to Y$ be single-valued mappings. Suppose that multi-valued mapping $M : Y \multimap Y$ be a generalized $H(.,.,.)$-$\varphi$-$\eta$-cocoercive operator in regards $Q$, $R$ and $S$ and range $(f) \cap$ dom $M \neq \emptyset$. We consider the following generalized set-valued variational like inclusion problem to find $u \in Y$, $v \in V(u)$ and $w \in W(u)$ such that

$$0 \in G(v,w) + M(f(u)). \tag{4.1}$$

If $Y$ is real Hilbert space and $M$ is maximal monotone operator, then the similar problem to (4.1) studied by Huang et al. [15].

**Lemma 4.1.** *Let us consider the mapping $\varphi : Y \to Y$ such that $\varphi(v+w) = \varphi(v) + \varphi(w)$ and $Ker(\varphi) = \{0\}$, where $Ker(\varphi) = \{v \in Y : \ \varphi(v) = 0\}$. If $(u,v,w)$, where $u \in Y$, $v \in V(u)$ and $w \in W(u)$ is a solution of problem (4.1) if and only if $(u,v,w)$ satisfies the following relation:*

$$f(u) = R^{H(.,.,.)-\eta}_{M,\lambda,\varphi} \ [H(Q(fu),R(fu),S(fu)) - \lambda \varphi o G(v,w)]. \tag{4.2}$$

The resolvent equation corresponding to generalized set-valued variational-like inclusion problem (4.1).

$$\varphi o \ G(v,w) + \lambda^{-1} J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t) = 0. \tag{4.3}$$

where $\lambda > 0$,

$$J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t) = \Big[I - H(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t)), R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t)), S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t)))\Big],$$

$I$ is the identity mapping and

$$H(Q,R,S)\Big[R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t)\Big] = H\Big(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t)), R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t)), S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t))\Big).$$

Now, we show that the problem (4.1) is equivalent to the resolvent equation problem (4.3).

**Lemma 4.2.** *If $(u,v,w)$ with $u \in Y$, $v \in V(u)$ and $w \in W(u)$ is a solution of problem (4.1) if and only if the resolvent equation problem (4.3) has a solution $(t,u,v,w)$ with $t,u \in Y$, $v \in V(u)$ and $w \in W(u)$, where*

$$f(u) \;=\; R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t), \tag{4.4}$$

*and $t = H(Q(fu),R(fu),S(fu)) - \lambda\varphi oG(v,w)$.*

**Proof:** Let $(u,v,w)$ be a solution of problem (4.1), and from Lemma 4.1 Using the fact that

$$J^{H(.,.,.)-\eta}_{M,\lambda,\varphi} = \Big[I - H\Big(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}), R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}), S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi})\Big)\Big],$$

$$\begin{aligned}
J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t) &= J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}\Big[H(Q(fu),R(fu),S(fu)) - \lambda\varphi oG(v,w)\Big]\\
&= \Big[I - H\Big(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}), R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}), S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi})\Big)\Big]\Big[H(Q(fu),R(fu),S(fu)) - \lambda\varphi oG(v,w)\Big]\\
&= \Big[H(Q(fu),R(fu),S(fu)) - \lambda\varphi oG(v,w)\Big]\\
&\quad - H\Big(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}), R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}), S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi})\Big)\Big(H(Q(fu),R(fu),S(fu)) - \lambda\varphi oG(v,w)\Big)\\
&= \Big[H(Q(fu),R(fu),S(fu)) - \lambda\varphi oG(v,w)\Big] - H(Q(fu),R(fu),S(fu))\\
&= -\lambda\varphi oG(v,w)
\end{aligned}$$

This implies that

$$\varphi oG(v,w) + \lambda^{-1} J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t) = 0.$$

Conversely, let $(t,u,v,w)$ is a solution of resolvent equation problem (4.3), then

$$\begin{aligned}
J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t) &= -\lambda\varphi oG(v,w)\\
\Big[I - H\Big(Q(R^{H(.,.,.)-\eta}_{M,\lambda\varphi}), R(R^{H(.,.,.)-\eta}_{M,\lambda\varphi}), S(R^{H(.,.,.)-\eta}_{M,\lambda\varphi})\Big)\Big](t) &= -\lambda\varphi oG(v,w)\\
t - H(Q(fu),R(fu),S(fu)) &= -\lambda\varphi oG(v,w).
\end{aligned}$$

This implies that

$$t = H(Q(fu),R(fu),S(fu)) - \lambda\varphi oG(v,w).$$

Hence $(u,v,w)$ is a solution of variational inclusion problem (4.1).

