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COMMON FIXED POINTS FOR GENERALIZED $(\alpha-\psi)$-MEIR-KEELER-KHAN MAPPINGS IN G-METRIC SPACES

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Abstract. In this paper, we present $(\alpha-\psi)$-Meir-Keeler-Khan contraction type mappings for two pairs of weakly compatible self-mappings in G-metric space.

Keywords: common fixed point; complete G-metric space; generalized (α,ψ) Meir-Keeler-Khan type contractions; weakly contractive mappings; α-admissible mappings.

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1. Introduction

In the transition from classical analysis to modern analysis, fixed point theory plays an important role. A main technique for non linear analysis is the Banach fixed theorem that guarantees the existence and uniqueness of complete metric space self-mappings and offers a realistic approach too finding fixed points. There are a lot of generalizations of Banach contraction theorem in complete metric space where the contractive nature of mappings is weakened. Many authors $[1,2,4,8-11,14-16,21,25-27]$ have extended this classical theory in several different directions. Meir-Keeler [15] gave a classical generalization. They studied the fixed point of the class of mappings satisfying the condition that for each $\varepsilon>0$, there exists $\delta(\varepsilon)>0$

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such that $\varepsilon \leq d(\rho_1,\rho_2) < \varepsilon + \delta(\varepsilon)$ implies $d(f\rho_1, f\rho_2) < \varepsilon$ for any $\rho_1,\rho_2 \in M$. Afterwards, this condition was extended and enhanced by many authors and fixed points results described [6, 12, 13, 17, 19, 20, 22, 23].

In 1996, the definition of weakly compatible mapping was introduced by Jungck and Rhodes[11] and showed that compatible mapping is weakly compatible but does not necessarily have inverse mapping.

In this paper, we research and define the fixed point results for four mappings based on Meir-Keeler-Khan type contraction in complete G-metric space with α-admissible weakly compatible mappings.

Although, we present some consequences of our new results. In this sequel, the following definitions will be used.

Let Ψ[22] be the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=}^{\infty} \psi^n(t) < +\infty$, for all $t > 0$, where ψ^n is the nth iterate of ψ.

Definition 1.1. [18] Let M be a non empty set, and $G : M \times M \times M \to R^+$ be a function satisfying the following properties:

(1) $G(\rho_1,\rho_2,\rho_3) = 0$ if $\rho_1 = \rho_2 = \rho_3$,
(2) $0 < G(\rho_1,\rho_1,\rho_2)$, for all $\rho_1,\rho_2 \in M$, with $\rho_1 \neq \rho_2$,
(3) $G(\rho_1,\rho_1,\rho_2) \leqslant G(\rho_1,\rho_2,\rho_3)$, for all $\rho_1,\rho_2,\rho_3 \in M$, with $\rho_3 \neq \rho_2$,
(4) $G(\rho_1,\rho_2,\rho_3) = G(\rho_1,\rho_3,\rho_2) = G(\rho_2,\rho_3,\rho_1)$ = ...(symmetry in all three variables),
(5) $G(\rho_1,\rho_2,\rho_3) \leq G(\rho_1,a,a) + G(a,\rho_2,\rho_3)$, for all $\rho_1,\rho_2,\rho_3,a \in M$, (rectangular inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on M, and the pair (M,G) is called a *G-metric space*.

Definition 1.2. [18] Let (M,G) be a G-metric space. A sequence ρ_n in M is said to be G-convergent if for $\varepsilon > 0$, there is an $\rho_1 \in M$ and $p \in N$ such that $n, p \geq N, G(\rho_1, \rho_{1_n}, \rho_{1_p}) < \varepsilon$.

Definition 1.3. [18] Let (M,G) be a G-metric space. A sequence ρ_n in M is said to be G-Cauchy if for each $\varepsilon > 0$, there exists $p \in N$ such that $G(\rho_{1_n}, \rho_{1_p}, \rho_{1_x}) < \varepsilon$ for all $n, p, x \geq N$.

Definition 1.4. [18] A G-metric space M is said to complete if every G-Cauchy sequence in M is G-convergent in M.

Lemma 1.5. [3] *Let $\psi \in \Psi$. Then*

(1) $\psi(t) < t$, *for all* $t > 0$,

(2) $\psi(0) = 0$.

Definition 1.6. [11] Let M has two self-maps S and f. If $S\rho_1 = f\rho_1$, for some $\rho_1 \in M$, then l is called coincidence point of S and f.

Definition 1.7. [11] Let M has two self mappings S and f. If they commute at coincidence point then S and f are weakly compatible. That is, if $S\rho_1 = f\rho_1$, for some $\rho_1 \in M$, then $Sf\rho_1 = fS\rho_1$.

