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NUMERICAL SOLUTION OF A FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND INVOLVING SOME DUAL SERIES EQUATIONS

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Abstract: In this article, the Laplace equation with nonhomogenous mixed boundary conditions, defined on a surface of semi-infinite cylinder is solved by using dual series equations (DSE). The DSE method is based on transforming the mixed problem to a Fredholm integral equation of second kind, where the resulted integral equation is equipped with kernel defined of infinite integral. We solved the equation by using numerical quadrature method and obtain the numerical solution for the Laplace equation.

Keywords: Laplace equation; dual series equations; Fredholm integral equation; quadrature method.

2010 AMS Subject Classification: 65M06, 65M22.

1. INTRODUCTION

Laplace equation is considered one of the most important equations in applied mathematics. In fact, many applications in electromagnetism, heat conduction, potential theory, solid mechanics, fluid mechanics, geometry and other areas involve the Laplace equation. Nevertheless, few analytical and numerical methods in solving the Laplace equation are known. On the other hand,

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many mixed problems related to Laplace equations such as nonstationary heat equations and Helmholtz equations with mixed boundary conditions are mentioned in many articles [1,5-11, 18].

In this paper, we consider the problem of solving the Laplace equation, which is defined over a semi-infinite solid cylinder with inner and outer shells (see figure 1). More precisely, the shells of the cylindrical object are divided into two parts with inner and outer circles of given radii. On the shells, the nonhomogeneous mixed boundary conditions of the third kind are given. According to the heat theory, these conditions represent the heat exchangers. Moreover, inside the larger circle, nonhomogeneous mixed boundary conditions of the third kind are different from those on the small circle, and on the boundaries of the cylindrical object the problem is based on first order homogenous conditions.

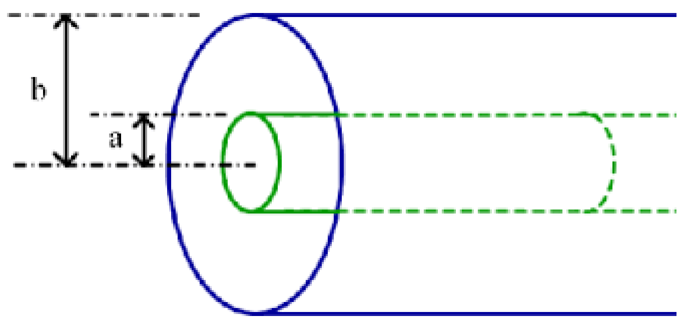

FIGURE 1: Semi-infinite solid cylinder with inner and outer shells

This article presents a method for solving the Laplace equation, where we used the dual series equations. The method is based on separation the variables, and then the solution is expressed as infinite series with unknown coefficients that need to be determined. By applying the mixed conditions on the cylindrical coordinates, we obtain a DSE containing an unknown function and Bessel's function of the first kind of order zero. So, for solving the DSE, we are applying the inverse formula of Fourier-Bessel transform. We then do some computations to reduce the DSE to a Fredholm integral equation of the second kind. Consequently, by solving the resulting Fredholm integral equation, the Laplace equation is then solved. We also use the numerical quadrature to solve the integral equation. In fact, many numerical methods could be used for solving the integral equations such as; *B*- spline wavelet method, the method of moments based on *B*-spline wavelets and variational iteration method [12,13], Taylor series expansion method, Rationalized Haar functions method [16], Haar wavelet method, Galerkin method, Adomian decomposition method [4], quadrature method [2] and other methods [15, 19]. Indeed, DSE could also be applied for several mixed problems with different boundary conditions.