Lemma 4.1 and Lemma 4.2 are very crucial from the numerical point of view. They permit us to suggest the following iterative scheme for finding the approximate solution of (4.3).

**Algorithm 4.3.** For any given $(t_0,u_0,v_0,w_0)$, we can choose $t_0,\ u_0\in Y$, $v_0\in V(u_0)$ and $w_0\in V(u_0)$ and $0<\varepsilon<1$ such that sequences $\{t_k\}$,$\{u_k\}$, $\{v_k\}$ and $\{w_k\}$ satisfy

$$\begin{cases} f(u_k)=R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t_k),\\ v_k\in V(u_k),\ \|v_k-v_{k+1}\|\le\ D(V(u_k),V(u_{k+1}))+\varepsilon^{k+1}\|u_k-u_{k+1}\|,\\ w_k\in W(u_k),\|w_k-w_{k+1}\|\ \le D(W(u_k),W(u_{k+1}))+\varepsilon^{k+1}\|u_k-u_{k+1}\|,\\ t_{k+1}\ =\ H(Q(fu_k),R(fu_k),S(fu_k))-\lambda\,\varphi oG(v_k,w_k), \end{cases}$$

where $\lambda>0$, $k\ge 0$, and $D(.,.)$ is the Hausdorff metric on $\mathrm{CB}(Y)$.

Next, we find the convergence of the iterative algorithm for the resolvent equation problem (4.3) corresponding generalized set-valued variational inclusion problem (4.1).

**Theorem 4.4.** *Let us consider the problem (4.1) with assumptions $M_1$-$M_4$ and $\varphi:Y\to Y$ be a single-valued mapping with $\varphi(v+w)=\varphi(v)+\varphi(w)$ and $Ker(\varphi)=\{0\}$. Assume that*

(i) *$V$ and $W$ are $\lambda_V$ and $\lambda_W$ continuous, respectively;*

(ii) *$\varphi oG$ is $(\nu,\eta)$-relaxed monotone in regards first component;*

(iii) *$\varphi oG$ is $\varepsilon_1,\ \varepsilon_2$-Lipschitz continuous in regards first and second component, respectively;*

(iv) *$H(Q,R,S)$ is $\kappa_1$, $\kappa_2$, $\kappa_3$-Lipschitz continuous in regards Q, R and S, respectively;*

(v) *$f$ is r-strongly monotone and $\lambda_f$-Lipschitz continuous;*

(vi) $0<\sqrt{\left\{\lambda_f^2\kappa^2+2\nu\lambda\lambda_V^2-2\varepsilon_1\lambda\lambda_V\left(\lambda_f\kappa+\tau\lambda_V\right)+\varepsilon_1^2\lambda^2\lambda_V^2\right\}}<\frac{(1-\sqrt{1-2r+\lambda_f^2})(\ell-m\lambda)}{\tau}-\varepsilon_2\lambda\lambda_W;$
where $\kappa=\kappa_1+\kappa_2+\kappa_3$

(vii) $\|R^{H(.,.)-\varphi-\eta}_{M^k(.,t_k)}(u)-R^{H(.,.)-\varphi-\eta}_{M^{k-1}(.,t_{k-1})}(u)\|\le\xi\|t_k-t_{k-1}\|,\ \forall\, t_k,t_{k-1}\in Y,\xi>0;$

*Then the iterative sequences $\{t_k\},\{u_k\},\{v_k\}$, and $\{w_k\}$ generated by Algorithm 4.3 converges strongly to the unique solution $(t,u,v,w)$ of the resolvent equation problem (4.3).*