On the other hand, Samet et al. [24] introduced the notion of $\alpha - \psi$ contractive mapping in a metric space using α-admissible mapping and proved the result of a fixed point in a complete metric space for $\alpha - \psi$ contractive mappings.

Definition 1.8. [24] Let $f : M \to M$ and $\alpha : M \times M \to [0,\infty)$ be two mappings. The mapping f is said to be an α-admissible if the following condition satisfied:

$$\forall \rho_1, j \in M, \quad \alpha(\rho_1, j) \geq 1 \quad \textit{implies} \quad \alpha(f\rho_1, fj) \geq 1. \tag{1.1}$$

For four self mappings Patel et al. [19] introduced α-admissible criterion.

Definition 1.9. Let $T, \Im, S, f : M \to M$ be four self-mappings of a non-empty set M and let $\alpha : T(M) \cup \Im(M) \cup \Im(M) \times T(M) \cup \Im(M) \cup \Im(M) \to [0,\infty)$ be a mapping. A pair (S, f) is called an α-admissible with respect to T and $\Im$, if for all $\rho_1, j \in M$, $\alpha(T\rho_1, \Im j, \Im j) \geq 1$ or $\alpha(\Im\rho_1, Tj, Tj) \geq 1$, implies

$$\alpha(S\rho_1, fj, fj) \geq 1 \quad \textit{and} \quad \alpha(f\rho_1, Sj, Sj) \geq 1. \tag{1.2}$$

Fisher [9] demonstrated the results provided by Khan [14] edition.

Theorem 1.10. [9] *Let f be a self map satisfying the following on a complete metric space (M,d) such that:*

$$d(f\rho_1, fj) \leq \mu \frac{d(\rho_1, fl)d(\rho_1, fj) + d(j, fj)d(j, f\rho_1)}{d(x, fj) + d(j, f\rho_1)}, \quad \mu \in [0,1] \tag{1.3}$$

if $d(\rho_1, fj) + d(j, f\rho_1) \neq 0$ $\quad d(f\rho_1, fj) = 0$, *if* $d(\rho_1, fj) + d(j, f\rho_1) = 0$. *Then f has a unique fixed point $x \in M$. Moreover, for every $x_0 \in M$, the sequence $\{f^n x_0\}$ converges to x.*

Definition 1.11. Let (M, G) be a G-metric space and $f : M \to M$ be a self mapping. f is called (α, ψ)-Meir-Keeler-Khan mapping, if there exists $\psi \in \Psi$ and $\alpha : M \times M \times M \to [0, \infty)$ satisfying the following conditions:

For each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \psi\left(\frac{G(\rho_1, f(\rho_1), f(\rho_1))G(\rho_1, f(j), f(j)) + G(j, f(j), f(j))G(j, f(\rho_1), f(\rho_1))}{G(\rho_1, f(j), f(j)) + G(j, f(\rho_1), f(\rho_1))}\right) < \varepsilon + \delta(\varepsilon)$$

implies

$$\alpha(\rho_1, j, j)G(f(\rho_1), f(j), f(j)) < \varepsilon. \tag{1.4}$$

2. Main Results

We introduced the class of common fixed point results for two pairs of weakly compatible self mappings in complete G-metric space satisfying (α, ψ)-Meir-Keeler-Khan type contractive through α-admissible mappings, in this section.

Definition 2.1. Let (M, G) be a complete G-metric space. The self-mappings $T, \Im, S, f : M \to M$ are said to be (α, ψ)-Meir-Keeler-Khan type, if there exists $\psi \in \Psi$ and $\alpha : T(M) \cup \Im(M) \cup \Im(M) \times T(M) \cup \Im(M) \cup \Im(M) \to [0, \infty)$ satisfying the following condition:

For each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that,

$$\varepsilon \leq \psi\left(\frac{G(T\rho_1, S\rho_1, S\rho_1)G(T\rho_1, fj, fj) + G(\Im j, fj, fj)G(\Im j, S\rho_1, S\rho_1)}{G(T\rho_1, fj, fj) + G(\Im j, S\rho_1, S\rho_1)}\right) < \varepsilon + \delta(\varepsilon)$$

$$\alpha(T\rho_1, \Im j, \Im j)G(S\rho_1, fj, fj) < \varepsilon. \tag{2.1}$$

Remark 2.2. It is easy to see that if $T, \Im, S, f : M \to M$ are (α, ψ)-Meir-Keeler-Khan type mappings, then

$$\alpha(T\rho_1, \Im j, \Im j)G(S\rho_1, fj, fj) \leq \psi\left(\frac{G(T\rho_1, S\rho_1, S\rho_1)G(T\rho_1, fj, fj) + G(\Im j, fj, fj)G(\Im j, S\rho_1, S\rho_1)}{G(T\rho_1, fj, fj) + G(\Im j, S\rho_1, S\rho_1)}\right) \tag{2.2}$$

for all $\rho_1, j \in M$.

