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CONTROL OF A REACTION-DIFFUSION SYSTEM: APPLICATION ON A SEIR EPIDEMIC MODE

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Abstract. In this work, we are interested in the study of a spatiotemporal SEIR epidemiological model, with no-flux boundary conditions. This model includes a constant inflow of new susceptible, exposed, infectious and recovered. In addition, it also incorporates a contact rate depending on the size of the population and another death related to the disease. Our objective is to characterize the optimal control pair, which minimizes exposed, infected individuals and the corresponding effort and treatment costs. We have demonstrated the existence of the state system solution and optimal control. The characterization of the optimal control pair is determined in terms of state functions and adjoint functions. The numerical resolution of the optimal system, has shown the effectiveness of our adopted strategy.

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1. INTRODUCTION

In epidemiology, mathematical modeling has become an important tool for analyzing the causes, dynamics and spread of epidemics. Indeed, mathematical models provide a better understanding of the mechanisms underlying the spread of emerging infectious diseases, and allow authorities to make decisions about effective control strategies. One of the most basic procedures in disease modeling is to use a model, in which the population is divided into different groups depending on the stage of infection, with assumptions about the nature and rate of transfer time from compartment to another.Several diseases that confer immunity against reinfection have been modeled using SIR,SIS, SEIR ... etc (see [1-10]). With S is the class of the susceptible population, which represents individuals not yet infected with the disease and who are susceptible to contracting the disease, I is the class of infected persons who represents the individuals infected with the disease, and who can transmit the disease to sensitive people. E is the class of people exposed and R is the class of recovered that represents people immunized against the disease. In the literature, several studies have been carried out on the SEIR models, for example, Greenhalgh [18] has examined the SEIR models integrating density dependence into the mortality rate. Cooke and van den Driessche [19] presented and studied the SEIRS models with two delays. Li and Muldowney [20] and Li et al. [21] investigated the overall dynamics of SEIR models with non-linear incidence. Li ,Smith H L,and al. [22] analyzed the overall dynamics of a SEIR model with vertical transmission and bilinear incidence. Zhang.al [25] studied the overall dynamics of a SEIR model with immigration from different compartments. The impact of the mobility of individuals plays an important role in the transmission of the disease, which makes it necessary to introduce the spatial factor in the SEIR models, in order to give a more realistic study [11-17][30]. Based on the SEIR model presented by Zhang.al, in which we introduce the spatial factor, and adopting a strategy in the form of a control problem, we formulate a spatial-temporal SEIR epidemiological model, as that system of parabolic partial differential equations coupled with no-flux boundary conditions. The main objective of this work is to set up an optimal control approach based on a combination of minimization of the number of latent and infected individuals and the therapeutic treatment, for the Reaction-Diffusion SEIR model. To achieve this goal, we characterize an optimal control pair in the form of an effort that reduces contact between infectious and susceptible individuals, and a treatment envisioned to combat the spread of the disease. We prove the existence of state system solutions and the existence of optimal control. Optimal control theory provides the characterization of the optimal control pair in terms of state and adjoint functions. The optimality system is solved numerically, using a forward-backward sweep method (FBSM) [23]. The numerical simulations of our control strategy show the effectiveness of the approach we have adopted. The document is organized as follows: In Section 2, we present the mathematical model and the associated optimal control problem. We prove the existence of a strong global solution for our system in section 3. In section 4, we show the existence of the optimal solution. The necessary optimality conditions are defined in section 5. As an application, the numerical results associated with our problem control are given in section 6. Finally, we conclude the document in section 7.

2. THE BASIC MATHEMATICAL MODEL AND OPTIMAL CONTROL PROBLEM

The model (1) used by zhang consists of four compartments SEIR: susceptible, exposed, infected and recovered. We note their densities at time t and at position x by.
$S(t,x), I(t,x), E(t,x), R(t,x)$, respectively:

$$(1) \qquad \begin{cases} \dfrac{\partial S}{\partial t} = (1-p-q-b)\Theta - \beta\lambda\frac{SI}{N} - dS \\[2ex] \dfrac{\partial E}{\partial t} = q\Theta + \beta\lambda\frac{SI}{N} - (d+\varepsilon)E \\[2ex] \dfrac{\partial I}{\partial t} = p\Theta - (d+\alpha+\gamma)I + \varepsilon E \\[2ex] \dfrac{\partial R}{\partial t} = b\Theta - dR + \gamma I \end{cases} \qquad (t,x) \in Q = [0,T] \times \Omega$$

With $N(t) = S(t) + I(t) + E(t) + R(t)$ is the total that population number at time t, and the parameters are defined as follows :

Parameter	Definition
β	Effective contact rate
d	Natural mortality rate
γ	Recovery rate
α	Rate that exposed individuals become infectious
ε	Disease induced death rate
Θ	Recruitement rate into the population
p	Rate of exposed individuals in new members of the population
q	Rate of infected individuals in new members of the population
b	Rate of recovred individuals in new members of the population
λ	The probability for an individual to take part in a contact.

Table 1:Parameter defintion

In model (1), in order to take into account the effect of the spatial factor, we introduce the Laplacian operator$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.In addition, we include two controls v_1 and v_2 which represent respectively the effort that reduces the contact between infectious and susceptible individuals and the rate of treatment of infectious individuals. We assume that $v_2(t,x)I$ individuals are removed from the infected class and added to the recovred class. The mathematical system with controls is given by the nonlinear differential equations

$$(2) \quad \begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S + (1-p-q-b)\Theta - (1-v_1(t,x))\beta\lambda\frac{SI}{N} - dS \\ \\ \dfrac{\partial E}{\partial t} = d_E \Delta E + q\Theta + (1-v_1(t,x))\beta\lambda\frac{SI}{N} - (d+\varepsilon)E \\ \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + p\Theta - (d+\alpha+\gamma+v_2(t,x))I + \varepsilon E \\ \\ \dfrac{\partial R}{\partial t} = d_R \Delta R + b\Theta - dR + \gamma I + v_2(t,x)I \end{cases} \qquad (t,x) \in Q = [0,T]\times\Omega$$

We denote by Ω a fixed and bounded domain in IR^2 with smooth boundary $\partial\Omega$ and η is the outward unit normal vector on the boundary. The initial conditions and no-flux boundary conditions are given by

(3) $$\frac{\partial S}{\partial \eta}=\frac{\partial E}{\partial \eta}=\frac{\partial I}{\partial \eta}=\frac{\partial R}{\partial \eta}=0\,,\quad (t,x)\in\Sigma=[0,T]\times\partial\Omega$$

(4) $$S(0,x)=S_0\geq 0,\, E(0,x)=E_0\geq 0,\, I(0,x)=I_0\geq 0,\, and\, R(0,x)=R_0\geq 0$$

Eligible controls are contained in the ensemble

(5) $$U_{ad}=\left\{(v_1,v_2)\in(L^{\propto}(Q))^2/0\leq v_1\leq v_1^{max}\leq 1\, and\, 0\leq v_2\leq v_2^{max}\leq 1\right\}$$

We seek to minimize the functional objective

(6) $$J(v)=\int_0^T\int_\Omega (K_1E(t,x)+K_2I(t,x))dxdt+\frac{\rho_1}{2}\|v_1\|^2_{L^2(Q)}+\frac{\rho_2}{2}\|v_2\|^2_{L^2(Q)}$$

for some positive constant v^{max}.

Where K_1,K_2, are constant weights.

For the parameter $\frac{\rho_1}{2}$,$\frac{\rho_2}{2}$, our objective is to find control functions such

$$J(v_1^*,v_2^*)=min\left\{J(v_1,v_2)\,,\,(v_1,v_2)\in U_{ad}\right\}$$

- We put $H(\Omega)=\left(L^2(\Omega)\right)^4$, and denote the space of all absolutely continuous functions $y:[0,T]\rightarrow H(\Omega)$ having the property that $\frac{\partial y}{\partial t}\in L^2([0,T],H(\Omega))$ by $W^{1,2}([0,T],H(\Omega))$. Moreover, define $\mathscr{L}(T,\Omega)=L^2\left([0,T],H^2(\Omega)\right)\cap L^\infty\left([0,T],H^1(\Omega)\right)$

3. Existence of Solution

In this section,we study the existence of a global strong solution, positivity, and boundedness of solutions for (2)-(4). Let $y=(y_1,y_2,y_3,y_4)=(S,E,I,R)$ the solution of the system (2)-(4) with $y^0=\left(y_1^0,y_2^0,y_3^0,y_4^0\right)=\left(S^0,E^0,I^0,R^0\right)$. A denotes the linear operator defned as foll