**Proof.** Using Algorithms 4.3 and $\lambda_V, \lambda_W$-$D$ Lipschitz continuity of $V, W$, we have

$$\|v_k - v_{k-1}\| \leq D(V(u_k), V(u_{k-1})) + \varepsilon^k \|u_k - u_{k-1}\| \leq \{\lambda_V + \varepsilon^k\} \|u_k - u_{k-1}\|, \tag{4.5}$$

$$\|w_k - w_{k-1}\| \leq D(W(u_k), W(u_{k-1})) + \varepsilon^k \|u_k - u_{k-1}\| \leq \{\lambda_W + \varepsilon^k\} \|u_k - u_{k-1}\|, \tag{4.6}$$

where $k = 1,\ 2,\ .....$

Now, we compute

$$\begin{aligned}
\|t_{k+1} - t_k\| &= \|H(Q(fu_k), R(fu_k), S(fu_k)) - H(Q(fu_{k-1}), R(fu_{k-1}), S(fu_{k-1})) \\
&\quad -\lambda(\varphi o G(v_k, w_k) - \varphi o G(v_{k-1}, w_{k-1}))\| \\
&\leq \|H(Q(fu_k), R(fu_k), S(fu_k)) - H(Q(fu_{k-1}), R(fu_{k-1}), S(fu_{k-1})) \\
&\quad -\lambda(\varphi o G(v_k, w_k) - \varphi o G(v_{k-1}, w_k))\| \\
&\quad +\lambda \|\varphi o G(v_{k-1}, w_k) - \varphi o G(v_{k-1}, w_{k-1}))\|.
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\|H(Q(fu_k), R(fu_k), S(fu_k)) - H(Q(fu_{k-1}), R(fu_{k-1}), S(fu_{k-1})) \\
&\qquad -\lambda(\varphi o G(v_k, w_k) - \varphi o G(v_{k-1}, w_k))\|^2 \\
&\leq \|H(Q(fu_k), R(fu_k), S(fu_k)) - H(Q(fu_{k-1}), R(fu_{k-1}), S(fu_{k-1}))\|^2 \\
&\quad -2\lambda[\varphi o G(v_k, w_k) - \varphi o G(v_{k-1}, w_k), \eta(v_k, v_{k-1})] \\
&\quad +2\lambda \|\varphi o G(v_k, w_k) - \varphi o G(v_{k-1}, w_k)\| \\
&\quad \times \Big\{ \|H(Q(fu_k), R(fu_k), S(fu_k)) - H(Q(fu_{k-1}), R(fu_{k-1}), S(fu_{k-1}))\| + \|\eta(v_n, v_{n-1})\| \Big\} \\
&\quad +\lambda^2 \|\varphi o G(v_k, w_k) - \varphi o G(v_{k-1}, w_k)\|^2.
\end{aligned} \tag{4.8}$$

Since $H(Q, R, S)$ is $\kappa_1, \kappa_2, \kappa_3$-Lipschitz continuous in regards $Q, R, S$, respectively, We have

$$\begin{aligned}
&\|H(Q(fu_k), R(fu_k), S(fu_k)) - H(Q(fu_{k-1}), R(fu_{k-1}), S(fu_{k-1}))\|^2 \\
&\leq \lambda_f^2 \kappa^2 \|u_k - u_{k-1}\|^2, \text{where } \kappa = \kappa_1 + \kappa_2 + \kappa_3
\end{aligned} \tag{4.9}$$

Since $\varphi o G$ is $(\nu, \eta)$-relaxed monotone, then we have Since $\varphi o G$ is $(\nu, \eta)$-relaxed monotone, then we have

$$[\varphi o G(v_k, w_k) - \varphi o G(v_{k-1}, w_k), \eta(v_k, v_{k-1})] \geq -\nu \{\lambda_V + \varepsilon^k\}^2 \|u_k - u_{k-1}\|^2. \tag{4.10}$$

As $\varphi oG(.,.)$ is $\varepsilon_1,\varepsilon_2$-Lipschitz continuous in the first, second arguments, respectively and using (4.5),(4.6), we have

$$\|\varphi oG(v_k,w_k)-\varphi oG(v_{k-1},w_k)\|\leq \varepsilon_1\|v_k-v_{k-1}\|\leq \varepsilon_1\{\lambda_V+\varepsilon^k\}\|u_k-u_{k-1}\|, \tag{4.11}$$

$$\|\varphi oG(v_{k-1},w_k)-\varphi oG(v_{k-1},w_{k-1})\|\leq \varepsilon_2\|w_k-w_{k-1}\|\leq\ \varepsilon_2\{\lambda_W+\varepsilon^k\}\|u_k-u_{k-1}\|. \tag{4.12}$$