In 2020, Arshad et al. [5] proved the following theorem in metric space and now we have extended their results in G-metric space.

Theorem 2.3. *Let (M,G) be a complete G-metric space and $T,\Im,S,f : M \to M$ be an (α,ψ)-Meir-Keeler-Khan type mappings such that $f(M) \subseteq T(M)$ and $S(M) \subseteq \Im(M)$. Assume that:*

(1) The pair (S,f) with respect to T and $\Im$ is α-admissible;

(2) There exists $\rho_{1_0} \in M$, such that $\alpha(T\rho_{1_0},S\rho_{1_0},S\rho_{1_0}) \geq 1$;

(3) One of $T,\Im,S$ and f is also continuous;

(4) (S,T) and $(f,\Im)$ are self-mappings that are weakly compatible pairs.

Then $\nu \in M$ is the common fixed point of $T,\Im,S$and f .

Proof. By assumption (2), there exists $\rho_{1_0} \in M$ such that $\alpha(T\rho_{1_0},S\rho_{1_0},S\rho_{1_0}) \geq 1$. Define the sequence $\{\rho_{1_n}\}$ and $\{j_n\}$ in M such that

$$j_{2n} = S\rho_{1_{2n}} = \Im\rho_{1_{2n+1}} \mathit{and} \quad j_{2n+1} = f\rho_{1_{2n+1}} = T\rho_{1_{2n+2}} \tag{2.3}$$

This is possible, since $f(M) \subseteq T(M)$ and $S(M) \subseteq \Im(M)$. Since (S,f) is $\alpha_{T,\Im}$-admissible, we have

$$\alpha(T\rho_{1_0},S\rho_{1_0},S\rho_{1_0}) = \alpha(T\rho_{1_0},\Im\rho_{1_1},\Im\rho_{1_1}) \geq 1$$

$$\text{implies } \alpha(S\rho_{1_0},f\rho_{1_1},f\rho_{1_1}) \geq 1 \quad \text{and } \alpha(f\rho_{1_0},S\rho_{1_1},S\rho_{1_1}) \geq 1,$$

which gives

$$\alpha(\Im\rho_{1_1},T\rho_{1_2},T\rho_{1_2}) \geq 1 \quad = \quad \alpha(j_0,j_1,j_1) \geq 1.$$

Again by using (1), we have

$$\begin{aligned} \alpha(\Im\rho_{1_1},f\rho_{1_1},f\rho_{1_1}) \quad &= \quad \alpha(\Im\rho_{1_1},T\rho_{1_2},T\rho_{1_2}) \geq 1 \quad \text{implies} \\ \alpha(f\rho_{1_1},S\rho_{1_2},S\rho_{1_2}) \geq 1 \quad &\mathit{and} \quad \alpha(S\rho_{1_1},f\rho_{1_2},f\rho_{1_2}) \geq 1, \end{aligned}$$

which gives,

$$\alpha(T\rho_{1_2},\Im\rho_{1_3},\Im\rho_{1_3}) \quad = \quad \alpha(j_1,j_2,j_2) \geq 1.$$

Persuasively, we get

$$\alpha(j_{2n},j_{2n+1},j_{2n+1}) \geq 1, \quad n = 0,1,2,... \tag{2.4}$$

That is $\alpha(T\rho_{1_{2n}},\Im\rho_{1_{2n+1}},\Im\rho_{1_{2n+1}})\geq 1$ and $\alpha(\Im\rho_{1_{2n+1}},T\rho_{1_{2n+2}},T\rho_{1_{2n+2}})\geq 1$. By (2.2) and (2.4), we have

$$\begin{aligned}
G(j_{2n},j_{2n+1},j_{2n+1}) &= G(S\rho_{1_{2n}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})\\
&\leq \alpha(T\rho_{1_{2n}},\Im\rho_{1_{2n+1}},\Im\rho_{1_{2n+1}})G(S\rho_{1_{2n}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})\\
&\leq \psi\Big(\tfrac{G(T\rho_{1_{2n}},S\rho_{1_{2n}},S\rho_{1_{2n}})G(T\rho_{1_{2n}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})+G(\Im\rho_{1_{2n+1}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})G(\Im\rho_{1_{2n+1}},S\rho_{1_{2n}},S\rho_{1_{2n}})}{G(T\rho_{1_{2n}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})+G(\Im\rho_{1_{2n+1}},S\rho_{1_{2n}},S\rho_{1_{2n}})}\Big)\\
&\leq \psi\Big(\tfrac{f(T\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})G(f\rho_{1_{2n-1}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})+G(S\rho_{1_{2n}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})G(S\rho_{1_{2n}},S\rho_{1_{2n}},S\rho_{1_{2n}})}{G(f\rho_{1_{2n-1}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})+G(S\rho_{1_{2n}},S\rho_{1_{2n}},S\rho_{1_{2n}})}\Big)\\
&\leq \psi\Big(\tfrac{G(f\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})G(f\rho_{1_{2n-1}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})}{G(f\rho_{1_{2n-1}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})}\Big)\\
&\leq \psi G(f\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})\\
&\leq \psi G(j_{2n-1},j_{2n},j_{2n}),\quad \text{for all } n\in N
\end{aligned}$$