(7) $$\begin{gathered}A:D(A)\subset H(\Omega)\longrightarrow H(\Omega)\\ Ay=(d_S\Delta y_1,d_E\Delta y_2,d_I\Delta y_3,d_R\Delta y_4)\in D(A)\,,\forall y=(y_1,y_2,y_3,y_4)\in D(A)\end{gathered}$$

with the domain of A is defined by

(8) $$D(A)=\left\{y\in\left(H^2(\Omega)\right)^3,\frac{\partial y_1}{\partial\eta}=\frac{\partial y_2}{\partial\eta}=\frac{\partial y_3}{\partial\eta}=\frac{\partial y_4}{\partial\eta}=0,a.e\,x\in\partial\Omega\right\}$$

Theorem 1. *Let Ω be a bounded domain from $\mathbb{R}^2$, with the boundary smooth enough, $y_i^0 \geq 0$ on Ω (for $i = 1,2,3,4$), the problem (2-4) has a unique (global) strong solution $y \in W^{1,2}([0,T] : H(\Omega))$ such that $y_i \in \mathscr{L}(T,\Omega) \cap L^\infty(Q)$ for $i = 1,2,3,4$. In addition y_1, y_2, y_3 and y_4 are nonnegative. Furthermore there exists $C > 0$ (independent of (v)) for all $t \in [0,T]$*

$$\left\| \frac{\partial y_i}{\partial t} \right\|_{L^2(Q)} + \|y_i\|_{L^2\left(0,T;H^2(\Omega)\right)} + \|y_i\|_{H^1(\Omega)} + \|y_i\|_{L^\infty(Q)} \leq C, \ for\ i = 1,2,3,4 \tag{9}$$

Proof. To prove the existence of a (global) strong solution for system(2)-(4), now we write system (2)-(4) as shown in ((6) see Appendix). Let

$$\begin{cases} \\ g_1(y(t)) = (1-p-q-b)\Theta - (1-v_1(t,x))\beta\lambda \frac{y_1 y_3}{N} - dy_1 \\ g_2(y(t)) = q\Theta + (1-v_1(t,x))\beta\lambda \frac{y_1 y_3}{N} - (d+\varepsilon)y_2, \quad t \in [0,T] \\ g_3(y(t)) = p\Theta - (d+\alpha+\gamma+v_2(t,x))y_3 + \varepsilon y_2 \\ g_4(y(t)) = b\Theta - dy_4 + (\gamma + v_2(t,x))y_3 \end{cases} \tag{10}$$

The system (10) represent the nonlinear term of (2) and we consider the function

$g(y(t)) = (g_1(y(t)), g_2(y(t)), g_3(y(t)), g_4(y(t)))$, then we can be rewrite the system (2)-(4) in the space $H(\Omega)$ as follows

$$\begin{cases} \dfrac{\partial y}{\partial t} = Ay + g(y(t)), \qquad t \in [0,T] \\ y(0) = y^0 \end{cases} \tag{11}$$

As the operator A defined in (7)-(8) is dissipating, self-adjoint and generates a C_0-semi-group of contractions on $H(\Omega)$[23], It is clear that function g is Lipschitz continuous in $y = (y_1, y_2, y_3, y_4)$ uniformly with respect to $t \in [0,T]$. Therefore, theorem (6) (see appendix) assures problem (2-4) admits a unique strong solution $y \in W^{1,2}([0,T], H(\Omega))$ with

$$y_1,\ y_2,\ y_3\ y_4 \in L^2\left([0,T], H^2(\Omega)\right) \tag{12}$$

In order to show that $y_i \in L^\infty(Q)$ for $i = 1, 2, 3, 4$, we denote $M = \max\left\{\|g_1\|_{L^\infty(Q)}, \left\|y_1^0\right\|_{L^\infty(\Omega)}\right\}$ and $\{S(t), t \geq 0\}$ is the C_0-semi-group generated by the operator $B :\ D(B) \subset L^2(\Omega) \longrightarrow$

$L^2(\Omega)$, where $By_1 = d_1\Delta y_1$ and $D(B) = \left\{ y_1 \in H^2(\Omega), \frac{\partial y_1}{\partial \eta} = 0, a.e\,\partial\Omega \right\}$. It is clear that the function $U_1(t,x) = y_1 - Mt - \left\|y_1^0\right\|_{L^\infty(\Omega)}$ satisfies the system

$$\begin{cases} \dfrac{\partial U_1}{\partial t}(t,x) = d_S \triangle U_1 + g_1(t, y(t)) - M, \quad t \in [0,T] \\ \\ U_1(0,x) = y_1^0 - \left\|y_1^0\right\|_{L^\infty(\Omega)} \end{cases} \tag{13}$$

Note that this system has a solution given by

$$U_1(t) = S(t)(y_1^0 - \left\|y_1^0\right\|_{L^\infty(\Omega)}) + \int_0^t S(t-s)(g_1(s,y(s)) - M)ds,$$

As $y_1^0 - \left\|y_1^0\right\|_{L^\infty(\Omega)} \leq 0$ and $g_1(s,y(s)) - M \leq 0$, we have $U_1(t,x) \leq 0, \forall (t,x) \in Q$. Similarly the function $U_2(t,x) = y_1 + Mt + \left\|y_1^0\right\|_{L^\infty(\Omega)}$ satisfies $U_2(t,x) \geq 0, \forall (t,x) \in Q$. Then

$$|y_1(t,x)| \leq Mt + \left\|y_1^0\right\|_{L^\infty(\Omega)}, \forall (t,x) \in Q$$

and analogously , we have

$$|y_i(t,x)| \leq Mt + \left\|y_i^0\right\|_{L^\infty(\Omega)} \forall (t,x) \in Q\, for\, i = 2,3,4 \tag{14}$$

Thus we have proved that

$$y_i \in L^\infty(Q) \forall (t,x) \in Q\, for\, i = 1,2,3,4. \tag{15}$$

By the first equation of (2), we obtain

$$\begin{aligned} &\int_0^t \int_\Omega \left|\frac{\partial y_1}{\partial s}\right|^2 dsdx + d_S^2 \int_0^t \int_\Omega |\triangle y_1|^2 dsdx - 2d_S \int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \triangle y_1 dsdx \\ &= \int_0^t \int_\Omega \left((1-p-q-b)\Theta - (1-u(t,x))\beta\lambda \tfrac{y_1 y_3}{N} - \mu y_1\right)^2 dsdx \end{aligned}$$

Using the regularity of y_1 and the Green's formula, we can write

$$2\int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \triangle y_1 dx = -\int_0^t \frac{\partial}{\partial s}\left(\int_\Omega \left|\nabla y_1^k\right|^2 dx\right) ds = -\int_\Omega |\nabla y_1|^2 dx + \int_\Omega \left|\nabla y_1^0\right|^2 dx$$

Then

$$\begin{aligned}&\int_0^t\int_\Omega\left|\frac{\partial y_1}{\partial s}\right|^2 dsdx+d_S^2\int_0^t\int_\Omega|\triangle y_1|^2 dsdx+d_S\int_\Omega|\nabla y_1|^2dx-d_S\int_\Omega\left|\nabla y_1^0\right|^2dx\\&\quad=\int_0^t\int_\Omega\left((1-p-q-b)\Theta-(1-v_1(t,x))\beta\lambda\tfrac{y_1y_3}{N}-\mu y_1\right)^2 dsdx\end{aligned}$$

Since $\|y_i\|_{L^\infty(Q)}$for $i=1,2,3,4$ are bounded independently of v_1,v_2 and $y_1^0\in H^2(\Omega)$, we deduce that

$$y_1\in L^\infty\left([0,T],H^1(\Omega)\right) \tag{16}$$

We make use of (12), (13), and (16), in order to get

$$y_1\in\mathscr{L}(T,\Omega)\cap L^\infty(Q)$$

and conclude that the inequality in (9) holds for $i=1$, similarly for y_2,y_3 and y_4.

In ordre to show the positiveness of y_i for $i=1,2,3,4$,we write the system (2) in the form:

$$\begin{cases}\dfrac{\partial y_1}{\partial t}=d_1\triangle y_1+H_1(y_1,y_2,y_3,y_4), \quad , \quad (t,x)\in Q,\\ \dfrac{\partial y_2}{\partial t}=d_2\triangle y_1+H_2(y_1,y_2,y_3,y_4)\\ \dfrac{\partial y_3}{\partial t}=d_3\triangle y_3+H_3(y_1,y_2,y_3,y_4)\\ \dfrac{\partial y_4}{\partial t}=d_4\triangle y_3+H_4(y_1,y_2,y_3,y_4)\end{cases} \tag{17}$$

It is easy to see that the functions $H_1(y_1,y_2,y_3,y_4),H_2(y_1,y_2,y_3,y_4),H_3(y_1,y_2,y_3,y_4)$ and $H_4(y_1,y_2,y_3,y_4)$, are continuously differentiable satisfying $H_1(0,y_2,y_3,y_4)=(1-p-q-b)\Theta\geq 0$, $H_2(y_1,0,y_3,y_4)=q\Theta+(1-v_1(t,x))\beta\lambda\frac{y_1y_3}{N}\geq 0, H_3(y_1,y_2,0,y_4)=p\Theta+\varepsilon y_2\geq 0$, $H_4(y_1,y_2,y_3,0)=b\Theta+(\gamma+v_2(t,x))y_3\geq 0$, for all $y_1,y_2y_3,y_4\geq 0$ (See [26]).This completes the proof. □

4. The Existence of the Optimal Solution

In this section, we will prove the existence of an optimal control for the problem (6) subject to reaction diffusion system (2)-(4) and $(v_1,v_2)\in U_{ad}$. The main result of this section is the following theorem.