By using M-3 and (4.9)-(4.12) in (4.8), we have

$$\begin{aligned}&\|H(Q(fu_k),R(fu_k),S(fu_k))-H(Q(fu_{k-1}),R(fu_{k-1}),S(fu_{k-1}))\\&\qquad -(\varphi oG(v_k,w_k)-\varphi oG(v_{k-1},w_k))\|\\&\leq\sqrt{\Big[\lambda_f^2\kappa^2+2\nu\lambda\{\lambda_V+\varepsilon^k\}^2+2\varepsilon_1\lambda\{\lambda_V+\varepsilon^k\}\Big\{\lambda_f\kappa+\tau\{\lambda_V+\varepsilon^k\}\Big\}+\varepsilon_1^2\lambda^2\{\lambda_V+\varepsilon^k\}^2\Big]}\\&\qquad\times\|u_k-u_{k-1}\|.\end{aligned} \tag{4.13}$$

Using (4.12) and (4.13) in (4.7), we get

$$\begin{aligned}&\|t_{k+1}-t_k\|\\&\leq\Big[\sqrt{\Big[\lambda_f^2\kappa^2+2\nu\lambda\{\lambda_V+\varepsilon^k\}^2+2\varepsilon_1\lambda\{\lambda_V+\varepsilon^k\}\Big\{\lambda_f\kappa+\tau\{\lambda_V+\varepsilon^k\}\Big\}+\varepsilon_1^2\lambda^2\{\lambda_V+\varepsilon^k\}^2\Big]}\\&\qquad+\varepsilon_2\lambda\{\lambda_W+\varepsilon^k\}\Big]\times\|u_k-u_{k-1}\|.\end{aligned} \tag{4.14}$$

By Lipschitz continuity of resolvent operator and condition (vii),(4.7), we have

$$\begin{aligned}\|u_k-u_{k-1}\|&=\Big\|u_k-u_{k-1}-(f(u_k)-f(u_{k-1}))+R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t_k)-R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(t_{k-1})\Big\|\\&\leq\|u_k-u_{k-1}-(f(u_k)-f(u_{k-1}))\|+\frac{\tau}{\ell-m\lambda}\|t_k-t_{k-1}\|\end{aligned} \tag{4.15}$$

$$\|u_k-u_{k-1}-(f(u_k)-f(u_{k-1}))\|^2\leq(1-2r+\lambda_f^2)\|u_k-u_{k-1}\|^2. \tag{4.16}$$

Using (4.16) in (4.15), we have

$$\begin{aligned}\|u_k-u_{k-1}\|&\leq\sqrt{1-2r+\lambda_f^2}\|u_k-u_{k-1}\|+\frac{\tau}{(\ell-m\lambda)}\|t_k-t_{k-1}\|\\\|u_k-u_{k-1}\|&\leq\Big[\frac{\tau}{(1-\sqrt{1-2r+\lambda_f^2})(\ell-m\lambda)}\Big]\|t_k-t_{k-1}\|.\end{aligned} \tag{4.17}$$

Using (4.17) in (4.14), equation (4.14) becomes

$$\|t_{k+1}-t_k\|\leq\Theta(\varepsilon^k)\|t_k-t_{k-1}\|,\ \text{where} \tag{4.18}$$

$$\Theta(\varepsilon^k) = \frac{\tau\sqrt{\left\{\lambda_f^2\kappa^2+2\nu\lambda\{\lambda_V+\varepsilon^k\}^2+2\varepsilon_1\lambda\{\lambda_V+\varepsilon^k\}\left(\lambda_f\kappa+\tau\{\lambda_V+\varepsilon^k\}\right)+\varepsilon_1^2\lambda^2\{\lambda_V+\varepsilon^k\}^2\right\}}+\tau\varepsilon_2\lambda\{\lambda_W+\varepsilon^k\}}{\left(1-\sqrt{1-2r+\lambda_f^2}\right)(\ell-m\lambda)}.$$

Since $0<\varepsilon<1$, this implies that $\Theta(\varepsilon^k)\to\Theta$ as $k\to\infty$, where

$$\Theta = \frac{\tau\Big[\sqrt{\left\{\lambda_f^2\kappa^2+2\nu\lambda\lambda_V^2+2\varepsilon_1\lambda\lambda_V\left(\lambda_f\kappa+\tau\lambda_V\right)+\varepsilon_1^2\lambda^2\lambda_V^2\right\}}+\varepsilon_2\lambda\lambda_W\Big]}{(1-\sqrt{1-2r+\lambda_f^2})(\ell-m\lambda)}.$$

It is given that $\Theta<1$, then $\{t_k\}$ is a Cauchy sequence in Banach space $Y$, then $t_k\to t$ as $k\to\infty$. From (4.17), $\{u_k\}$ is also Cauchy sequence in Banach space $Y$, then there exist $u$ such that $u_k\to u$.

