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## GRAPHS WITH LARGE DIAMETER AND THEIR TWO DISTANCE FORCING NUMBER

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**Abstract.** For a given simple graph  $G = (V, E)$ , the two distance forcing number  $Z_{2d}(G)$  is defined as the minimum cardinality among all  $Z_{2d}$ -sets in  $G$ . This paper examine about  $Z_{2d}(G)$  of some graphs with large diameter. Also we determine the 2-distance forcing number of some complement graphs.

**Keywords:** diameter; 2-distance forcing set; 2-distance forcing number.

**2010 AMS Subject Classification:** 05C12.

### 1. INTRODUCTION

In this article, we consider only connected, simple and finite graphs. The zero forcing number  $Z(G)$  of a graph  $G$  is the minimum cardinality of a zero forcing set in  $G$ . This parameter was defined and studied in detail [1]. In this paper, we discuss a generalization of zero forcing set based on the distance in graphs. More definitions based on 2-distance sets can be found in [8]. The 2-distance forcing number is defined in [8] as follows: In the initial step, each vertex in a graph  $G$  is given a black or white color. Once each vertex is given the black color, the process

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will start in which more white vertices can be coloured black. This process consists of some rules to be followed:

- Color change rule [8]: If a black colored vertex has at most one 2-distance white colored neighbor (If a vertex  $v$  of  $G$  lies at a distance at most two from the vertex  $u$  of  $G$ , then we say that  $v$  is a 2-distance neighbor of  $u$ ), then change the color of that white vertex to black. When the color change rule is applied to an arbitrary vertex  $v$  to alter the color of the vertex  $u$  to black, then we say that the vertex  $v$  forces the vertex  $u$  to black and we denote it as  $v \rightarrow u$  to black.
- An initially colored black vertex set  $Z_{2d}$  for which the entire vertices of the graph  $G$  can be colored black by applying the above color change rule is known as 2-distance forcing set of  $G$ . The minimum cardinality among all  $Z_{2d}$  sets in  $G$  is called the 2-distance forcing number of the graph  $G$  and is denoted by  $Z_{2d}(G)$  [8].
- Let  $u$  be an arbitrary vertex in  $G$ . The 2-distance degree of  $u$  is defined as the number of vertices which are at a distance at most two from  $u$  including the vertex  $u$ . The 2-distance degree of the vertex  $u$  is denoted by  $\text{deg}_{2d}(u)$ . For example, consider the graph depicted in Figure 1. In Figure 1,  $\text{deg}_{2d}(u_1) = 6, \text{deg}_{2d}(u_2) = 8, \text{deg}_{2d}(u_3) = 9, \text{etc.}$
- Consider the 2-distance degree of all vertices in a graph  $G$ . The minimum among them is called minimum 2-distance degree of the graph  $G$  and it is denoted by  $\delta_{2d}(G)$ . Similarly, the maximum among them is called the maximum 2-distance degree of the graph  $G$ . The maximum 2-distance degree of  $G$  is denoted by  $\Delta_{2d}(G)$ .

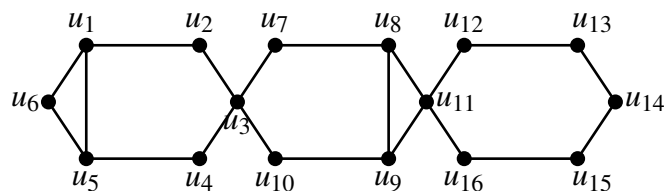


FIGURE 1. The graph  $G$

For more definitions on graphs, we refer to [5]. We start with the following preliminary result.

**Theorem 1.** *Let  $G$  be a connected graph of order  $n \geq 3$ . Then,  $Z_{2d}(G) \geq \delta_{2d}(G) - 1$  and this bound is sharp. Moreover, for any path  $P_n$ ,  $Z_{2d}(P_n) = \delta_{2d}(P_n) - 1$ .*

*Proof.* The proof is obvious. □

For a connected graph  $G$  of order  $n \geq 3$ , any  $Z_{2d}$  set forms a zero forcing set  $Z$  of  $G$ . Therefore, we have the following.

**Theorem 2.** *Let  $G$  be a connected graph of order  $n \geq 3$ . Then,  $Z(G) \leq Z_{2d}(G)$ .*

*Proof.* The proof is obvious. □

## 2. CERTAIN GRAPH CLASSES AND THEIR $Z_{2d}(G)$

First we consider the shadow graph with large diameter and find its 2-distance forcing number.

**Definition** [10]. The shadow graph  $D_2(G)$  of a connected graph  $G$  is constructed by taking two copies of  $G$ , say  $G_1$  and  $G_2$ , and join each vertex  $u_1$  in  $G_1$  to the neighbors of the corresponding vertex  $u_2$  in  $G_2$ .