Now,

$$\begin{aligned}
G(j_{2n-1},j_{2n},j_{2n}) &= G(f\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})\leq \alpha(\Im\rho_{1_{2n-1}},T\rho_{1_{2n}},T\rho_{1_{2n}})G(f\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})\\
&\leq \psi\Big(\tfrac{G(\Im\rho_{1_{2n-1}},f\rho_{1_{2n-1}},f\rho_{1_{2n-1}})G(\Im\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})+G(T\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})G(T\rho_{1_{2n-1}},f\rho_{1_{2n-1}},f\rho_{1_{2n-1}})}{G(T\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})+G(T\rho_{1_{2n-1}},f\rho_{1_{2n-1}},f\rho_{1_{2n-1}})}\Big)\\
&\leq \psi\Big(\tfrac{G(S\rho_{1_{2n-2}},f\rho_{1_{2n-1}},f\rho_{1_{2n-1}})G(S\rho_{1_{2n}},S\rho_{1_{2n}},S\rho_{1_{2n}})+G(f\rho_{1_{2n-1}},S\rho_{1_{2n}},S\rho_{1_{2n}})G(f\rho_{1_{2n-1}},f\rho_{1_{2n-1}},f\rho_{1_{2n-1}})}{G(S\rho_{1_{2n-2}},S\rho_{1_{2n}},S\rho_{1_{2n}})+G(f\rho_{1_{2n-1}},f\rho_{1_{2n-1}},f\rho_{1_{2n-1}})}\Big)\\
&\leq \psi G(S\rho_{1_{2n-2}},f\rho_{1_{2n-1}},f\rho_{1_{2n-1}})\leq \psi(j_{2n-2},j_{2n-1},j_{2n-1}).
\end{aligned}$$

That is

$$G(j_{2n},j_{2n+1},j_{2n+1})\leq \psi G(j_{2n-1},j_{2n},j_{2n})\leq \psi^2 G(j_{2n-2},j_{2n-1},j_{2n-1})$$

Proceeding in the same way, we get

$$G(j_{2n},j_{2n+1},j_{2n+1})\leq \psi^{2n}G(j_0,j_1,j_1).$$

Now, we write the above inequality as

$$G(j_n,j_{n+1},j_{n+1})\leq \psi^n G(j_0,j_1,j_1).$$

Now, we will deduce that $\{j_n\}$ ia a Cauchy sequence. By the properties of the function ψ, for any $\varepsilon > 0$ there exists $n(\varepsilon) \in N$ such that $\sum_{n \geq n(\varepsilon)} \psi^n(G(j_0, j_1, j_1)) < \varepsilon$. Let $n, p \in N$ with $n > p > n(\varepsilon)$, using the rectangular inequality, we get

$$G(j_p, j_n, j_n) \leq \sum_{k=p}^{n-1} G(j_p, j_{p+1}, j_{p+1})$$

$$\leq \psi^p(G(j_0, j_1, j_1))$$

$$\leq \sum_{p=n(\varepsilon)} \psi^p(G(j_0, j_1, j_1)) < \varepsilon.$$

So, we deduce that $\{j_n\}$ is a Cauchy sequence in a complete G-metric space (M, G). There exists $\nu \in M$ such that $\lim_{n\to\infty} j_n = \nu$ and sequentially, $S\rho_{1_{2n}}, \Im\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}, T\rho_{1_{2n+2}} \to \nu$, as $n \to \infty$. By assumption (3)

$$\lim_{n\to\infty} S\rho_{1_{2n}} = \lim_{n\to\infty} \Im\rho_{1_{2n+1}} = \lim_{n\to\infty} f\rho_{1_{2n+1}} = \lim_{n\to\infty} T\rho_{1_{2n+2}} = \nu.$$