2. FORMULATION OF THE PROBLEM

This article gives new methods to approximate the solutions of the linear Laplace equation. In particular, we consider the following problem

$$u_{rr}(r,z)+u_r(r,z)/r+u_{zz}(r,z)=0\ ,0<r<,\ \mathrm{b},\ 0<z<\infty \tag{1}$$

Subject to the boundary conditions;

$$u(a,z)=u(0,z)=0,\ \ 0\le z<\infty, \tag{2}$$

$$\alpha_1\,\mathrm{u}_z(r,0)-\beta_1\,\mathrm{u}(r,0)=-f_1(r),\quad 0\le r<a, \tag{3}$$

$$\alpha_2\,\mathrm{u}_z(r,0)-\beta_2\,\mathrm{u}(r,0)=-f_2(r),\quad a\le r<b. \tag{4}$$

where, $\alpha_i,\beta_i\ i=1,2$ are constant parameters, $f_i(r)$ are known continuous functions.

3. SOLUTION OF THE PROBLEM

This section provides the method for reducing the Laplace equation into a Fredholm integral equation of the second kind. Firstly, we separate variables in equation(1), and consider the function $u(r,z)$ is bounded as z approaches ∞ and $r=0$. More precisely, we use the following setting.

$$u(r,z)=\sum_{n=0}^{\infty}A_n\exp(-\lambda_n z/a\ )J_0(\lambda_n r/a) \tag{5}$$

where λ_n is the root of the Bessel function $J_0(\lambda_n r/a)$. Secondly, inserting the boundary conditions (3) and (4) into equation(5), we then obtain a pair of DSE with unknown coefficients A_n .

$$\sum_{n=0}^{\infty} A_n(\alpha_1\lambda+\beta_1)J_0(\lambda_n\rho)= f_1(\rho),\ \ \rho\in\Omega\ , \tag{6}$$

$$\sum_{n=0}^{\infty} A_n(\alpha_2\lambda+\beta_2)J_0(\lambda_n\rho)= f_2(\rho),\ \rho\in\overline{\Omega}\ . \tag{7}$$

$\rho = r/a,\ \alpha = b/a,\ \ \Omega:0<\rho<1\ ,\ \ \overline{\Omega}:1<\rho<\alpha\,.$

Consequently, equations (6) and (7) can be rewritten as

$$\sum_{n=1}^{\infty}(1-g_n)C_nJ_0(\lambda_n\rho)=f_1(\rho),\ \ \rho\in\Omega, \tag{8}$$

$$\sum_{n=1}^{\infty}C_nJ_0(\lambda_n\rho)= f_2(\rho),\ \rho\in\overline{\Omega}. \tag{9}$$

such that $C_n = A_n(\alpha_2\lambda+\beta_2)$, $g_n = \dfrac{\lambda_n\left(\alpha_2-\alpha_1\right)+\beta_2-\beta_1}{\alpha_2\lambda_n+\beta_2}$,

Equation (9) over the interval $(0,b)$ could be written as

$$\sum_{n=1}^{\infty}C_nJ_0(\lambda_n\rho)=\begin{cases} h(\rho)\ \ ,\rho\in\Omega, \\ f_2(\rho),\ \ \rho\in\overline{\Omega}\,. \end{cases} \tag{10}$$

where $h(\rho)$ is an unknown function on the interval (0, 1). Now, by using the inversion formula to equation(10), we have

$$C_n=\frac{2}{\alpha^2J_1^2\ (\lambda_n\alpha)}\left\{\int_0^1 yh(y)J_0(\lambda_ny)\,dy+\int_1^{\alpha} yf_2(y)J_0(\lambda_ny)\,dy\right\} \tag{11}$$

Subsequently, inserting equation (11) into (8) and interchanging the order of integration, we end up with Fredholm integral equation of the second kind with the unknown function $h(\rho)$

$$h(\rho)+\int_0^1 K(\rho,y)h(y)dy=F(\rho),\ \rho\in\Omega \tag{12}$$

where kernel and free term are defined as:

$$K(\rho,y)=\frac{2}{\alpha^2J_1^2(\lambda_n\alpha)}\sum_{n=0}^{\infty} yJ_0(\lambda_ny)J_0(\lambda_n\rho)g_n\ ,$$

$$F(\rho) = f_1(\rho) - \sum_{n=0}^{\infty} g_n \int_1^{\alpha} y J_0(\lambda_n y) J_0(\lambda_n \rho) f_1(y) dy \, .$$