From equation (4.5)-(4.7) and Algorithm 4.3, the sequences $\{v_k\}$ and $\{w_k\}$ are also Cauchy sequences in $Y$. Thus, there exist $v$ and $w$ such that $v_k\to v$ and $w_k\to w$ as $k\to\infty$. Next we will prove that $v\in V(u)$. Since $v_k\in V(u)$, then

$$\begin{aligned} d(v,\, V(u)) &\le \|v-v_k\| + d(v_k,\, V(u)) \\ &\le \|v-v_k\| + D(V(u_k),\, V(u)) \\ &\le \|v-v_k\| + \lambda_V\,\|u_k-u\|\to 0, \text{ as } k\to\infty, \end{aligned}$$

which gives $d(v,\, V(u))=0$. Due to $V(u)\in CB(Y)$, we have $v\in V(u)$. In the same manner, we easily show that $w\in W(u)$.

By the continuity of $R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}$, $Q$, $R$, $S$, $V$, $W$, $\varphi oG$, $f$, $\eta$ and $M$ and Algorithms 4.3, we know that $u$, $v$, $w$ and $k\to t$ satisfy

$$\begin{aligned} t_{k+1} &= [H(Q(fu_k),R(fu_k),S(fu_k))-\varphi oG(v_k,w_k)], \\ \to t &= [H(Q(fu),R(fu),S(fu))-\varphi oG(v,w)] \text{ as } k\to\infty \\ R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(t_k) &= f(u_k)\to f(u)=R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(t) \text{ as } k\to\infty. \end{aligned}$$

Now using the Lemma 4.2, we have

$$\varphi oG(v,w)+\lambda^{-1}(t-H(Q(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(t)),R(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(t)),S(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(t)))=0,$$

Thus we have

$$\varphi oG(v,w)+\lambda^{-1}J_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(t)=0. \tag{4.19}$$

Hence $(t,u,v,w)$ is a solution of the problem (4.3).

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

## REFERENCES

[1] R. Ahmad, M. Dilshad, Variational-Like Inclusions and Resolvent Equations Involving Infinite Family of Set-Valued Mappings, Fixed Point Theory Appl. 2011 (2011), 635030.

[2] R. Ahmad1, A.H. Siddiqi, M. Dilshad, Application of $H(.,.)$-cocoercive operators for solving a set-valued variational inclusion problem via a resolvent equation problem, Indian J. Ind. Appl. Math. 4 (2) (2013), 160-169.

[3] J.P. Aubin, A. Cellina, Differential inclusions, Springer-Verlag, Berlin, 1984.

[4] M.I. Bhut, B. Zahoor, Existence of solution and iterative approximation of a system of generalized variational-like inclusion problems in semi-inner product spaces, Filomat, 31 (19) (2017), 6051-6070.

[5] M.I. Bhat, B. Zahoor, Approximation solvability for a system of variational-like inclusions involving generalized $(H,\varphi)$-$\eta$-monotone operators, Int. J. Mod. Math. Sci. 15 (1) (2017), 30-49.

[6] S.S. Chang, J.K. Kim, K.H. Kim, On the existence and iterative approximation problems of solutions for set-valued variational inclusions in Banach spaces, J. Math. Anal. Appl. 268(1) (2002), 89-108.

[7] Y.-P. Fang and N.-J. Huang, $H$-monotone operator and resolvent operator technique for variational inclusions, Appl. Math. Comput. 145(2-3) (2003), 795-803.

[8] Y.P. Fang, N.J. Huang, Approximate solutions for nonlinear operator inclusions with $(H,\eta)$-monotone operator, Research Report, Sichuan University, 2003.

[9] Y.P. Fang, Y.J. Cho, J.K. Kim, $(H,\eta)$-accretive operator and approximating solutions for systems of variational inclusions in Banach spaces, preprint, 2004.

[10] H.R. Feng, X.P. Ding, A new system of generalized nonlinear quasi-variational-like inclusions with $A$-monotone operators in Banach spaces, J. Comput. Appl. Math. 225 (2009), 365-373.

[11] J.R. Giles, Classes of semi-inner product spaces, Trans. Amer. Math. Soc. 129 (1963), 436–446.

