Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 2, 1728-1742
https://doi.org/10.28919/jmcs/5352
ISSN: 1927-5307

THE EDGE GEODETIC VERTEX COVERING NUMBER OF A GRAPH

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Abstract. For a connected graph G of order $n \geq 2$, a set $S \subseteq V(G)$ is an *edge geodetic vertex cover* of G if S is both an edge geodetic set and a vertex covering set of G. The minimum cardinality of an edge geodetic vertex cover of G is defined as the *edge geodetic vertex covering number* of G and is denoted by $g_{1\alpha}(G)$. Any edge geodetic vertex cover of cardinality $g_{1\alpha}(G)$ is a $g_{1\alpha}$ - set of G. Some general properties satisfied by edge geodetic vertex cover are studied. The edge geodetic vertex covering number of several classes of graphs are determined. Connected graphs of order n with edge geodetic vertex covering number 2 is characterized. A few realization results are given for the parameter $g_{1\alpha}(G)$.

Keywords: geodesic; edge geodetic set; vertex covering set; edge geodetic vertex cover; edge geodetic vertex covering number.

2010 AMS Subject Classification: 05C12.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops and multiple edges. The *order* and *size* of G are denoted by n and m, respectively. For basic graph

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Received December 28, 2020

theoretic terminology we refer to Harary [6]. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u - v path in G. A u - v path of length $d(u,v)$ is called a *u - v geodesic*. It is known that this *distance* is a metric on the vertex set $V(G)$. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter*, *diam* G. The *neighborhood* of a vertex v of G is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is a *simplicial vertex or an extreme vertex* of G if the subgraph induced by its neighborhood $N(v)$ is complete. A *caterpillar* is a tree of order 3 or more, the removal of whose end vertices produces a path called the spine of the caterpillar.

A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinality $g(G)$ is a *minimum geodetic set* or a *geodetic basis* or a g - set of G. The geodetic number of a graph was introduced in [2,7] and further studied in [3 - 5]. A subset $S \subseteq V(G)$ is said to be a *vertex covering set* of G if every edge has at least one end vertex in S. A vertex covering set of G with minimum cardinality is called a *minimum vertex covering set* of G. The *vertex covering number* of G is the cardinality of any minimum vertex covering set of G. It is denoted by $\alpha(G)$. The vertex covering number of a graph was studied in [10]. A set S of vertices of G is a *geodetic vertex cover* of G if S is both a geodetic set and a vertex covering set of G. The minimum cardinality of a geodetic vertex cover of G is defined as the *geodetic vertex covering number* of G and is denoted by $g_\alpha(G)$. Any geodetic vertex cover of cardinality $g_\alpha(G)$ is a g_α - set of G. The geodetic vertex covering number was introduced and studied in [1].

An *edge geodetic cover* of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum order of its edge geodetic covers and any edge geodetic cover of order $g_1(G)$ is an *edge geodetic basis* of G. The *edge geodetic number* of a graph was studied in [8]. A subset $S \subseteq V(G)$ is a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. A set of vertices S in G is called an *edge geodetic dominating set* of G if S is both an edge geodetic set and a dominating set. The minimum cardinality of an edge geodetic dominating set of G is its *edge geodetic domination number* and is denoted by $\gamma_{g1}(G)$.

The following theorems will be used in the sequel.

Theorem 1.1 ([8]) Each extreme vertex of G belongs to every edge geodetic set of G. In particular, each end vertex of G belongs to every edge geodetic set of G.

Theorem 1.2 ([8]) If G has exactly one vertex v of degree $n-1$, then $g_1(G) = n-1$.

Theorem 1.3 ([8]) For any tree T, the edge geodetic number $g_1(T)$ equals the number of end vertices in T. In fact, the set of all end vertices of T is the unique edge geodetic basis of T.

Theorem 1.4 ([8]) Every edge geodetic cover in a connected graph is a geodetic cover.

Theorem 1.5 ([1]) Let T be a tree of order $n \geq 3$. Then $g_\alpha(T) = n-1$ if and only if T is either a star or a double star.

Theorem 1.6 ([9]) For the complete graph K_n $(n \geq 2), \gamma_{g1}(K_n) = n$.

Theorem 1.7 ([9]) For any tree T with $n \geq 3$ vertices $\gamma_{g1}(T) = n-1$ if and only if T is a star.