Theorem 2. *Under the hypotheses of theorem (1), the optimal control problem (2-6) admits an optimal solution* $(y^*,(v_1^*,v_2^*))$.

Proof. From Theorem 1, we know that, for every $(v_1,v_2)\in U_{ad}$, there exists a unique solution y to system (2-4) . Assume that

$$inf_{(v_1,v_2)\in U_{ad}} J((v_1,v_2)) > -\infty$$

Let $\{(u^n,v^n)\}\subset U_{ad}$ be a minimizing sequence such that

$$lim_{n\to\propto} J(v_1^n,v_2^n) = inf_{(u,v)\in U_{ad}} J(v_1,v_2)$$

where $\left(y_1^n,y_2^n,y_3^n,y_4^n\right)$ is the solution of system (2-4) corresponding to the control $\left(v_1^n,v_2^n\right)$ for $n=1,2,\ldots$. That is

$$\begin{cases} \dfrac{\partial y_1^n}{\partial t} = d_1\Delta y_1^n + (1-p-q-b)\Theta - (1-v_1^n)\beta\lambda \frac{y_1^n y_3^n}{(y_1^n+y_2^n+y_3^n+y_4^n)} - dy_1^n \\ \dfrac{\partial y_2^n}{\partial t} = d_2\Delta y_2^n + q\Theta + (1-v_1^n)\beta\lambda \frac{y_1^n y_3^n}{(y_1^n+y_2^n+y_3^n+y_4^n)} - (d+\varepsilon)y_2^n \\ \dfrac{\partial y_3^n}{\partial t} = d_3\Delta y_3^n + p\Theta - (d+\alpha+\gamma+v_2^n)y_3^n + \varepsilon y_2^n, (t,x)\in Q \\ \dfrac{\partial y_4^n}{\partial t} = d_4\Delta y_4^n + b\Theta - dy_4^n + (\gamma+v_2^n)y_3^n \end{cases} \tag{18}$$

$$\frac{\partial y_1^n}{\partial \eta} = \frac{\partial y_2^n}{\partial \eta} = \frac{\partial y_3^n}{\partial \eta} = \frac{\partial y_4^n}{\partial t} = 0 \qquad (t,x)\in\Sigma\,(t,x)\in\Sigma \tag{19}$$

$$y_i^n(0,x) = y_i^0 \; for\, i=1,2,3,4 \qquad x\in\Omega \tag{20}$$

and By theorem (1) using the estimate (9) of the solution y_i^n, there exists a constant $C>0$ such that for all $n\geq 1$,$t\in[0,T]$

$$\left\|\frac{\partial y_i^n}{\partial t}\right\|_{L^2(Q)} \leq C,\; \|y_i^n\|_{L^2\left(0,T;H^2(\Omega)\right)} \leq C,\; \|y_i^n\|_{H^1(\Omega)} \leq C,\; i=1,2,3,4 \tag{21}$$

$H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, so we deduce that $y_1^n(t)$ is compact in $L^2(\Omega)$.

Let's Show that $\left\{y_1^n(t), n\geq 1\right\}$ is equicontinuous in $C\left([0,T]:L^2(\Omega)\right)$. As $\dfrac{\partial y_1^n}{\partial t}$ is bounded in $L^2(Q)$, this implies that for all $s,t\in[0,T]$

$$\left|\int_\Omega (y_1^n)^2(t,x)\,dx - \int_\Omega (y_1^n)^2(s,x)\,dx\right| \leq K|t-s| \tag{22}$$

The Ascoli-Arzela Theorem(See [24]) implies that y_1^n is compact in $C\left([0,T]:L^2(\Omega)\right)$. Hence, selecting further sequences, if necessary, we have

$y_1^n \longrightarrow y_1^*$ in $L^2(\Omega)$, uniformly with respect to t and analogously, we have for $y_i^n \longrightarrow y_i^*$ in $L^2(\Omega)$ for $i=2,34,$ uniformly with respect to t.

From the boundedness of Δy_i^n in $L^2(Q)$, which implies it is weakly convergent in $L^2(Q)$ on a subsequence denoted again $\triangle y_i^n$ then for all distribution φ

$$\int_Q \varphi\Delta y_i^n = \int_Q y_i^n \triangle\varphi \to \int_Q y_i^* \triangle\varphi = \int_Q \varphi\Delta y_i^*$$

Which implies that $\triangle y_i^n \to \triangle y_i^*$ weakly in $L^2(Q)$,$i=1,\,2,\,3,\,4,$ In addition, the estimates (21) leads to

$\dfrac{\partial y_i^n}{\partial t} \to \dfrac{\partial y_i^*}{\partial t}$ weakly in $L^2(Q)$, $i=1,\,2,\,3,\,4,$

$y_i^n \to y_i^*$ weakly in $L^2\left(0,T;H^2(\Omega)\right)$, $i=1,\,2,\,3,\,4,$

$y_i^n \to y_i^*$ weakly star in $L^\infty\left(0,T;H^1(\Omega)\right)$, $i=1,\,2,\,3,\,4,$

We put $N(y)=\frac{\beta\lambda}{y_1+y_2+y_3+y_4}$,we now show that $y_1^n y_3^n \mapsto y_1^* y_3^*$ and $N(y^n)y_1^n y_3^n \mapsto N(y^*)y_1^* y_3^*$ strongly in $L^2(Q)$, we write

$$y_1^n y_3^n - y_1^* y_3^* = (y_1^n - y_1^*)\,y_3^n + (y_3^n - y_3^*)\,y_1^*$$

$$N(y^n)y_1^n y_3^n - N(y^*)y_1^* y_3^* = N(y^n)\,(y_1^n y_3^n - y_1^* y_3^*) + y_1^* y_3^*\,(N(y^n) - N(y^*))$$
$$N(y^n) - N(y^*) = \frac{\beta\lambda}{y_1^n + y_2^n + y_3^n + y_4^n} - \frac{\beta\lambda}{y_1^* + y_2^* + y_3^* + y_4^*}$$

and we make use of the convergences $y_i^n \longrightarrow y_i^*$ strongly in $L^2(Q)$, $i=1,3,$ and of the boundedness of y_1^n, y_3^n in $L^\infty(Q)$, then $y_1^n y_3^n \to y_1^* y_3^*$and $N(y^n)y_1^n y_3^n \mapsto N(y^*)y_1^* y_3^*$ strongly in $L^2(Q)$.

Since $v_1^n\, and\, v_2^n$ are bounded, we can assume that$v_1^n \to v_1^*\, and\;\; v_2^n \to v_2^*$ weakly in $L^2(Q)$ on a subsequence denoted again $v_1^n\, and\, v_2^n$. Since U_{ad} is a closed and convex set in $(L^2(Q))^2$, it is weakly closed, so $(v_1^*, v_2^*) \in U_{ad}$

We now show that

$$v_1^n N(y^n)y_1^n y_3^n \to v_1^* N(y^*)y_1^* y_3^* weakly\, in\, L^2(Q)$$

Writing with $i=1,2,3,4,$

$$v_1^n N(y^n)y_1^n y_3^n - v_1^* N(y^*)y_1^* y_3^* = (N(y^n)y_1^n y_3^n - N(y^*)y_1^* y_3^*)\,u^n + (v_1^n - v_1^*)\,N(y^*)y_1^* y_3^*$$

and making use of the convergences $N(y^n)y_1^n y_3^n \mapsto N(y^*)y_1^* y_3^*$ strongly in $L^2(Q)$, and $v_1^n \longrightarrow v_1^*$ weakly in $L^2(Q)$, one obtains that $u^n N(y^n)y_1^n y_3^n \to u^* N(y^*)y_1^* y_3^*$ weakly in $L^2(Q)$.