[12] S. Gupta, $H(.,.)$-$\varphi$-$\eta$-mixed accretive mapping with an application, J. Math. Comput. Sci., 10(6) (2020), 2327-2341.

[13] S. Gupta, M. Singh, Resolvent operator approach connected with $H(.,.)$-$\varphi$-$\eta$-mixed monotone mapping with an application, J. Math. Comput. Sci. 10(6) (2020), 3048-3064.

[14] S. Gupta, M. Singh, Variational-like inclusions involving infinite family of set-valued mappings governed by resolvent equations, J. Math. Comput. Sci. 11(1) (2021), 874-892.

[15] N.-J. Huang, M.R. Bai, Y.J. Cho, S.M. Kang, Generalized non-linear quasi mixed variational inequalities, Comput. Math. Appl. 40(2-3) (2000), 205-215.

[16] S. Husain, S. Gupta, V.N. Mishra, Graph convergence for the $H(.,.)$-mixed mapping with an application for solving the system of generalized variational inclusions, Fixed Point Theory Appl. 2013 (2013), 304.

[17] S. Husain, S. Gupta, V.N. Mishra, Generalized $H(.,.,.)$-$\eta$-cocoercive operators and generalized set-valued variational-like inclusions, J. Math. 2013 (2013), 738491.

[18] S. Husain, H. Sahper, S. Gupta, H(., ., .)- $\eta$-Proximal-Point Mapping with an Application, in: J.M. Cushing, M. Saleem, H.M. Srivastava, M.A. Khan, M. Merajuddin (Eds.), Applied Analysis in Biological and Physical Sciences, Springer India, New Delhi, 2016: pp. 351–372.

[19] S. Husain, S. Gupta, $H((.,.),(.,.))$-mixed cocoercive operators with an application for solving variational inclusions in Hilbert spaces, J. Funct. Space. Appl. 2013 (2013), 378364.

[20] S. Gupta, S. Husain, V.N. Mishra, Variational inclusion governed by $\alpha\beta$-$H((.,.),(.,.))$-mixed accretive mapping, Filomat, 31(20) (2017), 6529-6542.

[21] D. Koehler, A note on some operator theory in certain semi-inner-product space, Proc. Amer. Math. Soc. 30 (1971), 363-366.

[22] J. Lou, X.F. He, Z. He, Iterative methods for solving a system of variational inclusions $H$-$\eta$-monotone operators in Banach spaces, Comput. Math. Appl. 55 (2008), 1832-1841.

[23] X.P. Luo, N.J. Huang, $(H,\phi)$-$\eta$-monotone operators in Banach spaces with an application to variational inclusions, Appl. Math. Comput. 216 (2010), 1131-1139.

[24] G. Lumer, Semi-inner product spaces, Trans. Amer. Math. Soc. 100 (1961), 29-43.

[25] J.-W. Peng, D.L. Zhu, A new system of generalized mixed quasi-vatiational inclusions with $(H,\eta)$-monotone operators, J. Math. Anal. Appl. 327 (2007), 175-187.

[26] N.K. Sahu, R.N. Mohapatra, C. Nahak, S. Nanda, Approximation solvability of a class of $A$-monotone implicit variational inclusion problems in semi-inner product spaces, Appl. Math. Comput. 236 (2014), 109-117.

[27] N.K. Sahu, C. Nahak, S. Nanda, Graph convergence and approximation solvability of a class of implicit variational inclusion problems in Banach spaces, J. Indian Math. Soc. 81 (12) (2014), 155-172.

[28] R.U. Verma, Approximation solvability of a class of nonlinear set-valued inclusions involving $(A,\eta)$-monotone mappings, J. Math. Appl. Anal. 337 (2008), 969-975.

[29] R.U. Verma, General class of implicit variational inclusions and graph convergence on $A$-maximal relaxed monotonicity, J. Optim. Theory Appl. 155 (1) (2012), 196-214.

[30] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal., Theory Meth. Appl. 16(12) (1991), 1127-1138.

[31] Z.H. Xu, Z.B. Wang, A generalized mixed variational inclusion involving $(H(.,.),\eta)$-monotone operators in Banach spaces, J. Math. Res. 2(3) (2010), 47-56.

[32] Y.-Z. Zou, N.-J. Huang, $H(.,.)$-accretive operator with an application for solving variational inclusions in Banach spaces, Appl. Math. Comput. 204(2) (2008), 809-816.