Throughout the following G denotes a connected graph with at least two vertices.

2. The Edge Geodetic Vertex Cover of a Graph

Definition 2.1 Let G be a connected graph of order at least 2. A set S of vertices of G is an *edge geodetic vertex cover* of G if S is both an edge geodetic set and a vertex cover of G. The minimum cardinality of an edge geodetic vertex cover of G is defined as the *edge geodetic vertex covering number* of G and is denoted by $g_{1\alpha}(G)$. Any edge geodetic vertex cover of cardinality $g_{1\alpha}(G)$ is a $g_{1\alpha}$ - set of G.

Example 2.2 For the graph G given in Figure 2.1, $S = \{v_1, v_2, v_4\}$ is a minimum geodetic vertex cover of G so that $g_\alpha(G) = 3$ and $S' = \{v_1, v_2, v_3, v_4\}$ is a minimum edge geodetic vertex cover of G so that $g_{1\alpha}(G) = 4$. Thus the geodetic vertex covering number and the edge geodetic vertex covering number of a graph are different.

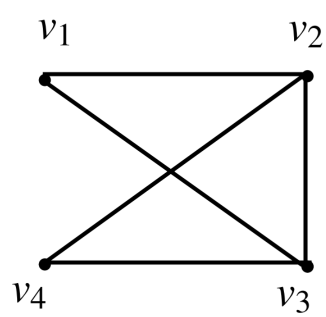

FIGURE 2.1. G

Remark 2.3 For the cycle $G = C_6$ given in Figure 2.2, $S = \{v_1, v_4\}$ is a minimum edge geodetic set of G so that $g_1(G) = 2$, S is also a minimum edge geodetic dominating set of G so that $\gamma_{g1}(G) = 2$, and $S' = \{v_1, v_3, v_5\}$ is a minimum edge geodetic vertex cover of G so that $g_{1\alpha}(G) = 3$. Hence the edge geodetic vertex covering number of a graph is different from the edge geodetic number and the edge geodetic domination number of a graph.

Theorem 2.4 For any connected graph G, $2 \leq max\{\alpha(G), g_1(G)\} \leq g_{1\alpha}(G) \leq n$.

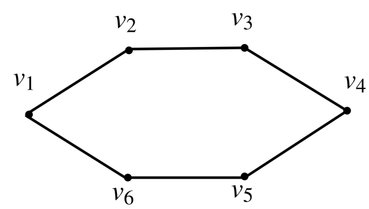

FIGURE 2.2. G

Proof. Any edge geodetic set of G needs at least two vertices and so $2 \leq max\{\alpha(G), g_1(G)\}$. From the definition of $g_{1\alpha}$ - set of G, we have $max\{\alpha(G), g_1(G)\} \leq g_{1\alpha}(G)$. Clearly $V(G)$ is an edge geodetic vertex cover of G so that $g_{1\alpha}(G) \leq n$. Thus $2 \leq max\{\alpha(G), g_1(G)\} \leq g_{1\alpha}(G) \leq n$.

Remark 2.5 The bounds in Theorem 2.4 are sharp. For the complete graph $K_n(n \geq 2), g_{1\alpha}(K_n) = n$. For the path $P_3, g_{1\alpha}(P_3) = 2$.

Theorem 2.6 Each extreme vertex of G belongs to every edge geodetic vertex cover of G. In particular, each end vertex of G belongs to every edge geodetic vertex cover of G.

Proof. Since every edge geodetic vertex cover of G is also an edge geodetic set, the result follows from Theorem 1.1.

Corollary 2.7 For any graph G with k extreme vertices, $max\{2, k\} \leq g_{1\alpha}(G) \leq n$.

Proof. The result follows from Theorem 2.4 and Theorem 2.6.

Corollary 2.8 Let $K_{1,n-1}(n \geq 3)$ be a star. Then $g_{1\alpha}(K_{1,n-1}) = n-1$.

Proof. Let x be the center and $S = \{v_1, v_2, ..., v_{n-1}\}$ be the set of all extreme vertices of $K_{1,n-1}(n \geq 3)$. It is clear that S is a minimum edge geodetic vertex cover of $K_{1,n-1}(n \geq 3)$. Hence $g_{1\alpha}(K_{1,n-1}) = n-1$.

Corollary 2.9 For the complete graph $K_n (n \geq 2), g_{1\alpha}(K_n) = n$.