**Theorem 3.** *The 2-distance forcing number of the shadow graph  $D_2(P_n)$  of the path  $P_n$ , where  $n \geq 3$  vertices, is  $n + 2$ .*

*Proof.* Let  $G_1$  and  $G_2$  be two copies of the path  $P_n$ . Denote the vertices of  $G_1$  by  $u_1, u_2, \dots, u_n$  and that of  $G_2$  by  $v_1, v_2, \dots, v_n$ . Consider the set  $Z_{2d} = \{u_1, u_2, \dots, u_n, v_1, v_2\}$  of black vertices. We shall show that this set  $Z_{2d}$  generates a 2-distance forcing set of  $D_2(P_n)$ . For, consider the black vertex  $u_1$ . Note that  $v_3$  is the only 2-distance white vertex of  $u_1$ . So the vertex  $u_1 \rightarrow v_3$  to black. Again, the black vertex  $u_2 \rightarrow v_4$  to black. Also,  $u_3 \rightarrow v_5$  to black,  $u_4 \rightarrow v_6$  to black,  $\dots, u_{n-2} \rightarrow v_n$  to black. Hence the set  $Z_{2d}$  generates a 2-distance forcing set of  $D_2(P_n)$ . The cardinality of the set  $Z_{2d}$  is  $n + 2$ . So,

$$(1) \quad Z_{2d}[D_2(P_n)] \leq n + 2.$$

To prove the reverse inequality, we assume that there exists a 2-distance forcing set consisting  $(n + 1)$  black vertices and we arrive at a contradiction. We consider the following cases.

**Case 1.** Consider the set  $Z_{2d} = \{u_1, u_2, \dots, u_n, v_1\}$  of black vertices. Then the further forcing is not possible because each black vertex has at least two 2-distance white neighbors. Therefore, the set  $Z_{2d}$  will never form a 2-distance forcing set for  $D_2(P_n)$ , a contradiction.

**Case 2.** Let  $Z_{2d} = \{v_1, v_2, \dots, v_n, u_1\}$  be a set of  $n + 1$  black vertices. By applying the same argument as in case 1, we can easily observe that the set  $Z_{2d}$  cannot give a derived coloring of  $D_2(P_n)$ , a contradiction.

**Case 3.** Consider the set  $Z_{2d}$  of  $n$  black vertices as the vertices of a path of length  $n - 1$ . Then each black vertex will have at least three white neighbors in its 2-distance neighborhood. If we add one more black vertex to the set  $Z_{2d}$  to form a 2-distance forcing set, then all black vertices will have at least two 2-distance white neighbors. Therefore, we cannot start the color changing rule from any of these black vertices. Hence

$$(2) \quad Z_{2d}(G) \geq n + 2.$$

From (1) and (2), the proof follows. □

**Definition** [11]. The middle graph of a graph  $G$ , denoted by  $M(G)$ , is the graph with vertex set  $V(G) \cup E(G)$  and in which two vertices are adjacent in  $M(G)$  if and only if, either they are adjacent edges in  $G$  or one is a vertex of  $G$  and the other is an edge incident with it.

**Theorem 4.** *Let  $G$  be the middle graph of the path  $P_n$ ,  $n \geq 4$ . Then,  $Z_{2d}(G) = n$ , where  $n$  is the number of vertices of the path  $P_n$ .*

*Proof.* Let  $A = \{v_1, v_2, \dots, v_n, v_1^1, v_2^1, v_{n-1}^1\}$  be the vertex set of  $G$ , where  $v_1, v_2, \dots, v_n$  are the vertices of  $P_n$  and  $v_1^1, v_2^1, v_{n-1}^1$  are the vertices corresponds to the edges  $e_1, e_2, \dots, e_{n-1}$  of the graph  $P_n$  in  $G$ . Our aim is to generate a 2-distance forcing set of  $G$  consisting of  $n$  black

vertices. For, color the vertex  $v_1$  to black. Then the 2-distance white neighbors of  $v_1$  are  $v_1^1$ ,  $v_2^1$  and  $v_2$ . To begin the color change rule, assign black color to at least two of these vertices. Let  $v_1^1, v_2$  to be black. Then  $v_1 \rightarrow v_2^1$  to black. Now consider the black vertex  $v_2$ . The white colored 2-distance vertices of  $v_2$  are  $v_3$  and  $v_3^1$ . Color  $v_3$  to black. Then  $v_2 \rightarrow v_3^1$  to black. Again, consider the black vertex  $v_3$ . Color  $v_4$  to black. Then  $v_3 \rightarrow v_4^1$  to black and so on. Apply this process iteratively, consider the black vertex  $v_{n-2}$ . Clearly the vertex  $v_{n-2}^1$  is black at this stage. The 2-distance white neighbors of  $v_{n-2}$  are  $v_{n-1}$  and  $v_{n-1}^1$ . Color  $v_{n-1}$  to black. Then the vertex  $v_{n-2} \rightarrow v_{n-1}^1$  to black. Then  $v_{n-1}^1 \rightarrow v_n$  to black. Thus we obtain a derived coloring of  $G$  using the set  $Z_{2d} = \{v_1, v_2, \dots, v_{n-1}, v_1^1\}$  of black vertices. The cardinality of the set  $Z_{2d}$  is  $n$ . Hence