By taking $n \to \infty$ i in (18-20),, we obtain that y^* is a solution of (2-4) corresponding to $(u,^* v^*) \in U_{ad}$. Therefore

$$\begin{aligned} J(y^*,(v_1^* v_2^*)) &= K_1 \int_0^T \int_\Omega y_2^*(t,x)\,dxdt + K_2 \int_0^T \int_\Omega y_3^*(t,x)\,dxdt + \frac{\rho_1}{2}\|v_1^*\|^2_{L^2(Q)} + \frac{\rho_2}{2}\|v_2^*\|^2_{L^2(Q)} \\ \leq &\quad lim_{n\to\infty} inf\left(K_1 \int_0^T \int_\Omega y_2^n(t,x)\,dxdt + K_2 \int_0^T \int_\Omega y_3^n(t,x)\,dxdt + \frac{\rho_1}{2}\|v_1^*\|^2_{L^2(Q)} + \frac{\rho_2}{2}\|v_2^*\|^2_{L^2(Q)}\right) \\ = &\quad lim_{n\to\infty}\left(K_1 \int_0^T \int_\Omega y_2^n(t,x)\,dxdt + K_2 \int_0^T \int_\Omega y_3^n(t,x)\,dxdt + \frac{\rho_1}{2}\|v_1^*\|^2_{L^2(Q)} + \frac{\rho_2}{2}\|v_2^*\|^2_{L^2(Q)}\right) \\ = &\quad inf_{(v_1,v_2)\in U_{ad}} J((y,(v_1,v_2)) \end{aligned}$$

This shows that J attains its minimum at $(y^*,(v_1^*,v_2^*))$, we deduce that $(y^*,(v_1^*,v_2^*))$ verifies problem (2-4) and minimizes the objectif functional (6). The proof is complet □

5. Necessary Optimality Conditions

Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in U_{ad}$,$v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} \in U_{ad}$ and $v^\varepsilon = v^* + \varepsilon v \in U_{ad}$, in this section, we show the optimality conditions to problem (2-4), and we find the characterization of optimal control. First , we need the Gateaux differentiability of the mapping $v \to y(v)$. For this reason, denoting by $y^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon) = (y_1,y_2,y_3,y_4)(v^\varepsilon)$ and $y^* = (y_1^*,y_2^*,y_3^*,y_4^*) = (y_1,y_2,y_3,y_4)(v^*)$ the solution

of (2-4) corresponding to v^ε and v^* respectively.

$$H = \begin{pmatrix} -\frac{\beta\lambda y_3^*(y_2^*+y_3^*+y_4^*)((1-v_1^\varepsilon)+d)}{(y_1^*+y_2^*+y_3^*+y_4^*)^2} & 0 & -\frac{\beta\lambda y_1^*(y_1^*+y_2^*+y_4^*)(1-v_1^\varepsilon)}{(y_1^*+y_2^*+y_3^*+y_4^*)^2} & 0 \\ \frac{\beta\lambda y_3^*(y_2^*+y_3^*+y_4^*)(1-v_1^\varepsilon)}{(y_1^*+y_2^*+y_3^*+y_4^*)^2} & -(d+\varepsilon) & \frac{\beta\lambda y_1^*(y_1^*+y_2^*+y_4^*)(1-v_1^\varepsilon)}{(y_1^*+y_2^*+y_3^*+y_4^*)^2} & 0 \\ 0 & \varepsilon & -(d+\gamma+\alpha+v_2^\varepsilon) & 0 \\ 0 & 0 & (v_2^\varepsilon+\gamma) & -d \end{pmatrix}$$

and $L = \begin{pmatrix} \beta\lambda\frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)} & 0 \\ -\beta\lambda\frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)} & 0 \\ 0 & -y_3^* \\ 0 & y_3^* \end{pmatrix}$.

Proposition 3. *The mapping $y : U_{ad} \to W^{1,2}([0,T]; H(\Omega))$ with $y_i \in \mathscr{L}(T,\Omega)$ for $i = 1,2,3,4$ is Gateaux differentiable with respect to v^*. For all direction $v \in U_{ad}$, $y'(v^*)v = Y$ is the unique solution in $W^{1,2}([0,T]; H(\Omega))$ with $Y_i \in \mathscr{L}(T,\Omega)$ of the following equation*

$$\left\{ \begin{array}{l} \dfrac{\partial Y}{\partial t} = AY + HY + Lv, \qquad t \in [0,T] \\ \\ Y(0) = 0 \end{array} \right. \tag{23}$$

Proof. Put $Y_i^\varepsilon = \dfrac{y_i^\varepsilon - y_i^*}{\varepsilon}$ for $i = 1,2,3,4.F(y_1,y_2,y_3,y_4) = \dfrac{y_1 y_3}{y_1+y_2+y_3+y_4}$,
$D_1^\varepsilon = \dfrac{F\left(y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon\right) - F\left(y_1^*, y_{2,}^\varepsilon, y_3^\varepsilon, y_4^\varepsilon\right)}{y_1^\varepsilon - y_1^*}$, and $D_3^\varepsilon = \dfrac{F\left(y_1^*, y_{2,}^\varepsilon, y_3^\varepsilon, y_4^\varepsilon\right) - F\left(y_1^*, y_3^*, y_3^*, y_4^*\right)}{y_3^\varepsilon - y_3^*}$.

We denote S^ε the system (2) corresponding to v^ε and S^* the system (2) corresponding to v^*, subtracting system S^ε from S^*, we have

$$\left\{ \begin{array}{l} \dfrac{\partial Y_1^\varepsilon}{\partial t} = d_1 \Delta Y_1^\varepsilon - (D_1^\varepsilon \beta\lambda(1-v_1^\varepsilon) + d) Y_1^\varepsilon - D_3^\varepsilon \beta\lambda(1-v_1^\varepsilon) Y_3^\varepsilon + v_1^* \beta\lambda \frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)} \\ \\ \dfrac{\partial Y_2^\varepsilon}{\partial t} = d_2 \Delta Y_2^\varepsilon + (D_1^\varepsilon \beta\lambda(1-v_1^\varepsilon)) Y_1^\varepsilon - (d+\varepsilon) Y_2^\varepsilon + (D_3^\varepsilon \beta\lambda(1-v_1^\varepsilon)) Y_3^\varepsilon - v_1^* \beta\lambda \frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)}, \quad (x,t) \in Q \\ \\ \dfrac{\partial Y_3^\varepsilon}{\partial t} = d_3 \Delta Y_3^\varepsilon + \varepsilon Y_2^\varepsilon - (d+\gamma+\alpha+v_2^\varepsilon) Y_3^\varepsilon - v_2^* y_3^* \\ \\ \dfrac{\partial Y_4^\varepsilon}{\partial t} = d_4 \Delta Y_4^\varepsilon + (v_2^\varepsilon + \gamma) Y_3^\varepsilon - d Y_4^\varepsilon + v_2^* y_3^* \end{array} \right. \tag{24}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial Y_1^\varepsilon}{\partial \eta} = \frac{\partial Y_2^\varepsilon}{\partial \eta} = \frac{\partial Y_3^\varepsilon}{\partial \eta} = \frac{\partial Y_4^\varepsilon}{\partial \eta} = 0 \qquad (x,t) \in \Sigma \tag{25}$$

$$Y_i^\varepsilon(0,x) = 0 \qquad x \in \Omega,\ for\, i = 1,2,3,4 \tag{26}$$

We prove that Y_i^ε are bounded in $L^2(Q)$ uniformly with respect to ε . For this end, denoting by

$Y^{\varepsilon} = \left(Y_1^{\varepsilon}, Y_2^{\varepsilon}, Y_3^{\varepsilon}, Y_4^{\varepsilon}\right),$

$$H^{\varepsilon} = \begin{pmatrix} -(D_1^{\varepsilon}\beta\lambda(1-v_1^{\varepsilon})+d) & 0 & -D_3^{\varepsilon}\beta\lambda(1-v_1^{\varepsilon}) & 0 \\ (D_1^{\varepsilon}\beta\lambda(1-v_1^{\varepsilon})) & -(d+\varepsilon) & (D_3^{\varepsilon}\beta\lambda(1-v_1^{\varepsilon})) & 0 \\ 0 & \varepsilon & -\left(d+\gamma+\alpha+v_2^{\varepsilon}\right) & 0 \\ 0 & 0 & (v_2^{\varepsilon}+\gamma) & -d \end{pmatrix},$$

$$\text{and } L = \begin{pmatrix} \beta\lambda\frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)} & 0 \\ -\beta\lambda\frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)} & 0 \\ 0 & -y_3^* \\ 0 & y_3^* \end{pmatrix}.$$

Then (24) given by

$$\begin{cases} \dfrac{\partial Y^{\varepsilon}}{\partial t} = AY^{\varepsilon} + H^{\varepsilon} Y^{\varepsilon} + Lv, & t \in [0,T] \\ Y^{\varepsilon}(0) = 0 \end{cases} \tag{27}$$