Since $f(M) \subseteq T(M)$, there exists $\beta_1 \in M$ such that $\nu = T\beta_1$. By (2.2) and (2.4), we have

$$G(S\beta_1, \nu, \nu) \leq G(S\beta_1, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) + G(f\rho_{1_{2n+1}}, \nu, \nu)$$

$$\leq \alpha(T\beta_1, \Im\rho_{1_{2n+1}}, \Im\rho_{1_{2n+1}})G(S\beta_1, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) + G(f\rho_{1_{2n+1}}, \nu, \nu)$$

$$\leq$$

$$\psi\Big(\frac{G(T\beta_1,S\beta_1,S\beta_1)G(Tu,f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})+G(\Im\rho_{1_{2n+1}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})G(\Im\rho_{1_{2n+1}},S\beta_1,S\beta_1)}{G(T\beta_1,f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})+G(\Im\rho_{1_{2n+1}},S\beta_1,S\beta_1)}\Big) + G(f\rho_{1_{2n+1}}, \nu, \nu)$$

$$\leq \psi\Big(\frac{G(\nu,S\beta_1,S\beta_1)G(\nu,f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})+G(S\rho_{1_{2n}},f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})G(S\rho_{1_{2n}},S\beta_1,S\beta_1)}{G(\nu,f\rho_{1_{2n+1}},f\rho_{1_{2n+1}})+G(S\rho_{1_{2n}},S\beta_1,S\beta_1)}\Big) + G(f\rho_{1_{2n+1}}, \nu, \nu).$$

Putting $\lim_{n\to\infty}$ in above inequality, we get

$$G(S\beta_1, \nu, \nu) \leq \psi\Big(\frac{G(\nu,S\beta_1,S\beta_1)G(\nu,\nu,\nu)+G(\nu,\nu,\nu)G(\nu,S\beta_1,S\beta_1)}{G(\nu,\nu,\nu)+G(\nu,S\beta_1,S\beta_1)}\Big) + G(\nu, \nu, \nu) = 0.$$

That is $S\beta_1 = \nu$. Thus $T\beta_1 = S\beta_1 = \nu$. Therefore $\beta 1$ is a coincidence point of T and S. Since the pair of mappings S and T are weakly compatible, we have

$$ST\beta_1 = TS\beta_1,$$

$$S\nu = T\nu.$$

Since $S(M) \subseteq \Im(M)$, there exists a point $\beta_2 \in M$ such that $\nu = \Im\beta_2$. By (2.2) and (2.4), we have

$$G(\nu, f\beta_2, f\beta_2) = G(S\beta_1, f\beta_2, f\beta_2) \leq \alpha(T\beta_1, \Im\beta_2, \Im\beta_2)G(S\beta_1, f\beta_2, f\beta_2)$$

$$\leq \psi\left(\frac{G(T\beta_1,S\beta_2,S\beta_2)G(T\beta_1,f\beta_2,f\beta_2)+G(\Im\beta_2,f\beta_2,f\beta_2)G(\Im\beta_2,S\beta_1,S\beta_1)}{G(T\beta_1,f\beta_2,f\beta_2)+G(\Im\beta_2,S\beta_1,S\beta_1)}\right)$$

$$\leq \psi\left(\frac{G(\nu,\nu,\nu)G(\nu,f\beta_2,f\beta_2)+G(\nu,f\beta_2,f\beta_2)G(\nu,\nu,\nu)}{G(\nu,f\beta_2,f\beta_2)+G(\nu,\nu,\nu)}\right) \leq \psi(0).$$

That is $G(\nu,f\beta_2,f\beta_2)=0$. Thus, $\nu = f\beta_2$. Therefore, $f\beta_2 = \Im\beta_2 = \nu$. So β_2 is coincident point of $\Im$ and f. Since, the pair of maps $\Im$ and f are weakly compatible.

$$\Im f\beta_2 = f\Im\beta_2,$$

$$\Im\nu = f\nu.$$

Now, we show that $S\nu = \nu$. By (2.2) and (2.4), we get

$$G(S\nu,\nu,\nu) = G(S\nu,f\beta_2,f\beta_2) \leq \alpha(T\nu,\Im\beta_2,\Im\beta_2)G(S\nu,f\beta_2,f\beta_2)$$

$$\leq \left(\frac{G(T\nu,S\nu,S\nu)G(T\nu,f\beta_2,f\beta_2)+G(\Im\beta_2,f\beta_2,f\beta_2)G(\Im\beta_2,S\nu,S\nu)}{G(T\nu,f\beta_2,f\beta_2)+G(\Im\beta_2,S\nu,S\nu)}\right)$$

$$\leq \psi\left(\frac{G(S\nu,S\nu,S\nu)G(S\nu,\nu,\nu)+G(\nu,\nu,\nu)G(\nu,S\nu,S\nu)}{G(\nu,\nu,\nu)+G(\nu,S\nu,S\nu)}\right)$$