$$(3) \quad Z_{2d}(G) \leq n.$$

To prove the result, it suffices to show that  $Z_{2d}(G) \geq n$ . For this, we claim that a set having  $(n-1)$  black vertices will never form a 2-distance forcing set of  $G$ . Consider the following cases.

**Case 1.** Let  $Z_{2d} = \{v_1, v_2, \dots, v_{n-1}\}$  be a set of black vertices. Then the color change rule is not possible, since each black vertex  $v_i$  ( $i=1, 2, \dots, n-1$ ) contains at least two 2-distance white neighbors.

**Case 2.** Suppose  $Z_{2d} = \{v_1^1, v_2^1, \dots, v_{n-1}^1\}$  be a black vertex set. In this case, each black vertex has at least three 2-distance white neighbors. Therefore, further forcing is not possible.

**Case 3.** Consider a set consisting of  $(n-1)$  black vertices of a path  $P_{n-1}$ . Then  $P_{n-1}$  will be of the following types.

**Type 1.** Choose the path  $P_{n-1}$  as  $v_1^1 v_2^1 \dots v_{n-1}^1$ . In this case, it is obvious that derived coloring of the graph  $G$  is not possible since each black vertex has at least three 2-distance white neighbors.

**Type 2.** Assume that  $n$  is odd. Select a path  $P_{n-1}$  of the type

$v_1 v_1^1 v_2 v_2^1, \dots, v_{\lfloor \frac{n}{2} \rfloor - 1} v_{\lfloor \frac{n}{2} \rfloor - 1}^1 v_{\lfloor \frac{n}{2} \rfloor} v_{\lfloor \frac{n}{2} \rfloor}^1$ . Note that we can consider only the last three vertices of this path for further forcing. But forcing from these vertices is not possible since each vertex has at least two 2-distance white neighbors.

Again, choose a path  $v_1^1 v_2 v_2^1 v_3, \dots, v_{\lfloor \frac{n}{2} \rfloor} v_{\lfloor \frac{n}{2} \rfloor}^1 v_{\lfloor \frac{n}{2} \rfloor + 1}$ . In this case, it is obvious that we cannot force all the remaining vertices of  $G$  to black.

**Type 3.** Suppose that  $n$  is even. Consider paths like  $v_1 v_1^1 v_2 v_2^1, \dots, v_{\lfloor \frac{n}{2} \rfloor - 1} v_{\lfloor \frac{n}{2} \rfloor - 1}^1 v_{\lfloor \frac{n}{2} \rfloor}$  and  $v_1^1 v_2 v_2^1 v_3, \dots, v_{\lfloor \frac{n}{2} \rfloor - 1} v_{\lfloor \frac{n}{2} \rfloor} v_{\lfloor \frac{n}{2} \rfloor}^1$ . It can be noted that neither of these paths generates a 2-distance forcing set of  $G$ .

From the above mentioned cases, we can observe that a set consisting of  $(n - 1)$  black vertices of a path  $P_{n-1}$  will never generate a 2-distance forcing set of the graph  $G$ . It will not happen in such other cases also. So from the above observations, we can conclude that

$$(4) \quad Z_{2d}(G) \geq n.$$

Therefore from (3) and (4),  $Z_{2d}(G) = n$ . □

**Definition [7].** The  $S^{\text{th}}$  Necklace graph  $N_s$  is defined as a 3-regular graph that can be constructed from a  $3s$ -cycle by appending  $s$  central vertices. Each extra vertex is adjacent to 3-sequential cycle vertices. The order of a  $S^{\text{th}}$  Necklace graph  $N_s$  is  $4s$  and the diameter is  $\lfloor \frac{3s}{2} \rfloor$ .

**Theorem 5.** *If  $G$  is the  $S^{\text{th}}$  Necklace graph  $N_s$ , then  $Z_{2d}(G) = s + 4$ .*

*Proof.* We generate a 2-distance forcing set of  $G$  as follows. First we color all  $s$  vertices of  $G$  as black. Each  $s$  vertex will have five 2-distance vertices. Without loss of generality, start the color change rule from any one of the  $s$  vertices. To begin the color change rule from any one of the  $s$  vertices, we have to color at least four 2-distance vertices of that  $s$  vertex to black. Then clearly these  $s + 4$  black vertices generate a 2-distance forcing set of  $G$  and we can easily observe that with  $s + 3$  black vertices, we cannot form a 2-distance forcing set of  $G$ . Hence,  $Z_{2d}(G) = s + 4$ .