$(S(t), t \geq 0)$be the semi-group generated by A, then the solution of (27) can be expressed as

$$Y^{\varepsilon}(t) = \int_0^t S(t-s) H^{\varepsilon}(s) Y^{\varepsilon}(s)\, ds + \int_0^t S(t-s) Lv(s)\, ds, \tag{28}$$

On the other hand the coefficients of the matrix H^{ε} are bounded uniformly with respect to ε, using Gronwall's inequality, we have

$$\|Y_i^{\varepsilon}\|_{L^2(Q)} \leq \Gamma \tag{29}$$

where $\Gamma > 0$ $(i = 1,2,3)$. Then

$$\|y_i^{\varepsilon} - y_i^*\|_{L^2(Q)} = \varepsilon \|Y_i^{\varepsilon}\|_{L^2(Q)} (4.7) \tag{30}$$

Hence $y_i^{\varepsilon} \to y_i^*$ in $L^2(Q)$, $i = 1,2,3$.

$$\text{Denoting by } H = \begin{pmatrix} -(D_1^*\beta\lambda(1-v_1^*)+d) & 0 & -D_3^*\beta\lambda(1-v_1^*) & 0 \\ (D_1^*\beta\lambda(1-v_1^*)) & -(d+\varepsilon) & (D_3^*\beta\lambda(1-v_1^*)) & 0 \\ 0 & \varepsilon & -(d+\gamma+\alpha+v_2^*) & 0 \\ 0 & 0 & (v_2^*+\gamma) & -d \end{pmatrix},$$

where $D_1^* = \dfrac{\partial F\left(y_1^*, y_2^*, y_3^*, y_4^*\right)}{\partial y_1}$, $D_3^* = \dfrac{\partial F\left(y_1^*, y_2^*, y_3^*, y_4^*\right)}{\partial y_3}$, and $Y = (Y_1, Y_2, Y_3, Y_4)$. Hence, then system (24-26) can be written in the form

$$\text{(31)} \qquad \begin{cases} \dfrac{\partial Y}{\partial t} = AY + HY + Lv, \qquad t \in [0, T] \\ \\ Y(0) = 0 \end{cases}$$

and its solution can be expressed as

$$\text{(32)} \qquad Y(t) = \int_0^t S(t-s) H(s) Y(s)\, ds + \int_0^t S(t-s) Lv(s)\, ds,$$

By (28) and (32) we deduce that

$$\text{(33)} \qquad Y^{\varepsilon}(t) - Y(t) = \int_0^t S(t-s) H^{\varepsilon}(s) \left(Y^{\varepsilon} - Y\right) + Y(s) \left(H^{\varepsilon}(s) - H(s)\right) ds$$

Thus all the coefficients of the matrix H^{ε}tend to the corresponding coefficients of the matrix H in $L^2(Q)$, An application of Gronwall's Inequality yields to $Y_i^{\varepsilon} \to Y_i$ in $L^2(Q)$ as $\varepsilon \to 0$, for $i = 1, 2, 3, 4$. □

Let v^* be an optimal control of (2-5), $y^* = \left(y_1^*, y_2^*, y_3^*, y_4^*\right)$ be the optimal state, Z is the matrix defined by $Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $K = (0, K_1, K_2, 0)$, $Z*$ is the adjoint matrix associated to Z, H^* is the adjoint matrix associated to H and $p = (p_1, p_2, p_{3,}, p_4)$ is the adjoint variable,we can write the dual system associated to system (2-5):

$$\text{(34)} \qquad \begin{cases} -\dfrac{\partial p}{\partial t} - Ap - H^* p = Z^* Z K, \quad t \in [0, T] \\ \\ p(T, x) = 0 \end{cases}$$

Lemma 4. *Under hypotheses of theorem (1) , if $(y^*, (v_1, v_2))$ is an optimal pair, then there exists a unique strong solution $p \in W^{1,2}([0, T]; M(\Omega))$ to the system (34) with $p_i \in \mathscr{L}(T, \Omega)$ for $i = 1, 2, 3, 4$.*

Proof. Like in Theorem (1), by making the change of variable $s = T - t$ and the change of functions $q_i(s,x) = p_i(T-s,x) = p_i(t,x),(t,x) \in Q,\ i = 1,2,3,4.$ we can easily prove the existence of the solution to this lemma . □

To obtain the necessary conditions for the optimal control problem, applying standard optimality techniques, analyzing the objective functional and utilizing relationships between the state and adjoint equations,we obtain a characterization of the control optimal.

Theorem 5. *Let* u^**be an optimal control of (2)-(5) and let* $y^* \in W^{1,2}([0,T];H(\Omega))$ *with* $y_i^* \in \mathscr{L}(T,\Omega)$ *for* $i = 1,2,3,4$ *be the optimal state, that is* y^* *is the solution to (2)-(5) with the control* $v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix}$. $\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$ *Then,*

$$v_1^* = min\left(v_1^{max}, max\left(0, \frac{\beta\lambda \frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)}(p_2 - p_1)}{\rho_1}\right)\right) \tag{35}$$

$$v_2^* = min\left(v_2^{max}, max\left(0, \frac{y_3^*(p_3 - p_4)}{\rho_2}\right)\right) \tag{36}$$

Proof. We suppose $v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix}$ is an optimal control and $y^* = (y_1^*, y_2^*, y_3^*, y_4^*) = (y_1, y_2, y_3, y_4)(v^*)$ are the corresponding state variables. Consider $v^\varepsilon = v^* + \varepsilon h \in U_{ad}$,$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ and corresponding state solution $y^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon) = (y_1, y_2, y_3, y_4)(v^\varepsilon)$, we have

(37)

$$
\begin{aligned}
J'(v^*)(h) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(J(v^\varepsilon)-J(v^*)\right)\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_0^T\int_\Omega K_1\left(y_2^\varepsilon-y_2^*\right)(t,x)\,dxdt+\int_0^T\int_\Omega K_2\left(y_3^\varepsilon-y_3^*\right)(t,x)\,dxdt\right.\\
&+ \left.\frac{\rho_1}{2}\int_0^T\int_\Omega\left(\left(v_1^\varepsilon\right)^2-(v_1^*)^2\right)(t,x)\,dxdt+\frac{\rho_2}{2}\int_0^T\int_\Omega\left(\left(v_2^\varepsilon\right)^2-(v_2^*)^2\right)(t,x)\,dxdt\right)\\
&= \lim_{\varepsilon\to 0}\left(\int_0^T\int_\Omega K_1\left(\frac{y_2^\varepsilon-y_2^*}{\varepsilon}\right)(t,x)\,dxdt+\int_0^T\int_\Omega K_2\left(\frac{y_3^\varepsilon-y_3^*}{\varepsilon}\right)(t,x)\,dxdt\ \right)\\
&+ \left.\frac{\rho_1}{2}\int_0^T\int_\Omega\left((\varepsilon h_1)^2+2hv_1^*\right)(t,x)\,dxdt+\frac{\rho_2}{2}\int_0^T\int_\Omega\left((\varepsilon h_2)^2+2hv_2^*\right)(t,x)\,dxdt\right)\\
&= \int_0^T\int_\Omega K_1Y_2(t,x)\,dxdt+\int_0^T\int_\Omega K_2Y_3(t,x)\,dxdt+\rho_1\int_0^T\int_\Omega(h_1v_1^*)(t,x)\,dxdt\\
&\quad+\rho_2\int_0^T\int_\Omega(h_2v_2^*)(t,x)\,dxdt\\
&= \int_0^T\langle ZK,ZY\rangle_{H(\Omega)}\,dt+\int_0^T\langle\rho v^*,h\rangle_{L^2(\Omega)}\,dt
\end{aligned}
$$

We use (23) and (34), we have

$$
\begin{aligned}
\int_0^T\langle ZK,ZY\rangle_{H(\Omega)}\,dt &= \int_0^T\langle Z^*ZK,Y\rangle_{H(\Omega)}\,dt\\
&= \int_0^T\left\langle-\frac{\partial p}{\partial t}-Ap-H^*p,Y\right\rangle_{H(\Omega)}dt\\
&= \int_0^T\left\langle p,\frac{\partial Y}{\partial t}-AY-HY\right\rangle_{H(\Omega)}dt\\
&= \int_0^T\langle p,Lh\rangle_{H(\Omega)}\,dt \qquad (38)\\
&= \int_0^T\langle L^*p,h\rangle_{L^2(\Omega)}\,dt
\end{aligned}
$$

Since J is Gateaux differentiable at v^* and U_{ad} is convex, as the minimum of the objective functional is attained at v^* it is seen that $J'(v^*)(u-v^*)\geq 0$ for all $u\in U_{ad}$.