$$G(S\nu,\nu,\nu) = 0.$$

So, $G(S\nu,\nu,\nu)=0$. Thus, $S\nu = \nu$. Hence,

$$S\nu = T\nu = \nu.$$

Now, we show that $f\nu = \nu$. By using (2.2) and (2.4), we get

$$G(\nu,f\nu,f\nu) = G(S\nu,f\nu,f\nu) \leq \alpha(T\nu,\Im\nu,\Im\nu)G(S\nu,f\nu,f\nu)$$

$$\leq \psi\left(\frac{G(T\nu,S\nu,S\nu)G(T\nu,f\nu,f\nu)+G(\Im\nu,f\nu,f\nu)G(\Im\nu,S\nu,S\nu)}{G(T\nu,f\nu,f\nu)+G(\Im\nu,S\nu,S\nu)}\right)$$

$$\leq \psi\left(\frac{G(\nu,\nu,\nu)G(\nu,f\nu,f\nu)+G(f\nu,f\nu,f\nu)G(f\nu,\nu,\nu)}{G(\nu,f\nu,f\nu)+G(f\nu,\nu,\nu)}\right) = \psi(0) = 0$$

Thus, $G(\nu,f\nu,f\nu)=0$. That is, $\nu = f\nu$. Therefore, $f\nu = \Im\nu = \nu$. Thus, $S\nu = T\nu = f\nu = \Im\nu = \nu$.

Hence $T,\Im,S$ and f have a common fixed point ν. □

Theorem 2.4. *Let (M,G) be a complete G-metric space and $T,\Im,S,f : M \to M$ be an (α,ψ)-Meir-Keeler-Khan type mappings such that $f(M) \subseteq T(M)$ and $S(M) \subseteq \Im(M)$. Assume that:*

(1) The pair (S,f) is α-admissible with respect to T and $\Im$;

(2) There exists $\rho_{1_0} \in M$ such that $\alpha(T\rho_{1_0},S\rho_{1_0},S\rho_{1_0}) \geq 1$;

(3) If $\{j_n\}$ is a sequence in M such that $\alpha(j_n, j_{n+1}, j_{n+1}) \geq 1$ for all $n \in N$ and $j_n \to \nu \in M$ as $n \to \infty$, then $\alpha(j_n, \nu, \nu) \geq 1$, for all $n \in N$.

Then ν is the common fixed point of $T, \Im, S$ and f such that $\nu \in M$ given $(f, \Im)$ are weakly compatible pairs of self-mappings.

Proof. We obtain the sequence $\{j_n\}$ in M by following the proof of Theorem 2.3 which is defined by:

$$j_{2n} = S\rho_{1_{2n}} = \Im\rho_{1_{2n+1}} \text{ and } \quad j_{2n+1} = f\rho_{1_{2n+1}} = T\rho_{1_{2n+2}},$$

for all $n \geq 0$, which converges to some $\nu \in M$. Sequentially,

$$S\rho_{1_{2n}}, \quad \Im\rho_{1_{2n+1}}, \quad f\rho_{1_{2n+1}}, \quad T\rho_{1_{2n+2}} \to \nu,$$

as $n \to \infty$. Since $f(M) \subseteq T(M)$, there exists $\beta 1 \in M$ such that $\nu = T\beta_1$. By (3) and (2.4), we have

$$\begin{aligned} G(S\beta_1, \nu, \nu) = G(S\beta_1, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) &\leq \alpha(T\beta_1, \Im\rho_{1_{2n+1}}, \Im\rho_{1_{2n+1}}) G(S\beta_1, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) \\ &\leq \psi\left(\frac{G(T\beta_1, S\beta_1, S\beta_1) G(T\beta_1, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) + G(\Im\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) G(\Im\rho_{1_{2n+1}}, S\beta_1, S\beta_1))}{G(T\beta_1, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) + G(\Im\rho_{1_{2n+1}}, S\beta_1, S\beta_1)}\right) \\ &\leq \psi\left(\frac{G(\nu, S\beta_1, S\beta_1) G(\nu, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) + G(S\rho_{1_{2n}}, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) G(S\rho_{1_{2n}}, S\beta_1, S\beta_1))}{G(\nu, f\rho_{1_{2n+1}}, f\rho_{1_{2n+1}}) + G(S\rho_{1_{2n}}, S\beta_1, S\beta_1)}\right) \end{aligned}$$

Putting $\lim_{n\to\infty}$ in above inequality we end up with

$$G(S\beta_1, \nu, \nu) \leq \psi\left(\frac{G(\nu, S\beta_1, S\beta_1) G(\nu, \nu, \nu) + G(\nu, \nu, \nu) G(\nu, S\beta_1, S\beta_1))}{G(\nu, \nu, \nu) + G(\nu, S\beta_1, S\beta_1)}\right) \leq 0.$$