We then consider a particular case of a DSE (6) and (7), $\alpha_1 = \beta_2 = 0, \alpha_2 = \beta_1 = 1, f_2(r) = 0$ $f_1(r) = f_0 = const, \; f_1(r) = f_0 = const$, where the kernel and the free functions are integrable functions, then the DSE becomes

$$\sum_{n=1}^{\infty} C_n J_0(\lambda_n \rho) = f_0, \quad \rho \in \Omega \qquad (13)$$

$$\sum_{n=1}^{\infty} C_n \lambda_n J_0(\lambda_n \rho) = 0 \quad , \rho \in \overline{\Omega} \qquad (14)$$

If the inversion formula is applied for equation(13), then we will have an equation with the unknowns c_n and the function $h(y)$ as follows

$$C_n = \frac{2}{\alpha^2 J_1^2 \; (\lambda_n \alpha)} \int_0^1 y h(y) J_0(\lambda_n y)(\lambda_n y) \, dy \qquad (15)$$

$$h(y) = -y^{-1} \frac{d}{dy} \int_y^1 \frac{\phi(t) dt}{\sqrt{t^2 - y^2}} \qquad (16)$$

By Substituting equation (16) into equation(15), and by some simple calculations we will have an integral equation of the second kind

$$\psi(x) = \int_0^1 K(x,t) \psi(t) \, dt = \frac{2}{\pi} f_0, \quad 0 \le x < 1 \qquad (17)$$

$$K(x,t) = \frac{4}{\pi^2} \int_0^{\infty} \frac{K_0(\alpha y)}{I_0(\alpha y)} ch(ty) ch(xy) dy \qquad (18)$$

where $K_0(x), \; I_0(x)$ are known as modified Bessel's function, $ch(.)$ is a cosine hyperbolic function and $\psi(x) = x\phi(x)$, for more details see [15, 16].

4. NUMERICAL SOLUTION OF A FREDHOLM INTEGRAL EQUATION

In the previous section, the mixed boundary value problem is reduced into a Fredholm integral equation of the second kind. In what follows, we discuss the numerical solution for the integral equation(17), which gives rise to the solution for the Laplace equation. In fact, we will apply a quadrature method to derive a numerical solution for the integral equation. Because $f(x)$ is continuous on the interval $(0,1)$, and the kernel $k(x,t)$ is continuous on the square $0 < x,t < 1$, an appropriate numerical method could be used for approximating the solution. By substituting a node x_i into the integral equation to obtain n equations [2],

$$\psi(x_i) + \int_0^1 K(x_i,t)\psi(t)dt = \frac{2}{\pi} f_0, \quad 0 \le x < 1 . \tag{19}$$

After substituting the nodes and positive weights indicated for the chosen method, the value of each of these integrals can be expressed in the form

$$\int_0^1 K(x,t)\psi(t)dt = \sum_{j=1}^{n} w_j K(x_i,x_j)\psi(x_j) + E(x_j) \ . \tag{20}$$

Replacing the definite integrals with the finite sums in (20) to produce the n equations

$$\psi(x_i) + \sum_{j=1}^{n} w_j K(x_i,x_j)\psi(x_j) = \frac{2}{\pi} f_0 + E(x_j) \ . \tag{21}$$

After discarding the error term in (21), we obtain the system

$$\psi_i + \sum_{j=1}^{n} w_j K_{ij}\psi_j = \frac{2}{\pi} f_0 \tag{22}$$

of n equations with n unknowns ψ_i. By replacing the exact values $\psi(x_i)$ with the approximate values ψ_j, the linear system (22) becomes