We take $h=u-v^*$ and we use (37)-(38) then $J'(v^*)(u-v^*)=\int_0^T\langle L^*p+\alpha u^*,(u-v^*)\rangle_{L^2(\Omega)}\,dt$. We conclude that $J'(v^*)(u-v^*)\geq 0$ equivalent to $\int_0^T\langle L^*p+\alpha v^*,(u-v^*)\rangle_{L^2(\Omega)}\,dt\geq 0$ for all

$u \in U_{ad}$. By standard arguments varying u, we obtain $\rho v^* = -L^* p$ Then

$$\rho v^* = \begin{pmatrix} \rho_1 v_1^* \\ \rho v_2^* \end{pmatrix} = \begin{pmatrix} -\beta\lambda \frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)} & +\beta\lambda \frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)} & 0 & 0 \\ 0 & 0 & y_3^* & -y_3^* \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

As $v^* \in U_{ad}$, we have

$$v_1^* = min\left(v_1^{max}, max\left(0, \frac{\beta\lambda \frac{y_1^* y_3^*}{(y_1^*+y_2^*+y_3^*+y_4^*)}(p_2 - p_1)}{\rho_1}\right)\right), \; v_2^* = min\left(v_2^{max}, max\left(0, \frac{y_3^*(p_3 - p_4)}{\rho_2}\right)\right)$$

□

6. Numerical Results

In this section, we present the results obtained by the numerical resolution of the optimality system ((2-4),(34)(35)), based on a discrete iterative scheme that converges following a test appropriate to the forward-backward scanning method (FBSM) [23]. We adopt two situations for the resolution: the first is that the disease starts with the middle of domain $\Omega(1)$, and in the second situation the disease begins with the lower corner $\Omega(2)$. A rectangular area of 30 km × 40 km is considered, and the parameter values and the initial values are given in Table 1.The upper limits of the optimality condition are considered to be $v_1^{max} = 1, v_2^{max} = 1,$. The constant weighting values in the objective function are $K_1 = 0.1, K_2 = 0.1$, $\rho_1 = \rho_2 = 2$.

Notations	Value	Description$(Units)$
$S_0(x,y)$	$4\, for\ (x,y) \in \Omega_i \quad i=1,2$ $10\, for\ (x,y) \notin \Omega_i$	$Initial\, susceptible\, population$ $(people/km^2)$
$E_0(x,y)$	$3\, for\ (x,y) \in \Omega_i \quad i=1,2$ $0\, for\ (x,y) \notin \Omega_i$	$Initial\, exposed\, population$ $(people/km^2)$
$I_0(x,y)$	$2\, for\ (x,y) \in \Omega_i \quad i=1,2$ $0\, for\ (x,y) \notin \Omega_i$	$Initial\, infected\, population$ $(people/km^2)$
$R_0(x,y)$	$1\, for\ (x,y) \in \Omega_i \quad i=1,2$ $0\, for\ (x,y) \notin \Omega_i$	$Initial\, recovered\, population$ $(people/km^2)$
λ	1	$The\, probability\, for\, an\, individual\, to\, take\, part\, in\, a\, contact$
α	0.01	$Rate\, that\, exposed\, individuals\, become\, infectious$
β	1	$Effective\, contact\, rate$
γ	0.04	$Recovery\, rate$
Θ	0.112	$Recruitement\, rate\, into\, the\, population$
d_S	0.05	$diffusion\, rate\, for\, susceptible$ km^2/day
d_E	0.5	$diffusion\, rate\, for\, susceptible$ km^2/day
d_I	0.5	$diffusion\, rate\, for\, infected$ km^2/day
d_R	0.1	$diffusion\, rate\, for\, recovered$ km^2/day
d	0.01	$Natural\, mortality\, rate$
p	0.01	$Rate\, of\, exposed\, individuals\, in\, new\, members\, of\, the\, population$
q	0.001	$Rate\, of\, infected\, individuals\, in\, new\, members\, of\, the\, population$
b	0.001	$Rate\, of\, recovred\, individuals\, in\, new\, members\, of\, the\, population$
ε	0.09	$Disease\, induced\, death\, rate$
t	$[1,55]$	$time\, period$ (day)

TABLE 1. Initial conditions and parameters values

To illustrate the effectiveness of the control and its impact on the spread of the disease, we have chosen three scenarios:

-In the first scenario, we illustrate the dynamics of the system without intervention.

-In the second scenario, from the 48 day of onset of the disease, we simultaneously apply the strategy that reduces contact between infectious and susceptible individuals, and that of treatment.

-In the third scenario, we repeat the second scenario, but from the first days of the onset of the epidemic.

6.1. Simulations without optimal control. For the time interval from t = 1 to t = 55 days, when the disease starts at the corner or in the middle, the numerical results (Figures 1,2,3 and 4) in the absence of controls show a spread of the epidemic to all sides and the number of infected people is rising rapidly.