Thus $S\beta_1 = \nu$, therefore, $T\beta_1 = S\beta_1 = \nu$. Therefore β_1 is a coincidence point of T and S. Since the pair of mappings S and T are weakly compatible, we have

$$ST\beta_1 = TS\beta_1,$$

$$S\nu = T\nu.$$

Similarly, as $S(M) \subseteq \Im(M)$, we obtain $G(\nu, f\beta_2, f\beta_2) = 0$. Thus, $\nu = f\beta_2$. Therefore $f\beta_2 = \Im\beta_2 = \nu$. So, β_2 is coincident point of $\Im$ and f. Since, the pair of maps $(\Im, f)$ are weakly compatible so,

$$\Im f\beta_2 = f\Im\beta_2,$$

$$\Im\nu = f\nu.$$

We can easily show that $S\nu = \nu$ and $f\nu = \nu$ and the proof is completed. $\square$

We will assume the following hypothesis for the uniqueness of the fixed points of a generalized (α,ψ)-Meir-Keeler-Khan type contractive mapping.

(H) For all common fixed points l and j of $T,\Im,S$ and f, there exists $\beta_2 \in M$ such that $\alpha(\rho_1,\beta_2,\beta_2) \geq 1$ and $\alpha(j,\beta_2,\beta_2) \geq 1$.

Theorem 2.5. *We get the uniqueness of common fixed point of S,T,f and $\Im$, by adding hypothesis (H) to the statement of Theorem* 2.3 *or* 2.4.

Proof. Theorem 2.3 (respectively Theorem 2.4) shows the existence of a fixed point. To prove its uniqueness assume that we have some x which is another common fixed point of $T,\Im,S$ and f such that $\nu \neq x$. By hypothesis (H), there exists $\beta_2 \in M$ such that $\alpha(T\nu,\beta_2,\beta_2) \geq 1$ and $\alpha(\Im x,\beta_2,\beta_2) \geq 1$. Define a sequence $\{\beta_{2_n}\}$ in M by

$$\beta 2_0 = S\beta 2_0 = \Im_1, \beta 2_{2n} = S\beta 2_{2n} = \Im\beta 2_{2n+1}$$

and

$$\beta 2_1 = f\beta 1_1 = T\beta 1_2, \beta 1_{2n+1} = f\beta 1_{2n+1} = T\beta 1_{2n+2},$$

for all $n \geq 0$. Since the pair (S,f) is $\alpha_{T,\Im}$-admissible, we get

$$\alpha(\nu,\beta 2_{2n},\beta 2_{2n}) \geq 1 \text{ and } \alpha(x,\beta 2_{2n},\beta 2_{2n}) \geq 1, \text{ for all } n.$$

Now, by Remark 2.2, we have

$$\begin{aligned}
G(\nu,\beta 2_{2n+1},\beta 2_{2n+1}) &= G(S\nu, f\beta 2_{2n+1}, f\beta 2_{n+1}) \\
&\leq \alpha(T\nu,\Im\beta 2_{2n+1},\Im\beta 2_{n+1})G(S\nu, f\beta 2_{2n+1}, f\beta 2_{2n+1}), \\
&\leq \psi\Big(\tfrac{G(T\nu,S\nu,S\nu)G(T\nu,f\beta 2_{2n+1},f\beta 2_{2n+1})+G(\Im\beta 2_{2n+1},f\beta 2_{2n+1},f\beta 2_{2n+1})G(\Im\beta 2_{2n+1},S\nu,S\nu)}{G(T\nu,f\beta 2_{2n+1},f\beta 2_{2n+1})+G(\Im\beta 2_{2n+1},S\nu,S\nu)}\Big) \\
&\leq \psi\Big(\tfrac{G(S\nu,S\nu,S\nu)G(S\nu,f\beta 2_{2n+1},f\beta 2_{2n+1})+G(\Im\beta 2_{2n+1},f\beta 2_{2n+1},f\beta 2_{2n+1})G(\Im\beta 2_{2n+1},S\nu,S\nu)}{G(S\nu,f\beta 2_{2n+1},f\beta 2_{2n+1})+G(\Im\beta 2_{2n+1},S\nu,S\nu)}\Big)
\end{aligned}$$