Proof. Since every vertex of the complete graph $K_n (n \geq 2)$ is an extreme vertex, the vertex set of K_n is the unique edge geodetic vertex cover of K_n. Thus, $g_{1\alpha}(K_n) = n$.

Now we proceed to characterize graphs G for which $g_{1\alpha}(G) = 2$.

Theorem 2.10 Let G be a connected graph of order $n \geq 2$. Then $g_{1\alpha}(G) = 2$ if and only if there exists an edge geodetic set $S = \{u, v\}$ of G such that $d(u, v) \leq 2$.

Proof. Let $g_{1\alpha}(G) = 2$. Let $S = \{u, v\}$ be a minimum edge geodetic vertex cover of G. We prove that $d(u, v) \leq 2$. Suppose not, then $d(u, v) \geq 3$. Then there exists at least one edge which is not incident with any of the vertices u and v. Hence S is not an edge geodetic vertex cover of G, which is a contradiction. Thus $d(u, v) \leq 2$.

Conversely, assume that there exists an edge geodetic set $S = \{u, v\}$ of G such that $d(u, v) \leq 2$. It is clear that every edge of G is incident with either u or v. Hence S is a minimum edge geodetic vertex cover of G. Thus $g_{1\alpha}(G) = 2$.

Theorem 2.11 Let G be a connected graph with $g_1(G) \geq n - 1$. Then $g_{1\alpha}(G) = g_1(G)$.

Proof. Let G be a connected graph with $g_1(G) \geq n - 1$. By Theorem 2.4, $g_1(G) \leq g_{1\alpha}(G) \leq n$. If $g_1(G) = n$, then $g_{1\alpha}(G) = n$ and so $g_1(G) = g_{1\alpha}(G)$. If $g_1(G) = n - 1$, then let $S = \{x_1, x_2, ..., x_{n-1}\}$ be a minimum edge geodetic set of G with $|S| = n - 1$. Let $x \notin S$ be a vertex of G. Then any edge $xx_i (1 \leq i \leq n - 1)$ lies on a geodesic joining some pair of vertices of S and every edge of G has at least one end point in S. Hence S is a minimum edge geodetic vertex cover of G and so $g_1(G) = g_{1\alpha}(G)$.

Remark 2.12 The converse of Theorem 2.11 need not be true. For the cycle $C_4 : v_1, v_2, v_3, v_4, v_1$; $S = \{v_1, v_4\}$ is both a g_1 - set of C_4 and a $g_{1\alpha}$ - set of C_4. Hence $g_1(C_4) = g_{1\alpha}(C_4) = 2$ but $g_1(C_4) < n-1$.

Theorem 2.13 Let G be a connected graph of order $n \geq 2$. Then $g_{1\alpha}(G) = g_1(G)$ if and only if there exists a minimum edge geodetic set S of G such that $V(G) - S$ is either empty or an independent set.

Proof. Let $g_{1\alpha}(G) = g_1(G)$. Let $S = \{v_1, v_2, ..., v_k\}$ be a minimum edge geodetic vertex cover of G. Then S is a minimum edge geodetic set of G. If $n = k$, then $V(G) - S$ is empty. Let $n > k$. Now claim that $V(G) - S$ is an independent set. If not, there exist two vertices $u, v \in V(G) - S$ such that $uv \in E(G)$. Then the edge uv has none of its end vertices in S, which is a contradiction.

Conversely, assume that there exists a minimum edge geodetic set S of G such that $V(G) - S$ is either empty or an independent set. Let $S = \{v_1, v_2, ..., v_k\}$ so that $|S| = g_1(G)$. Suppose $V(G) - S$ is empty. Then $n = k$ and $S = V(G)$. Hence S is a minimum edge geodetic vertex cover of G so that $g_{1\alpha}(G) = g_1(G)$. If not, let $V(G) - S$ be independent. Then every edge of G has at least one end in $V(G) - (V(G) - S) = S$ and hence S is a vertex cover of G. Thus S is a minimum edge geodetic vertex cover of G. Thus $g_{1\alpha}(G) = |S| = g_1(G)$.

Theorem 2.14 If G is a connected graph of order $n \geq 2$ with $diam\ G \leq 2$, then $g_{1\alpha}(G) = g_1(G)$.