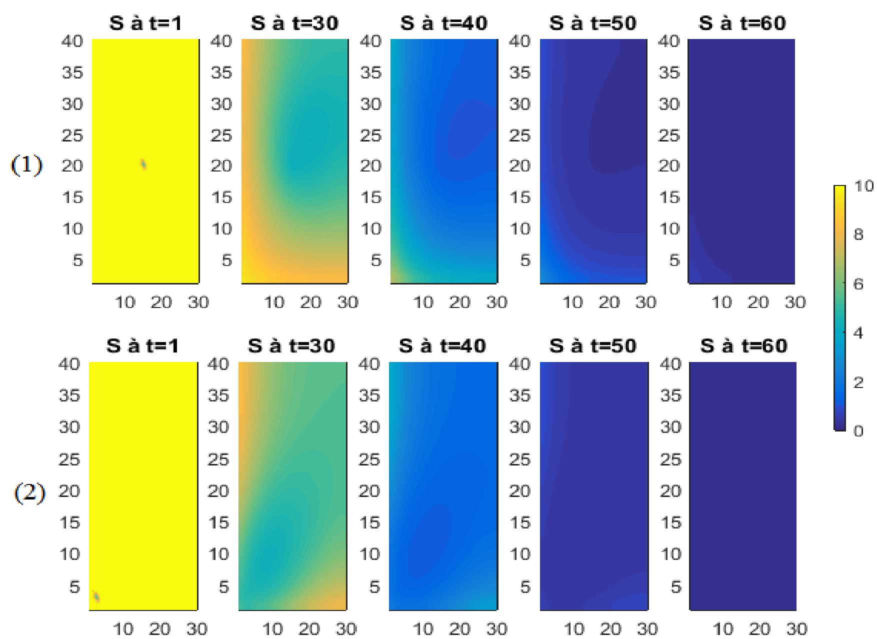

FIGURE 1. Susceptible behavior within Ω without control

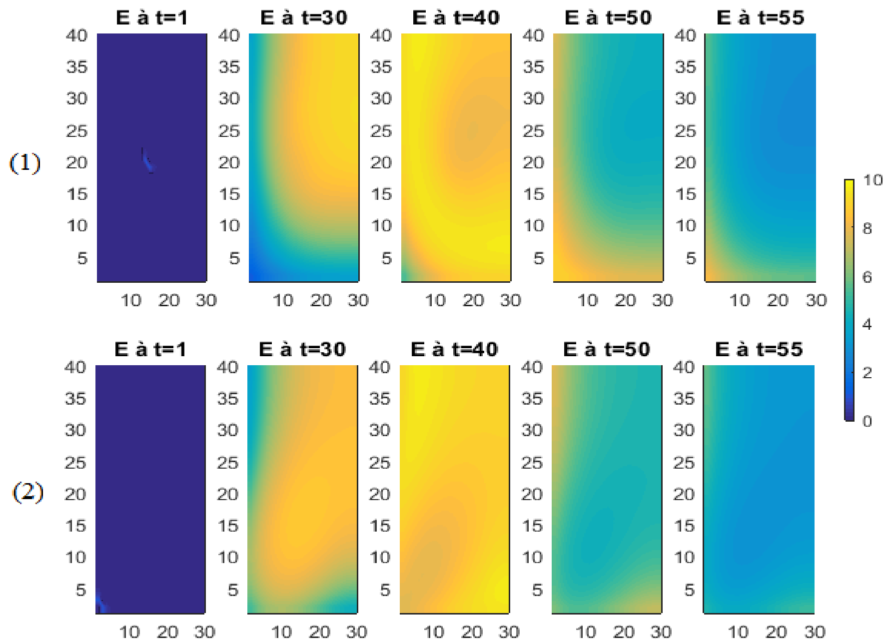

FIGURE 2. Exposed behavior within Ωwithout control

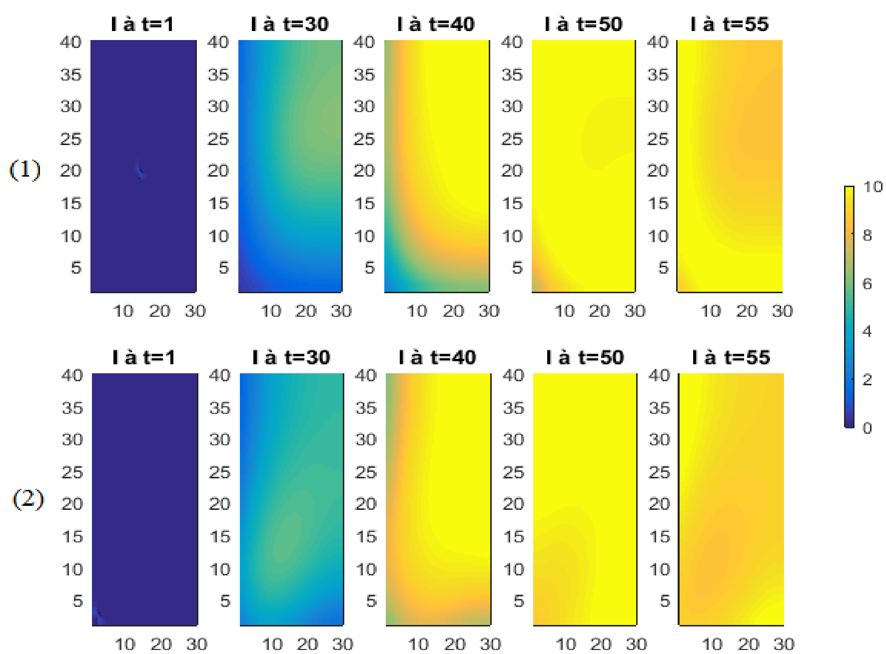

FIGURE 3. Infectied behavior within Ω without control

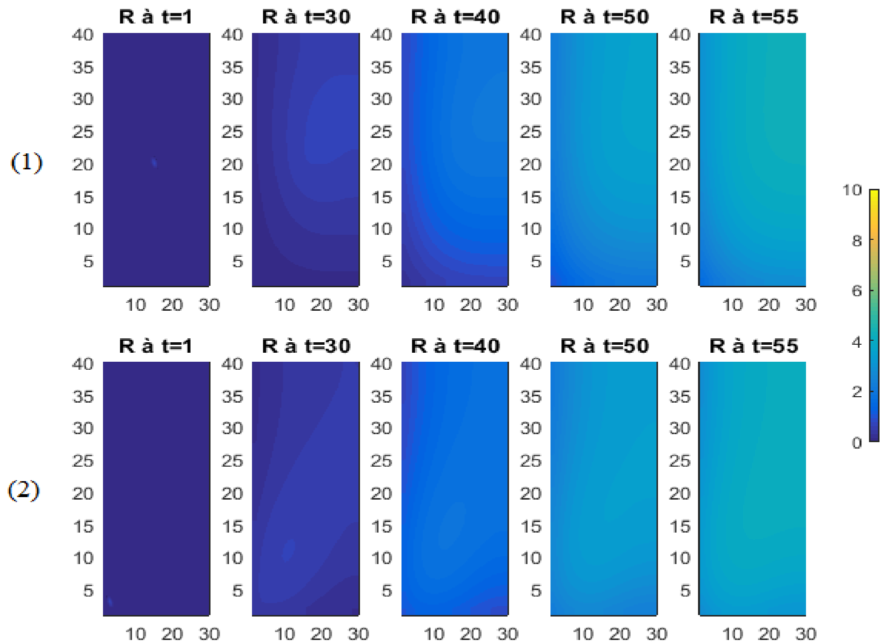

FIGURE 4. Recovered behavior within Ω without control

6.2. The control strategy starts from the 48 day. In this subsction, we present the numerical results obtained during the application of the second scenario: the controls are introduced after 48 days of the onset of the disease. Figures 5, 6 and 7 show the impact of the controls adopted against the spread of the epidemic, since the number of infected decreased from a density of 10 infected to 2 infected (Fig 6), and the number of recovered increased by a density of 1 recovered to 5 recovered (Fig 7). But despite these results, the disease is not completely gone, as there are still infected individuals, which can be a major source of the spread of the epidemic.

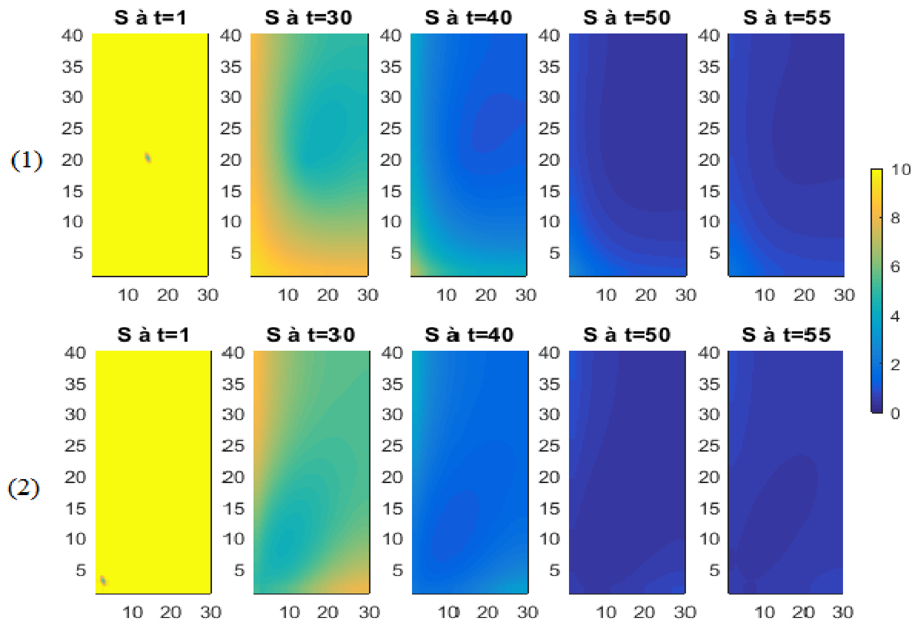

FIGURE 5. Susceptible behavior within Ω with control (control strategy after 48 days)

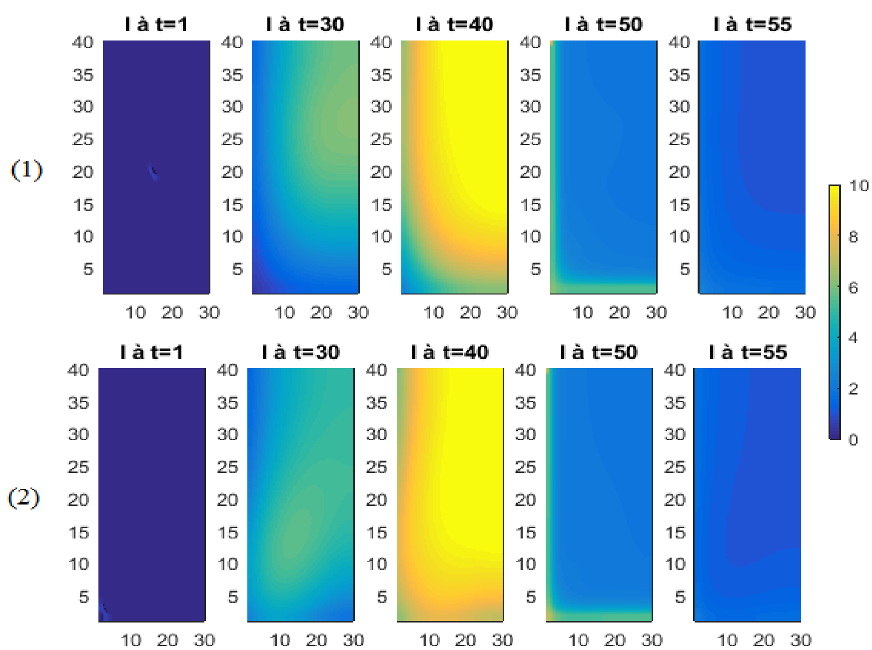

FIGURE 6. Infectied behavior within Ω with control (control strategy after 48 days)

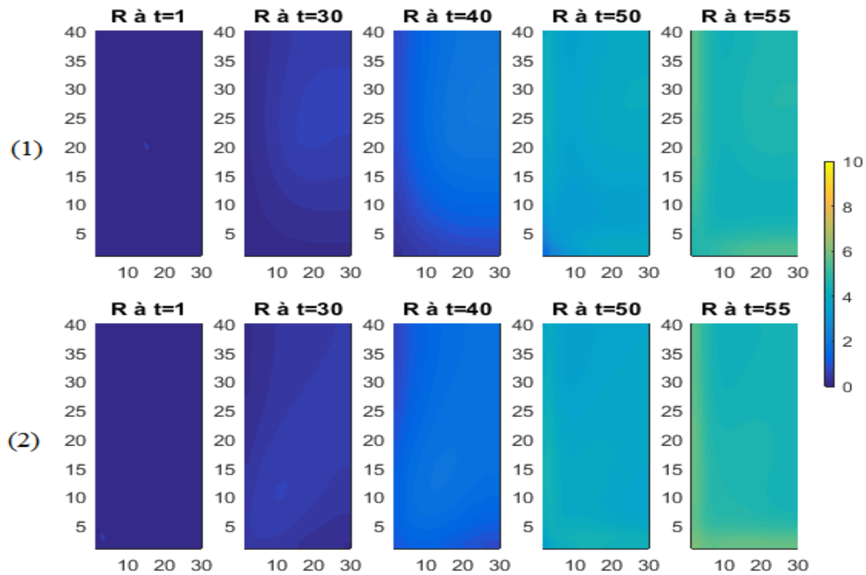

FIGURE 7. Recovered behavior within Ω with control (control strategy after 48 days)

6.3. The control strategy starts from the first day. Figures 8, 9 and 10 present the numerical results obtained when applying our control strategy from the first day of the disease. The effectiveness of our approach is clear, since the number of infected and exposed individuals has remained almost zero, due to the combination of efforts to reduce contact between susceptible individuals and infected individuals, and treatment performed on the infected.