□

**Definition** [13]. The triangular snake graph  $T_n$  can be viewed as the graph formed by replacing each edge of the path  $P_n$  by a triangle  $C_3$ , thus adding  $(n - 1)$  vertices and  $2(n - 1)$  edges.

**Theorem 6.** *Let  $G$  be the triangular snake graph with at least 2 triangles. Then  $Z_{2d}(G) = k + 2$ , where  $k$  is the number of triangles in  $G$  and  $P_n$ ,  $n \geq 3$ , is the path used to replace the edges.*

*Proof.* Let  $u_1, u_2, \dots, u_n$  be the vertices of the path  $P_n$  in  $G$ . Represent the  $(n - 1)$  additional vertices in  $G$  by  $v_1, v_2, \dots, v_{n-1}$ . We prove the result by induction on the number of triangles in  $G$ .

Assume that  $k = 2$ . Then clearly  $n = 3$  in  $G$ . Represent the vertices in  $G$  by  $u_1, u_2, u_3, v_1, v_2$ . Let  $Z_{2d} = \{u_1, u_2, u_3, v_1\}$  be a set of black vertices. Then the vertex  $u_1 \rightarrow v_2$  to black. So the set  $Z_{2d}$  forms a 2-distance forcing set of  $G$ . Here  $|Z_{2d}| = 4$ . Therefore,  $Z_{2d}(G) = 4 = k + 2$ .

Again, assume that  $k = 3$ . In this case  $n = 4$  in  $G$ . Let  $u_1, u_2, u_3, u_4, v_1, v_2, v_3$  be the vertices of  $G$ . Suppose that  $Z_{2d} = \{u_1, u_2, u_3, u_4, v_1\}$  be a set of black vertices. Clearly the vertex  $u_1 \rightarrow v_2$  to black. Then the vertex  $u_2 \rightarrow v_3$  to black. Thus the set  $Z_{2d}$  generates a 2-distance forcing set of  $G$ . Here the cardinality of the set  $Z_{2d}$  is 5. Hence  $Z_{2d}(G) = 5 = k + 2$ .

Let us assume that the result is true for the graph  $G$  with  $k - 1$  triangles, where  $k \geq 5$ . Let  $A = \{u_n, v_{n-1}\}$ . The induced subgraph  $\langle G[V - A] \rangle$  is a triangular snake graph with  $k - 1 < k$  triangles. Assume that the result is true for  $\langle G[V - A] \rangle$ , that is,  $Z_{2d}(\langle G[V - A] \rangle) = k - 1 + 2 = k + 1$ .

Let  $W$  be a minimum 2-distance forcing set of  $\langle G[V - A] \rangle$  with  $|W| = k + 1$ . Color the vertex  $u_n$  in  $A$  as black. Then it is easy to observe that the vertex  $v_{n-1}$  will be colored to black. Therefore by induction hypothesis,  $Z_{2d}(G) = Z_{2d}(\langle G[V - A] \rangle) + |\{u_n\}| = k + 1 + 1 = k + 2$ .

□

**Definition** [14]. The  $n$ -Sunlet graph  $S_n$  is the graph on  $2n$  vertices got by attaching  $n$  pendant edges to a cycle graph  $C_n$ . Sunlet graphs are also called Crown graphs. The diameter of a Sunlet graph is  $\lfloor \frac{n}{2} \rfloor + 2$ .

**Theorem 7.** *Let  $G$  denotes the Sunlet Graph  $S_n$ . Then*

$$Z_{2d}(G) = \begin{cases} 4 & \text{if } n = 3 \\ n & \text{if } n \geq 4 \end{cases}$$

where  $n$  is the number of vertices of the cycle  $C_n$ .

*Proof. Case 1.* Assume that  $n = 3$ . Let  $u_1, u_2, u_3, v_1, v_2, v_3$  be the vertices of the Sunlet graph  $G$ , where  $v_1, v_2, v_3$  are the vertices joined to  $u_1, u_2, u_3$  of the cycle  $C_3$  respectively. Also let  $Z_{2d} = \{u_1, u_2, v_1, v_2\}$  be a set of black vertices. Then clearly the black vertex  $v_1 \rightarrow u_3$  to black. Now the vertex  $u_2 \rightarrow v_3$  to black. Thus the set  $Z_{2d}$  forms a 2-distance forcing set of  $G$ . Cardinality of the set  $Z_{2d}$  is 4. Also we can easily observe that with three black vertices the derived coloring is not possible. Hence  $Z_{2d}(G) = 4$ .