Proof. Let G be a connected graph of order $n \geq 2$ with $diam\ G \leq 2$. Let S be a minimum edge geodetic set of G so that $|S| = g_1(G)$. If $diam\ G = 1$, then $G = K_n$ and so $g_1(G) = g_{1\alpha}(G) = n$. Let $diam\ G = 2$. We show that S is a vertex cover of G. Suppose not, then there exists an edge $xy \in E(G)$ such that both $x, y \notin S$. Since S is a g_1 - set of G, the edge xy lies in a $u - v$ geodesic $P : u = u_0, u_1, ..., u_i = x, y = u_{i+1}, ..., u_k = v$ for some $u, v \in S$. Then $d(u, v) \geq 3$, which is a contradiction. Hence S is a minimum edge geodetic vertex cover of G and so $g_{1\alpha}(G) = |S| = g_1(G)$.

Remark 2.15 The converse of Theorem 2.14 need not be true. For the graph G given in Figure 2.3, $S = \{v_1, v_2, v_3, v_5\}$ is both a g_1 - set and a $g_{1\alpha}$ - set of G. Hence $g_{1\alpha}(G) = g_1(G) = 4$. But $diam\ G = 3$.

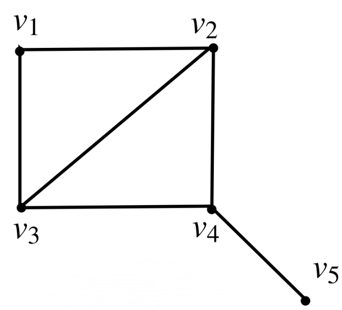

FIGURE 2.3. G

Theorem 2.16 For the cycle $C_n (n \geq 4)$, $g_{1\alpha}(C_n) = \lceil \frac{n}{2} \rceil$.

Proof. Let $C_n : v_1, v_2, ..., v_n, v_1$ be a cycle of order n. It is clear that $S = \{v_1, v_3, v_5, ..., v_{2\lceil \frac{n}{2} \rceil - 1}\}$ is a minimum edge geodetic vertex cover of C_n and so $g_{1\alpha}(C_n) = \lceil \frac{n}{2} \rceil$.

Theorem 2.17 Let G be a connected graph of order $n \geq 3$. If G has exactly one vertex v of degree $n-1$, then $g_{1\alpha}(G) = n-1$.

Proof. Let v be the unique vertex of G with degree $n-1$. Then by Theorem 1.2, $g_1(G) = n-1$. Let $S = V(G) - \{v\}$ be a minimum edge geodetic set of G. It is clear that every edge of G has at least one end vertex in S. Thus S is a minimum edge geodetic vertex cover of G. Hence $g_{1\alpha}(G) = n-1$.

Corollary 2.18 If G has exactly one vertex v of degree $n-1$, then $g_{1\alpha}(G) = n-1$ and G has a unique minimum edge geodetic vertex cover containing all the vertices of G other than v.

Corollary 2.19 For the wheel $W_n = K_1 + C_{n-1} (n \geq 5), g_{1\alpha}(W_n) = n-1$.

Corollary 2.20 Let $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. Then $g_{1\alpha}(G) = n-1$.

Theorem 2.21 For any tree T of order $n \geq 2, g_{1\alpha}(T) = g_1(T)$ if and only if T is a star.

Proof. Let T be a tree of order $n \geq 2$. Let S be the set of all end vertices of T. Then by Theorem 1.3, S is the unique g_1 - set of T. Let $g_{1\alpha}(T) = g_1(T)$. We prove that T is a star. If not, then *diam* $T \geq 3$ and so T has at least one edge other than the end edges. Let S' be the set of all edges of T which are not end edges. Then clearly no edge of S' has its end vertices in S. Hence S is not a vertex cover of T. Since, by Theorem 2.6, S is a subset of any edge geodetic vertex cover of T, $g_{1\alpha}(T) > |S| = g_1(T)$, which is a contradiction.

Conversely, if T is a star, then *diam* $T = 2$. Then by Theorem 2.14, $g_{1\alpha}(T) = g_1(T)$.

Theorem 2.22 Let T be a tree of order $n \geq 3$. Then $g_{1\alpha}(T) = n-1$ if and only if T is either a star or a double star.