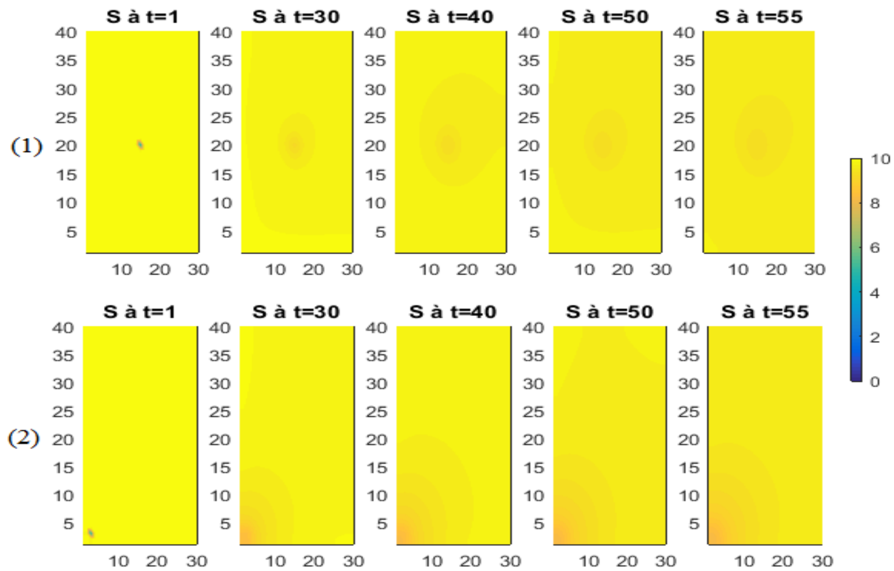

FIGURE 8. Susceptible behavior within Ω with control (control strategy starts from the first day)

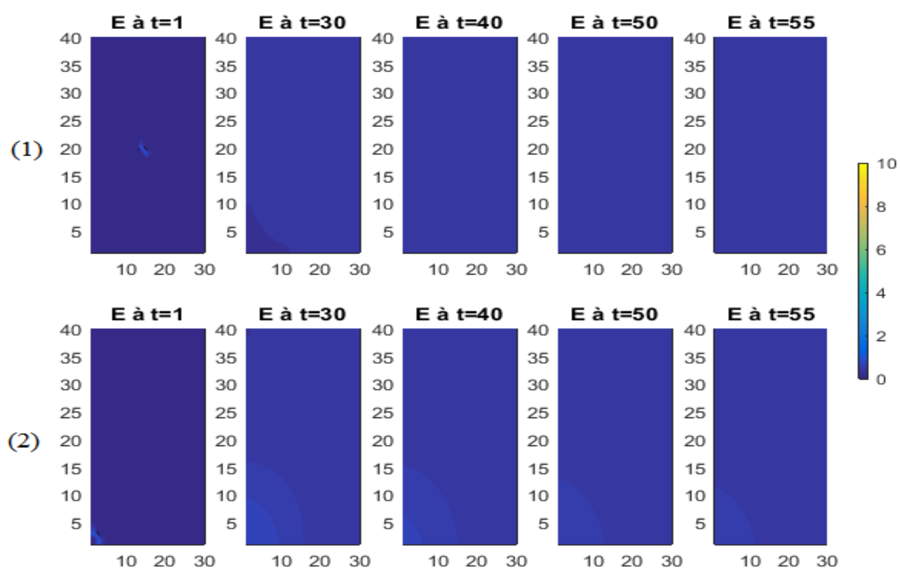

FIGURE 9. Exposed behavior within Ω with control (control strategy starts from the first day)

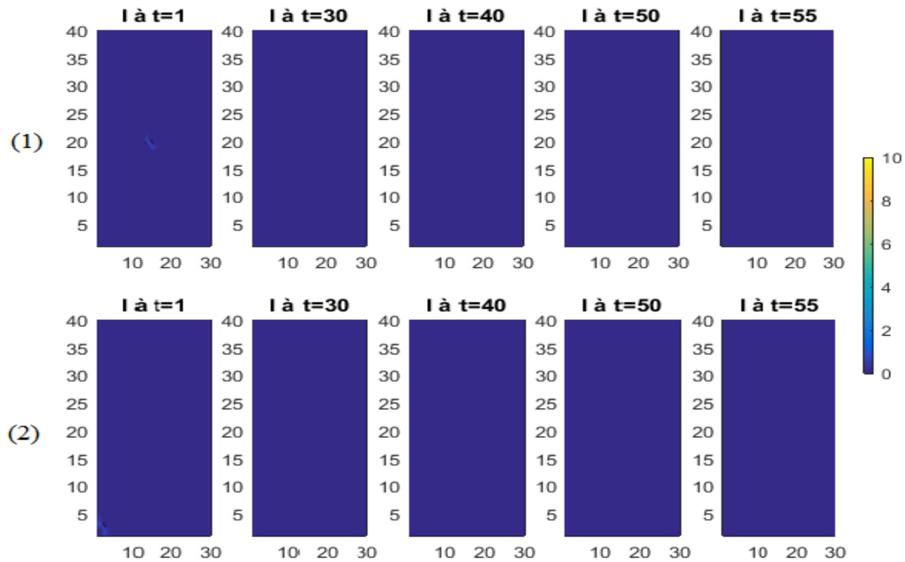

FIGURE 10. Infected behavior within Ω with control (control strategy starts from the first day)

7. CONCLUSION

In this article, we present a theoretical work that can be used in the study of several infectious diseases in the form of SEIR model, we have also expanded this model to take into account the spatial spread of the disease, we studied a couple Optimal control for this SEIR spatio-temporal model, the first control has the role of reducing contact between susceptible and infected, and the second is in the form of a therapeutic treatment. Theoretically, we demonstrate the existence of optimal controls and the solution of the state system. The characterization of the optimal control torque is determined in terms of state functions and adjoint functions. The numerical resolution is based on the forward / backward scanning method (FBSM). The numerical results have shown that the introduction of control from the first day of the disease, which reduces the contact between the susceptible and the infected, and that of the treatment plus the corresponding cost, constitutes the best optimal strategy for obtaining better results. This strategy has made it possible to block the spread of the epidemic.

8. APPENDIX

First recall a general existence result which we use in the sequel (Proposition 1.2, p. 175, [28]; see also[27,29]. Consider the initial value problem

$$\begin{cases} \frac{\partial z}{\partial t} = Az(t) + g(t, z(t)), & t \in [0, T] \\ z(0) = z_0 \end{cases} \tag{39}$$

where A is a linear operator defined on a Banach space X, with the domain $D(A)$ and $g : [0,T] \times X \to X$ is a given function. If X is a Hilbert space endowed with the scalar product $(\cdot,\cdot)$, then the linear operator A is called dissipative if $(Az,z) \leq 0$, $(\forall z \in D(A))$.

Theorem 6. *X be a real Banach space, $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup of linear contractions $S(t), t \geq 0$ on X, and $g : [0,T] \times X \to X$ be a function measurable in t and Lipschitz continuous in $x \in X$, uniformly with respect to $t \in [0,T]$.*

(i) If $z_0 \in X$, then problem (39) admits a unique mild solution, i.e. a function $z \in C([0,T];X)$ which verifies the equality $z(t) = S(t)z_0 + \int_0^t S(t-s)g(s,z(s))ds, (\forall t \in [0,T]$.

(ii) If X is a Hilbert space, A is self-adjoint and dissipative on X and $z_0 \in D(A)$, then the mild solution is in fact a strong solution and $z \in W^{1,2}([0,T];X) \cap L^2(0,T;D(A))$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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