**Case 2.** Assume that  $n \geq 4$ . Let  $A = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$  be the vertex set of the Sunlet graph  $G$ , where  $u_1, u_2, \dots, u_n$  are the vertices of the cycle  $C_n$  and  $v_1, v_2, \dots, v_n$  are the vertices joined to  $u_1, u_2, \dots, u_n$  respectively. We claim that the set  $Z_{2d} = \{u_1, u_n, v_1, v_2, \dots, v_{n-2}\}$  of black vertices generates a 2-distance forcing set of  $G$ . For, the pendant vertex  $v_1$  has three 2-distance vertices  $u_1, u_n$  and  $u_2$  of which  $u_1$  and  $u_n$  are black. So the black vertex  $v_1 \rightarrow u_2$  to black. Consider the black vertex  $v_2$ . The 2-distance vertices of  $v_2$  are  $u_1, u_2$  and  $u_3$ . Hence  $v_2 \rightarrow u_3$  to black, since  $u_1$  and  $u_2$  are black. Again consider the black vertex  $v_3$ . Then  $v_3 \rightarrow u_4$  to black, since  $u_4$  is the only white colored 2-distance vertex of  $v_3$ . Proceeding like this, consider the black vertex  $v_{n-2}$ . Clearly the vertex  $v_{n-2} \rightarrow u_{n-1}$  to black. Also  $u_{n-2} \rightarrow v_{n-1}$  to black,  $u_n \rightarrow v_n$  to black. Hence we obtain a derived coloring of  $G$  with the set  $Z_{2d}$ . Cardinality of the set  $Z_{2d}$  is  $n$ .

Therefore,

$$(5) \quad Z_{2d}(G) \leq n.$$



In order to prove the reverse part, we assert that any set  $Z_{2d}$  containing  $(n-1)$  black vertices will not form a 2-distance forcing set of the graph  $G$ . For this, we consider the following cases.

**Case 1.** Let  $Z_{2d} = \{v_1, v_2, \dots, v_{n-1}\}$  be a set of  $(n-1)$  black vertices. Then further forcing is not possible, since each black vertex will have three 2-distance white neighbors.

**Case 2.** Let  $Z_{2d}$  be the set of vertices of a path  $P_{n-1}$  on  $(n-1)$  black vertices. Then the path  $P_{n-1}$  will be of the following types.

Assume that  $n$  is even.

**Type 1.** Consider the path  $u_1u_2 \dots u_{n-1}$ . Then further forcing is not possible since each black vertex has at least three 2-distance white neighbors.

**Type 2.** Select the path  $u_1u_2 \dots u_{n-3}u_{n-2}v_{n-2}$ . Then only one more forcing is possible from the vertex  $v_{n-2}$  because all other black vertices have at least two 2-distance white neighbors. So in this case, derived coloring for  $G$  is not possible.

**Type 3.** Choose a path  $v_1u_1u_2 \dots u_{n-4}u_{n-3}u_{n-2}$ . Clearly only one forcing from the vertex  $v_1$  is possible because all other black vertices have at least two white neighbors in their 2-distance neighborhood. Hence we cannot generate a 2-distance forcing set of  $G$ .

We can also observe that any set of  $(n-1)$  black vertices constructed in any other way than we mentioned above will never form a 2-distance forcing set of  $G$ . We also note that the cases are the same when  $n$  is odd.

There fore, we can conclude that a set of  $(n-1)$  black vertices cannot form a 2-distance forcing set of  $G$ . Hence

$$(6) \quad Z_{2d}(G) \geq n.$$

So from (5) and (6), the result follows. □

**Theorem 8.** *Let  $G$  be a connected graph with maximum degree  $\Delta(G) = 2$ . Then  $2 \leq Z_{2d}(G) \leq 4$ .*

*Proof.* Suppose that the maximum degree of  $G$  is 2. Then  $G$  is either a cycle  $C_n (n \geq 3)$  or a path  $P_n, (n \geq 3)$ . Now we have from [8] that if  $G$  is a cycle  $C_n$ , then  $Z_{2d}(G) = 4$  and if  $G$  is a path  $P_n$ , then  $Z_{2d}(G) = 2$ . Hence in this case,  $2 \leq Z_{2d}(G) \leq 4$ . □

Let  $F_p^k$  be the graph obtained by taking  $k$ -copies of the cycle  $C_p$  and joining the  $k$  copies of the cycle  $C_p$  with a common vertex  $v_p$  of each cycle  $C_p$ .

**Theorem 9.** *Let  $G$  be the graph  $F_p^k, p \geq 5$  and  $k \geq 2$ . Then,  $Z_{2d}(G) = 3k$ .*

*Proof.* Represent the  $k$  copies  $C_1, C_2, \dots, C_k$  of the cycle  $C_p$  in  $F_p^k$  as follows.

$$C_1 = v_1^1, v_2^1, \dots, v_p, v_1^1$$

$$C_2 = v_1^2, v_2^2, \dots, v_p, v_1^2$$

...