Proof. Let $g_{1\alpha}(T) = n-1$. Let $P: v_0, v_1, v_2, ..., v_d$ be a diametral path of T. Since $n \geq 3$, we have $d \geq 2$. If $d \geq 4$, then $S = V(T) - \{v_1, v_3\}$ is an edge geodetic vertex cover of T and so $g_{1\alpha}(T) \leq n-2$, which is a contradiction. Hence $d = 2$ or 3 and so T is either a star or a double star. Converse is clear.

Theorem 2.23 Let T be a caterpillar with diameter d. Then $g_{1\alpha}(T) = \lceil \frac{d}{2} \rceil + k - 1$, where k is the number of end vertices of T.

Proof. Let T be a caterpillar. Let $P: v_0, v_1, v_2, ..., v_d$ be a diametral path of T and let k be the number of end vertices of T. If d is even, then $S = \{v_0, v_2, v_4, ..., v_{d-2}, v_d\}$ and if d is odd, then $S = \{v_0, v_2, v_4, ..., v_{d-1}, v_d\}$. Then $|S| = \lceil \frac{d}{2} \rceil + 1$ and every edge of P has at least one end in S. Since any edge of P lies on the $v_0 - v_d$ geodesic, S is an edge geodetic vertex cover of the diametral path P. Since T is a caterpillar, $S' = (V(T) - V(P)) \cup \{v_0, v_d\}$ is the set of all end vertices of T. Then by Theorem 2.6, every edge geodetic vertex cover of T contains S'. Now, it is clear that $S'' = (S - \{v_0, v_d\}) \cup S'$ is a minimum edge geodetic vertex cover of T and so $g_{1\alpha}(T) = \lceil \frac{d}{2} \rceil - 1 + k = \lceil \frac{d}{2} \rceil + k - 1$.

Remark 2.24 The converse of Theorem 2.23 need not be true. For the tree given in Figure 2.4, $d = 6,\ k = 3$ and $S = \{v_0, v_2, v_4, v_6, v_8\}$ is a minimum edge geodetic vertex cover of T. Hence $g_{1\alpha}(T) = 5 = \lceil \frac{d}{2} \rceil + k - 1$. But T is not a caterpillar.

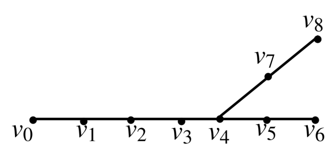

FIGURE 2.4. G

Theorem 2.25 Every edge geodetic vertex cover of a connected graph G is a geodetic vertex cover of G.

Proof. Let G be a connected graph and let S be an edge geodetic vertex cover of G. By Theorem 1.4, S is a geodetic vertex cover of G.

Corollary 2.26 For any connected graph G, $g_\alpha(G) \leq g_{1\alpha}(G)$.

Remark 2.27 For the graph G given in Figure 2.1, $g_\alpha(G) = 3$ and $g_{1\alpha}(G) = 4$ so that $g_\alpha(G) < g_{1\alpha}(G)$. Also for any non-trivial tree T, $g_\alpha(G) = g_{1\alpha}(G)$.

Theorem 2.28 For any two positive integers a and b with $3 \leq a \leq b$, there exists a connected graph G with $g_\alpha(G) = a$ and $g_{1\alpha}(G) = b$.

Proof. We prove this theorem by considering two cases.

Case (i). $3 \leq a = b$. Take $G = K_{1,n-1} (n \geq 4)$. Then by Theorem 1.5 and Theorem 2.22, $g_\alpha(G) = g_{1\alpha}(G) = n-1$.

Case (ii) $3 \leq a < b$. Let G be the graph obtained from the path on three vertices $P : u_1, u_2, u_3$ by adding $b-2$ new vertices $v_1, v_2, ..., v_{b-a+1}, w_1, w_2, ..., w_{a-3}$ and joining each v_i $(1 \leq i \leq b-a+1)$ with u_1, u_2, u_3 and joining each w_i $(1 \leq i \leq a-3)$ with u_2 and the graph G is shown in Figrue 2.5. Let $S = \{w_1, w_2, ..., w_{a-3}\}$. It is clear that S is not a geodetic vertex cover of G. Also, it is clear that $S' = S \cup \{u_1, u_2, u_3\}$ is a minimum geodetic vertex cover of G and hence $g_\alpha(G) = a-3+3 = a$. Also, since u_2 is the only full degree vertex of G, by Theorem 2.17, $g_{1\alpha}(G) = a-3+2+b-a+1 = b$.