$$\left(I - KW\right)\psi = f \ . \tag{23}$$

In this matrix equation, we set $K = K_{ij} = K(x_i,x_j)$, $\psi = (\psi_1,\psi_2,...,\psi_n)^T$, f is constant vector, $f = (\frac{2}{\pi} f_0, ..., \frac{2}{\pi} f_0)^T$. The matrix $W = W_{ij}$ is a diagonal matrix such that $W_{ii} = w_i$ and $W_{ij} = 0$ $i \ne j$. Assuming that the matrix $I - KW$ is invertible, the solution of (23) is of the form $\psi = \left(I - KW\right)^{-1} f$. The functions w_j are the weight functions that are evaluated by using Simpson's method [10]. For instance, if $n = 6$, then we have $w_1 = w_6 = 1/15$, $w_2 = w_4 = 4/15$, $w_3 = w_5 = 2/15$.

In the following tables, we obtain the values for the kernel K_{ij} at the grid points

$$\{(x_i,t_i)=x_i=0,\ 0.2,...,1,t_i=0,\ 0.2,...,1\}$$

with different values of α .

Table 1: The numerical values for approximation $K(x,t)\approx K_{ij}$ with $\alpha=1$

x_i \ t_i	0	0.2	0.4	0.6	0.8	1
0	0.554299	0.559599	0.576206	0.606488	0.655363	0.732559
0.2	0.559599	0.565252	0.583043	0.615785	0.669523	0.756917
0.4	0.576206	0.583043	0.604831	0.646079	0.717338	0.844305
0.6	0.606488	0.615785	0.646079	0.706385	0.82086	1.06509
0.8	0.655363	0.669523	0.717338	0.82086	1.05414	1.80111
1	0.732559	0.756917	0.844305	1.06509	1.80111	

Table 2: The numerical values for approximation $K(x,t)\approx K_{ij}$ with $\alpha=10$

x_i \ t_i	0	0.2	0.4	0.6	0.8	1
0	0.0554299	0.0554351	0.0554508	0.0554771	0.0555139	0.0555613
0.2	0.0554351	0.0554403	0.0554561	0.0554824	0.0555192	0.0555666
0.4	0.0554508	0.0554561	0.0554719	0.0554982	0.0555351	0.0555826
0.6	0.0554771	0.0554824	0.0554982	0.0555246	0.0555616	0.0556093
0.8	0.0555139	0.0555192	0.0555351	0.0555616	0.0555988	0.0556467
1	0.0555613	0.0555666	0.0555826	0.0556093	0.0556467	0.0556949

Table 3: The numerical values for approximation $K(x,t) \approx K_{ij}$ with $\alpha = 100$

x_i \ t_i	0	0.2	0.4	0.6	0.8	1
0	0.00554299	0.00554299	0.00554301	0.00554303	0.00554307	.00554312
0.2	0.00554299	0.005543	0.00554301	0.00554304	0.00554307	0.00554312
0.4	0.00554301	0.00554301	0.00554303	0.00554305	0.00554309	.00554314
0.6	0.00554303	0.00554304	0.00554305	0.00554308	0.00554312	.00554316
0.8	0.00554307	0.00554307	0.00554309	0.00554312	0.00554315	0.0055432
1	0.00554312	0.00554312	0.00554314	0.00554316	0.0055432	0.00554325

By using the values of the above tables, the system of equations (22) could be solved with different values of α. Indeed, when the values of α approaches ∞ the values for the $K(x,t)$ are vanished, hence $\psi(x) = \dfrac{2}{\pi} f_0$, this means that the DSE (13) and (14) give rise to dual integral equations of the form

$$\int_0^\infty C(\lambda) J_0(\lambda\rho) d\lambda = f_0, \quad \rho \in \Omega, \quad \int_0^\infty \lambda C(\lambda) J_0(\lambda\rho) d\lambda = 0, \quad \rho \in \overline{\Omega}$$

Such that $C(\lambda) = \int_0^1 \phi(t) \cos \lambda t dt$, and we find that $\phi(t) = \dfrac{2}{\pi} f_0$.