...

...

$$C_k = v_1^k, v_2^k, \dots, v_p, v_1^k$$

We have the 2-distance forcing number of a cycle  $C_n$  on  $n \geq 5$  vertices is 4 [8]. Consider the cycle  $C_1$  in  $G$ . Let  $A = \{v_p, v_1^1, v_2^1, v_3^1\}$  be a set of four adjacent black vertices of the cycle  $C_1$  in  $G = F_p^k$ . Then clearly the set  $A$  generates a 2-distance forcing set of the cycle  $C_1$  in  $F_p^k$ . That is,  $Z_{2d}(C_1) = 4$ . We observe that the vertex  $v_p$  is also a black vertex of the cycle  $C_2$ . Consider the set  $B = \{v_p, v_1^2, v_2^2, v_3^2\}$  of black vertices of the cycle  $C_2$ . Then the set  $B$  forms a 2-distance forcing set for the cycle  $C_2$  in  $F_p^k$ . Again, suppose  $C = \{v_p, v_1^3, v_2^3, v_3^3\}$  be a set of black vertices of the cycle  $C_3$ . Clearly the set  $C$  generates a 2-distance forcing set for the cycle  $C_3$  in  $F_p^k$ . The case is similar for other cycles  $C_4, C_5 \dots, C_{k-2}$ . Hence to get the derived coloring of the cycles

$C_2, C_3, \dots, C_{k-1}$ , we need in total  $3(k-2)$  black vertices. Now consider the cycle  $C_k$ . Also consider the vertex  $v_1^k$  in  $C_k$  which is adjacent to the vertex  $v_p$ . Color the vertex  $v_1^k$  as black. Then the vertex  $v_1^1$  in  $C_1$  forces the vertex  $v_{p-1}^k$ , which is adjacent to the vertex  $v_p$ , in  $C_k$  to black. Hence we need one more black vertex  $v_2^k$  in  $C_k$  to get the derived coloring of the cycle  $C_k$ . So with  $4 + 3(k-2) + 2 = 3k$  black vertices, we can force all the vertices of the graph  $G = F_p^k$  to black. Therefore,

$$(7) \quad Z_{2d}(G) \leq 3k.$$

To prove the converse part, we claim that with  $(3k-1)$  number of black vertices, we cannot obtain a derived coloring of  $G$ . For, assume that we have a 2-distance forcing set consisting of  $(3k-1)$  black vertices of  $G$ . Since the the vertex  $v_p$  will be colored black after getting the derived coloring of the cycle  $C_1$  using four adjacent black vertices and  $v_p$  is common for all cycles, we need at least 3 adjacent black vertices ( Note that these vertices together with the vertex  $v_p$  must form an induced subgraph of  $G$ ) for each of the cycles  $C_2, C_3, \dots, C_{k-1}$  to get the derived coloring of them. So in total, we need at least  $3(k-2) = 3k-6$  black vertices to color the remaining white vertices of the cycles  $C_2, C_3, \dots, C_{k-1}$ . Now, we have only  $[(3k-1) - (4 + 3k-6)] = 1$  black vertex. With this single black vertex, we cannot form a derived coloring of the cycle  $C_k$ , a contradiction to our assumption. Therefore,

$$(8) \quad Z_{2d}(G) \geq 3k.$$

Hence,  $Z_{2d}(G) = 3k$ . □

### 3. 2-DISTANCE FORCING NUMBER OF ROOTED PRODUCT OF GRAPHS

**Definition** [4]. Let  $G$  be a simple graph with the vertices labelled by  $v_1, v_2, \dots, v_n$  and let  $\mathbb{H}$  be a collection of  $n$  rooted simple graphs  $H_1, H_2, \dots, H_n$ . Then the rooted product,  $G \odot H$ , is the graph generated by identifying the root of  $H_j$  with the  $j^{th}$  vertex of  $G$ ,  $1 \leq j \leq n$ .

In this section, we deal with the 2-distance forcing number of rooted product of some graphs.

**Theorem 10.** *Let  $G$  be the rooted product  $P_n \odot P_m$  of a path  $P_n$  and the rooted path  $P_m$  rooted with the pendant vertex of  $P_m$ ,  $n \geq 2$ ,  $m \geq 4$ . Then,  $Z_{2d}(G) \leq 2(n - 1)$ , where  $n$  is the number of vertices of the path  $P_n$ .*

*Proof.* Assume that  $u_1, u_2, \dots, u_n$  be the vertices of the path  $P_n$  and  $P_m^1, P_m^2, \dots, P_m^n$  be the  $n$  copies of the path  $P_m$  rooted at the vertices  $u_1, u_2, \dots, u_n$  of the path  $P_n$  respectively. Denote the paths  $P_m^1, P_m^2, \dots, P_m^n$  in  $G$  as follows.

$$P_m^1 = p_1^1, p_2^1, \dots, p_m^1$$

$$P_m^2 = p_1^2, p_2^2, \dots, p_m^2$$

...