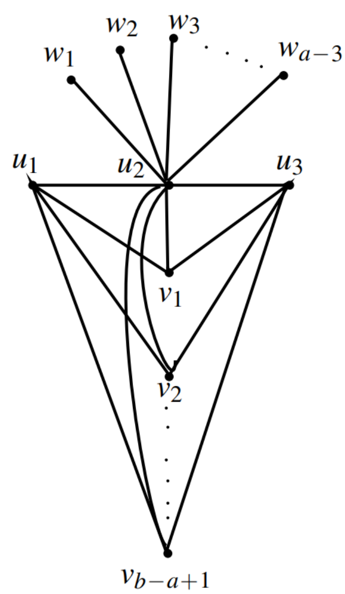

FIGURE 2.5. G

Theorem 2.29 For any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with $g_1(G) = a$ and $g_{1\alpha}(G) = b$.

Proof. We prove this theorem by considering two cases.

Case (i). $2 \leq a = b$. Take $G = K_{1,n-1} (n \geq 3)$. Then by Theorems 1.3 and 2.21, $g_1(G) = g_{1\alpha}(G) = n-1$.

Case (ii). $2 \leq a < b$. Let $C_i : v_i, x_i, v_{i+1}, y_i, v_i$ $(1 \leq i \leq b-a-1)$ be a cycle of order 4. Let H be the graph obtained from C_i $(1 \leq i \leq b-a-1)$ by identifying the vertices v_{i+1} of C_i and v_{i+1} of C_{i+1} $(1 \leq i \leq b-a-2)$. Let G be the graph obtained from H by adding new vertices $u_1, u_2, ..., u_a$ and joining u_1 with v_1 and joining each u_i $(2 \leq i \leq a)$ with v_{b-a}. The graph G is shown in Figure 2.6.

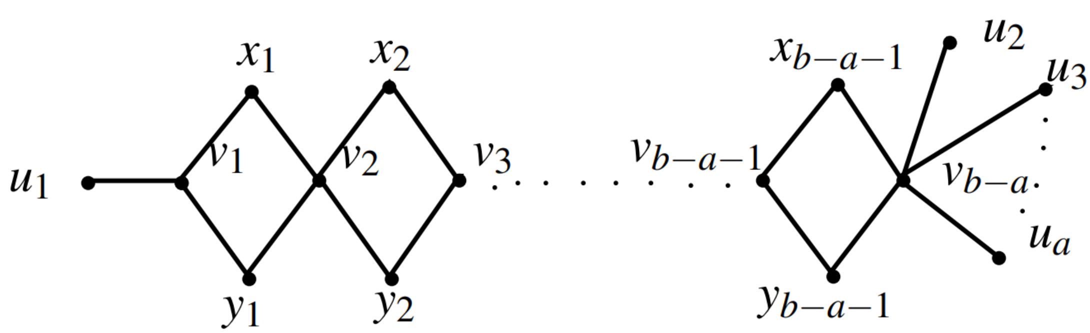

FIGURE 2.6. G

Let $S = \{u_1, u_2, ..., u_a\}$ be the set of all simplicial vertices of G. By Theorem 1.1, any edge geodetic set of G contains S. It is clear that S is the unique minimum edge geodetic set of G and so $g_1(G) = |S| = a$. Also, we observe that $S_1 = S \cup \{v_1, v_2, ..., v_{b-a}\}$ is a minimum edge geodetic vertex cover of G and so $g_{1\alpha}(G) = |S_1| = b$.

Theorem 2.30 Let G be a connected graph of order $n \geq 2$. Then every edge geodetic vertex cover of G is an edge geodetic dominating set of G.

Proof. Let S be an edge geodetic vertex cover of G. Then S is both an edge geodetic set and a vertex cover of G. Since S is a vertex cover of G, every edge of G has at least one end in S and hence every vertex in $V(G) - S$ has at least one neighbour in S so that S is a dominating set of G. Hence S is an edge geodetic dominating set of G.

Corollary 2.31 For any connected graph G, $2 \leq \gamma_{g1}(G) \leq g_{1\alpha}(G) \leq n$.