The solution of the above dual equations exists explicitly [16, 18]. Numerically, in the integral equation(17), $K(x,t)$ is integrable in the square $\{(x,t), 0 < x, t < 1\}$

$$\int_0^1\int_0^1 K^2(x,t) dxdt = \int_0^1\int_0^1 \left(\frac{4}{\pi^2} \int_0^\infty \frac{K_0(\alpha y)}{I_0(\alpha y)} ch(ty) ch(xy) dy \right)^2 dxdt$$

$$\leq \frac{4}{\pi^2} \int_0^1\int_0^1\int_0^\infty \left(\frac{K_0(\alpha y)}{I_0(\alpha y)} ch(ty) ch(xy) \right)^2 dydxdt$$

The following tables show the values for the integral $\int_0^1\int_0^1 K^2(x,t) dxdt$ with different values of α

Table 4: **The numerical values for** $\int_0^1\int_0^1 K^2(x,t)dxdt$ **with** $\alpha = 1$

x_i \ t_i	0	0.2	0.4	0.6	0.8	1
0	0.386857	0.388115	0.392054	0.39924	0.410856	0.429286
0.2	0.388115	0.389414	0.393495	0.400976	0.413188	0.432901
0.4	0.392054	0.393495	0.398054	0.406569	0.420955	0.445734
0.6	0.39924	0.400976	0.406569	0.417441	0.43733	0.477695
0.8	0.410856	0.413188	0.420955	0.43733	0.472988	0.584252
1	0.429286	0.432901	0.445734	0.477695	0.584252	

Table 5: The numerical values for $\int_0^1\int_0^1 K^2(x,t)dxdt$ **with** $\alpha = 10$

x_i \ t_i	0	0.2	0.4	0.6	0.8	1
0	0.0386857	0.038687	0.0386907	0.0386969	0.0387057	0.0387169
0.2	0.038687	0.0386882	0.0386919	0.0386982	0.0387069	0.0387182
0.4	0.0386907	0.0386919	0.0386957	0.0387019	0.0387107	0.0387219
0.6	0.0386969	0.0386982	0.0387019	0.0387082	0.0387169	0.0387282
0.8	0.0387057	0.0387069	0.0387107	0.0387169	0.0387257	0.038737
1	0.0387169	0.0387182	0.0387219	0.0387282	0.038737	0.0387483

Table 6: The numerical values for $\int_0^1\int_0^1 K^2(x,t)dxdt$ **with** $\alpha = 100$

x_i \ t_i	0	0.2	0.4	0.6	0.8	1
0	0.00386857	0.00386857	0.00386858	0.00386858	0.00386859	0.0038686
0.2	0.00386857	0.00386857	0.00386858	0.00386858	0.00386859	0.0038686
0.4	0.00386858	0.00386858	0.00386858	0.00386859	0.0038686	0.00386861
0.6	0.00386858	0.00386858	0.00386859	0.00386859	0.0038686	0.00386861
0.8	0.00386859	0.00386859	0.0038686	0.0038686	0.00386861	0.00386862
1	0.0038686	0.0038686	0.00386861	0.00386861	0.00386862	0.00386863

The graphs of the kernel function $K(x,t)$ with different values of α are presented in the figures 3 and 4 respectively.