By rectangular inequality, we have

$$\begin{aligned}
G(\Im\beta 2_{2n+1}, f\beta 2_{2n+1}, f\beta 2_{2n+1}) &\leq \alpha(G(S\nu, f\beta 2_{2n+1}, f\beta 2_{2n+1}) + G(\Im\beta 2_{2n+1}, S\nu, S\nu)), \\
&\leq \psi\Big(\tfrac{G(\Im\beta 2_{2n+1},f\beta 2_{2n+1},f\beta 2_{2n+1})G(\Im\beta 2_{2n+1},S\nu,S\nu)}{G(S\nu,f\beta 2_{2n+1},f\beta 2_{2n+1})+G(\Im\beta 2_{2n+1},S\nu,S\nu)}\Big), \\
&\leq \psi(\Im\beta 2_{2n+1}, S\nu, S\nu),
\end{aligned}$$

$$\leq \psi G(\nu, \beta 2_{2n}, \beta 2_{2n}).$$

Iteratively, the inequality implies that

$$G(\nu, \beta 2_{2n+1}, \beta 2_{2n+1}) \leq \psi^{2n+1}(G(\nu, \beta 2_0, \beta 2_0)]), \qquad \text{for all } n.$$

Putting $n \to \infty$, in above inequality, we get

$$\lim_{n\to\infty} G(\beta 2_{2n}, \nu, \nu) = 0. \tag{2.5}$$

$$\lim_{n\to\infty} G(\beta 2_{2n}, x, x) = 0. \tag{2.6}$$

From (2.5), (2.6) we get $\nu = x$. □

There is an example to support Theorem 2.3

Example 2.6. Let $M = [1,20]$ and (M,G) be a G-metric space. Define $T, \Im, S$ and f as follows:

$$S(\rho_1) = 1 \text{ for all } l.$$

$$f(\rho_1) = \{1, \qquad \text{if } \rho_1 \in [1,7) \cup [8,20]$$
$$\rho_1 + 1, \qquad \text{if } \rho_1 \in [7,8]$$

$$T(\rho_1) = \{\rho_1, \qquad \text{if } \rho_1 \in [1,8]$$
$$8, \qquad \text{if } \rho_1 \in [8,20]$$

$$\Im(\rho_1) = \{1, \qquad \text{if } \rho_1 = 1,$$
$$5 \qquad \text{if } \rho_1 \in (1,7) \cup [8,20]$$
$$\rho_1 + 3, \qquad \text{if } \rho_1 \in [7,8).$$

Note that, $f(M) \subseteq T(M)$ and $S(M) \subseteq \Im(M)$, we note $Sl = Tl$ for which $l = 1$ implies $STl = TSl$ and $fl = \Im l$ implies $f\Im l = \Im fl$, thus the pairs $\{S,T\}$ and $\{f,\Im\}$ are weakly compatible.

Now, consider $\varepsilon = \frac{1}{4}$ and suppose that $\psi(t) = \frac{t}{4}$ then $T, \Im, S$ and f satisfy the (α,ψ)-Meir-Keeler contractive condition with the mapping $\alpha : T(M) \cup \Im(M) \cup \Im(M) \times T(M) \cup \Im(M) \cup \Im(M) \to [0,\infty)$ defined by

$$\alpha(p,q,r) = \{1, \qquad \text{if } p,q,r \in [1,7) \cup [11,20] \cup [11,20]$$
$$\frac{1}{5}, \qquad \text{otherwise.}$$

Clearly, $\rho_1 = 1$ is the common fixed point. Indeed, hypothesis(2) is satisfied when $\rho_{1_0} = 1 \in M$ with $\alpha(1,1,1) \geq 1$. Then all the requirements of Theorem 2.3 are fullfilled.

Corollary 2.7. *Let (M,G) be a complete G-metric space and let $f : M \to M$ be an (α,ψ)-Meir-Keeler-Khan mapping. Consider that:*

(1) f is an α-admissible mapping,

(2) There exists some $\rho_{1_0} \in M$ such that $\alpha(\rho_1+0, f(\rho_{1_0}), f\rho_{1_0}) \geq 1$,

(3) f is also continuous.

Then $f(p) = p$ for $p \in M$.

Proof. Immediately by taking $S = f = \Im = T$ in the Theorem 2.3. □

Corollary 2.8. *Let (M,G) be a complete G-metric space and let $f : M \to M$ be an (α,ψ)-Meir-Keeler-Khan mapping. Consider that:*

(1) f is an α-admissible mapping,

(2) There exists some $\rho_{1_0} \in M$ such that $\alpha(\rho_1+0, f(\rho_{1_0}), f\rho_{1_0}) \geq 1$,

(3) If $\{\rho_{1_n}\}$ is a sequence in M such that $\alpha(\rho_{1_n}, \rho_{1_{n+1}}, \rho_{1_{n+1}}) \geq 1$ for all $n \in N$ and $\rho_{1_n} \to \rho_1 \to M$ as $n \to \infty$, then $\alpha(\rho_{1_n}, \rho_1, \rho_1) \geq 1$, for all $n \in N$. Then $f(p) = p$ for $p \in M$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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