Remark 2.32 For the graph K_2, $\gamma_{g1}(K_2) = 2$. For the graph G given in Figure 2.2, $\gamma_{g1}(G) = 2$ and $\mathrm{g}_{1\alpha}(G) = 3$ so that $\gamma_{g1}(G) < g_{1\alpha}(G)$. For the complete graph K_n, by Theorem 1.6 and Corollary 2.9, $\gamma_{g1}(K_n) = g_{1\alpha}(K_n) = n$.

Theorem 2.33 Let S be a minimum edge geodetic dominating set of a connected graph G. Then S is an edge geodetic vertex cover of G if and only if $V(G) - S$ is an independent set of G.

Proof. Let S be a minimum edge geodetic dominating set of a connected graph G. If S is an edge geodetic vertex cover of G, then every edge of G has at least one end in S. Hence no pair of vertices of $V(G) - S$ are adjacent so that $V(G) - S$ is independent.

Conversely, let $V(G) - S$ be an independent set of G. Then every edge of G has at least one end in S so that S is also a vertex cover of G. Hence S is an edge geodetic vertex cover of G.

Theorem 2.34 For any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with $\gamma_{g1}(G) = a$ and $\mathrm{g}_{1\alpha}(G) = b$.

Proof. We prove this theorem by considering two cases.

Case (i). $2 \leq a = b$. Take $G = K_{1,a}$ $(a \geq 2)$. Then by Theorems 1.7 and 2.22, $\gamma_{g1}(G) = g_{1\alpha}(G) = a$.

Case (ii) $2 \leq a < b$. Let $K_{1,b-a+1}$ be the star with center x and $U = \{v_1, v_2, \ldots, v_{b-a+1}\}$ the set of end vertices of $K_{1,b-a+1}$, and let $K_{1,b-1}$ be the star with center y and $W = \{w_1, w_2, \ldots, w_{b-a+1}, u_1, u_2, \ldots, u_{a-2}\}$ the set of end vertices of $K_{1,b-1}$. Let G be the graph obtained from the stars $K_{1,b-a+1}$ and $K_{1,b-1}$ by joining the vertices v_i of $K_{1,b-a+1}$ and w_i of $K_{1,b-1}$ $(1 \leq i \leq b-a+1)$. The graph G is shown in Figure 2.7.

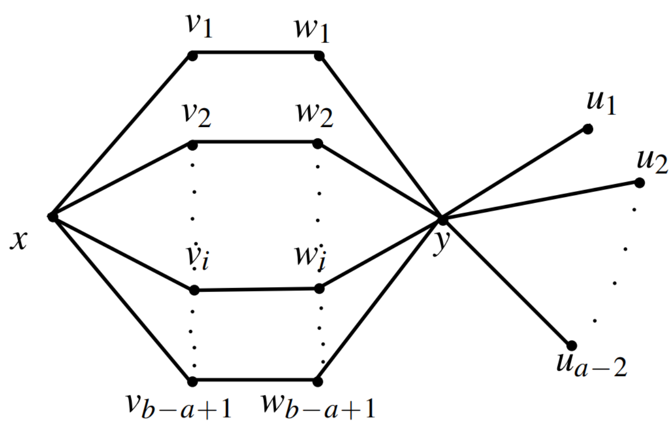

FIGURE 2.7. G

Let $S = \{u_1, u_2, ..., u_{a-2}\}$ be the set of all simplicial vertices of G. Then by Theorem 1.1, every edge geodetic set of G contains S. It is clear that S is not an edge geodetic set of G, $S' = S \cup \{x\}$ is a minimum edge geodetic set of G, and $S'' = S' \cup \{y\}$ is a minimum edge geodetic dominating set of G. Hence $\gamma_{g1}(G) = a$. Since the edges $v_i w_i$ $(1 \leq i \leq b-a+1)$ are not covered by any of the vertices of S'', S'' is not an edge geodetic vertex cover of G. Let $S''' = S' \cup \{w_1, w_2, ..., w_{b-a+1}\}$. Clearly S''' is an edge geodetic vertex cover of G of minimum cardinality. Hence $g_{1\alpha}(G) = a-1+b-a+1 = b$.

3. CONCLUSION

In this paper, we initiated the study on "The edge geodetic vertex covering number of a graph" and established some results related to this parameter. We characterized graphs G for which the edge geodetic number and the edge geodetic vertex covering number are equal. The results presented in this paper can be used for future study on the connected edge geodetic vertex covering number, the upper edge geodetic vertex covering number and forcing edge geodetic vertex covering number of a graph and so on.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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