...

...

$$P_m^n = p_1^n, p_2^n, \dots, p_m^n$$

Let us root the vertex  $p_1^1$  of the path  $P_m^1$  at  $u_1$ , the vertex  $p_1^2$  of the path  $P_m^2$  at  $u_2$ , ..., the vertex  $p_1^n$  of the path  $P_m^n$  at  $u_n$ . That is,  $p_1^1 = u_1, p_1^2 = u_2, \dots, p_1^n = u_n$ . We generate a 2-distance forcing set of the graph  $G$  as follows.

Let  $A_1 = \{p_{m-1}^1, p_m^1\}$ ,  $A_2 = \{p_{m-1}^2, p_m^2\}$ , ...,  $A_{n-1} = \{p_{m-1}^{n-1}, p_m^{n-1}\}$ . Suppose that  $Z_{2d} = A_1 \cup A_2 \cup \dots \cup A_{n-1}$ . Assign black color to all vertices of the set  $Z_{2d}$ . Since  $Z_{2d}(P_t) = 2$  for a path  $P_t$ ,  $t \geq 3$  (See [8]), clearly the set  $A_1$  generates a 2-distance forcing set of the path  $P_m^1$ . In a similar manner, the set  $A_2$  forms a 2-distance forcing set of the path  $P_m^2$ . Proceeding like this, the set  $A_{n-1}$  forms a 2-distance forcing set of the path  $P_m^{n-1}$ . Now consider the black vertex  $p_2^{n-1}$  of the path  $P_m^{n-1}$ . We can easily see that the vertex  $p_2^{n-1}$  forces the vertex  $u_n$  to black. Then the black vertex  $u_{n-1}$  forces the vertex  $p_2^n$  of the path  $P_m^n$  to black. Now the black vertices  $u_n$  and  $p_2^n$  generate a 2-distance forcing set of the path  $P_m^n$ . Thus the set  $Z_{2d}$  generates a 2-distance forcing set of the graph  $G$ . Since the cardinality of the set  $Z_{2d}$  is  $2(n - 1)$ . Therefore,  $Z_{2d}(G) \leq 2(n - 1)$ .

□

We strongly believe that the above bound is sharp.

**Theorem 11.** *Let  $G$  denotes the rooted product  $P_n \odot C_m$  of a path  $P_n$  and the rooted cycle  $C_m$ ,  $n \geq 2, m \geq 5$ . Then,  $Z_{2d}(G) \leq 3n$ , where  $n$  is the number of vertices of the path  $P_n$ .*

*Proof.* Represent the vertices of the path  $P_n$  by  $u_1, u_2, \dots, u_n$ . Let  $C_m^1, C_m^2, \dots, C_m^n$  be the  $n$  copies of the cycle  $C_m$  rooted at the vertices  $u_1, u_2, \dots, u_n$  of the path  $P_n$  respectively. Denote the cycles  $C_m^1, C_m^2, \dots, C_m^n$  in  $G$  as follows.

$$C_m^1 = v_1^1, v_2^1, \dots, v_m^1, v_1^1$$

$$C_m^2 = v_1^2, v_2^2, \dots, v_m^2, v_1^2$$

...

...

...

$$C_m^n = v_1^n, v_2^n, \dots, v_m^n, v_1^n$$

Let the vertex  $v_1^1$  of the cycle  $C_m^1$  be rooted at the vertex  $u_1$  of  $P_n$ , the vertex  $v_1^2$  of the cycle  $C_m^2$  be rooted at the vertex  $u_2$  of  $P_n, \dots$ , the vertex  $v_1^n$  of the cycle  $C_m^n$  be rooted at the vertex  $u_n$  of  $P_n$ . That is,  $u_1 = v_1^1, u_2 = v_1^2, \dots, u_n = v_1^n$ . Let  $A_1 = \{u_1, v_2^1, v_3^1, v_4^1\}, A_2 = \{v_2^2, v_3^2, v_4^2\}, A_3 = \{v_2^3, v_3^3, v_4^3\}, \dots, A_{n-1} = \{v_2^{n-1}, v_3^{n-1}, v_4^{n-1}\}, A_n = \{v_2^n, v_3^n\}$ . Also let  $Z_{2d} = A_1 \cup A_2 \cup \dots \cup A_n$ . Color all vertices of the set  $Z_{2d}$  as black. Then we claim that the set  $Z_{2d}$  generates a 2-distance forcing set of  $G$