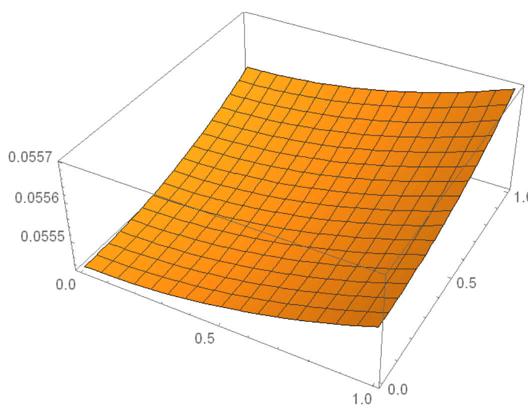

FIGURE 2: The graph of the function $K(x,t)$ with $\alpha = 1$

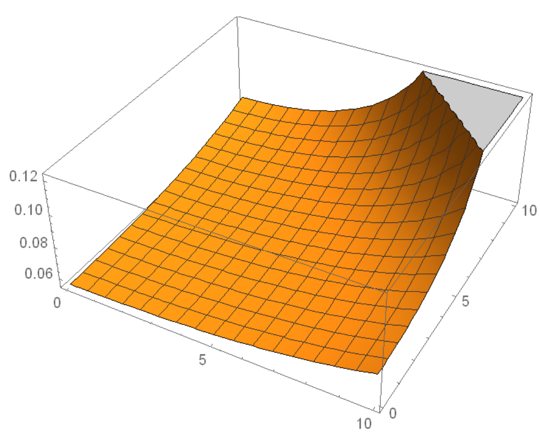

FIGURE 3: The graph of the function $K(x,t)$ with $\alpha = 100$

Finally, we consider a particular case of the DSE (6) and (7), $\alpha_1 = \beta_2 = 1, \alpha_2 = \beta_1 =, f_2(r) = 0$ $f_1(r) = q_0 = const$. So, equations (6) and (7) become as

$$\sum_{n=1}^{\infty} \lambda_n C_n J_0(\lambda_n \rho) = q_0, \quad \rho \in \Omega \qquad (24)$$

$$\sum_{n=1}^{\infty} C_n J_0(\lambda_n \rho) = 0 \quad, \rho \in \overline{\Omega} \qquad (25)$$

In [1], the last DSE (24) and (25) were reduced to a Fredholm integral equation of the form

$$\phi(x) + \int_0^1 M(x,t)\phi(t)dt = xq_0, \quad 0 \le x < 1 \qquad (26)$$

$$M(x,t) = \frac{4}{\pi^2}\int_0^{\infty} \frac{K_1(\alpha y)}{I_1(\alpha y)} sh(ty)sh(xy)dy \qquad (27)$$

The integral equation (26) with kernel (27) can be solved in a similar manner by quadrature method which is mentioned above. The numerical values of the kernel (27) are shown in tables 7 and 8 with different values of α .

TABLE 7: The numerical values for $M(x,t) = \frac{4}{\pi^2}\int_0^{\infty} \frac{K_1(\alpha y)}{I_1(\alpha y)} sh(ty)sh(xy)dy$ **with** $\alpha = 1$

x_i \ t_i	0	0.2	0.4	0.6	0.8	1
0	0	0	0	0	0	0
0.2	0	0.0414252	0.085575	0.135975	0.198303	0.28356
0.4	0	0.085575	0.177401	0.283878	0.419535	0.615947
0.6	0	0.135975	0.283878	0.46096	0.701522	1.09986
0.8	0	0.198303	0.419535	0.701522	1.14129	2.18322
1	0	0.28356	0.615947	0.09986	2.18322	

TABLE 8: The numerical values for $\frac{4}{\pi^2}\int_0^{\infty}\frac{K_1(\alpha y)}{I_1(\alpha y)}sh(ty)sh(xy)dy$ **with** $\alpha = 10$

x_i \ t_i	0	0.2	0.4	0.6	0.8	1
0	0	0	0	0	0	0
0.2	0.	0.0000405901	0.0000812046	0.000121868	0.000162605	0.000203441
0.4	0.	0.0000812046	0.000162458	0.00024381	0.00032531	0.000407006
0.6	0.	0.000121868	, 0.0003659	0.000610819	0.00024381	0.000488211
0.8	0.	0.000162605	0.00032531	0.000488211	0.000651409	0.000815003
1	0.	0. 00203441	0.000407006	0.000610819	0.000815003	0.00101969

5. CONCLUSION

In this work, we have examined a DSE method to solve a mixed boundary value problem, whose solution is solved by converting the problem to a Fredholm integral equation of the second kind. Integral equations were then reduced to a system of algebraic equations, and this system can be easily solved by a numerical quadrature method. It would be interesting to further study nonlinear or nonhomogeneous equations and other applications in various areas.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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