For, Since  $Z_{2d}(C_n) = 4$  for  $n \geq 5$  (See[8]), clearly the set  $A_1$  generates a 2-distance forcing set of the cycle  $C_m^1$  in  $G$ . Then the vertex  $v_m^1$  of the cycle  $C_m^1$  forces the vertex  $u_2$  of the path  $P_n$  to black, since  $u_2$  is the only 2-distance white neighbor of  $v_m^1$ . Since  $u_2$  is also a black vertex of the cycle  $C_m^2$ , we can observe that the set  $A_2$  together with the black vertex  $u_2$  forms a 2-distance forcing set of the cycle  $C_m^2$  in  $G$ . Then the black vertex  $v_m^2$  of  $C_m^2$  forces the vertex  $u_3$  of the path  $P_n$  to black, because  $u_3$  is the only 2-distance white neighbor of the vertex  $v_m^2$ . Now consider the cycle  $C_m^3$  in  $G$ . Since  $u_3$  is also a black vertex of the cycle  $C_m^3$ , we can see that the set  $A_3$  together with the black vertex  $u_3$  generates a 2-distance forcing set of the cycle  $C_m^3$  in  $G$ . Proceeding like this, Consider the cycle  $C_m^{n-1}$ . In  $C_m^{n-1}$ , the vertex  $u_{n-1}$  is already colored to black by the vertex

$v_m^{n-2}$  of the cycle  $C_m^{n-2}$ . There fore, the set  $A_{n-1} = \{v_2^{n-1}, v_3^{n-1}, v_4^{n-1}\}$  together with the black vertex  $u_{n-1}$  will form a 2-distance forcing set of the cycle  $C_m^{n-1}$  in  $G$ . Finally, consider the cycle  $C_m^n$  in  $G$ . Then clearly the vertex  $u_n$  of the cycle  $C_m^n$  will be forced to black by the black vertex  $v_m^{n-1}$  of the cycle  $C_m^{n-1}$ . Then the black vertex  $u_{n-1}$  forces the vertex  $v_m^n$  of the cycle  $C_m^n$  to black because the vertices  $u_n$  and  $v_2^n$  are black vertices in  $C_m^n$ . Then we can easily observe that the set  $A_n$  with the black vertices  $u_n$  and  $v_m^n$  generates a 2-distance forcing set of the cycle  $C_m^n$ . Thus the set  $Z_{2d}$  generates a 2-distance forcing set of  $G$ . Cardinality of the set  $Z_{2d}$  is  $4+3(n-2)+2=3n$ . Therefore, we have  $Z_{2d}(G) \leq 3n$ . This completes the proof.  $\square$

We strongly believe that the above bound is sharp.

**Theorem 12.** *Let  $G$  be the graph representing the rooted product  $C_n \odot P_t$  of a cycle  $C_n$  and the path  $P_t$  rooted at the pendant vertex of  $P_t$  ( $n, t \geq 4$ ). Then  $Z_{2d}(G) \leq 2n - 4$ , where  $n$  is the number of vertices of the cycle  $C_n$ .*

*Proof.* Let  $P_t^1, P_t^2, \dots, P_t^n$  be the  $n$  copies of the path  $P_t$  rooted at the vertices  $u_1, u_2, \dots, u_n$  of the cycle  $C_n$  respectively. Denote the paths  $P_t^1, P_t^2, \dots, P_t^n$  in  $G$  as follows.

$$\begin{aligned}
 P_t^1 &= p_1^1, p_2^1, \dots, p_t^1 \\
 P_t^2 &= p_1^2, p_2^2, \dots, p_t^2 \\
 &\dots \\
 &\dots \\
 &\dots \\
 P_t^n &= p_1^n, p_2^n, \dots, p_t^n
 \end{aligned}$$

Let the vertex  $p_1^1$  of the path  $P_t^1$  be rooted at the vertex  $u_1$ ,  $p_1^2$  of the path  $P_t^2$  at  $u_2$ , ..., the vertex  $p_1^n$  of the path  $P_t^n$  at  $u_n$ . That is,  $p_1^1 = u_1, p_1^2 = u_2, \dots, p_1^n = u_n$ . Let  $A_1 = \{p_t^1, p_{t-1}^1\}$ ,  $A_2 = \{p_t^2, p_{t-1}^2\}, \dots, A_{n-2} = \{p_t^{n-2}, p_{t-1}^{n-2}\}$ . Also let  $Z_{2d} = A_1 \cup A_2 \cup \dots \cup A_{n-2}$ . Color all vertices of the set  $Z_{2d}$  as black. Then we assert that the set  $Z_{2d}$  forms a 2-distance forcing set of the graph  $G$ .

For, consider the path  $P_t^1$  rooted at the vertex  $u_1$ . Since  $Z_{2d}(P_n) = 2$  ( $n \geq 3$ ) (See[8]), clearly the set  $A_1$  generates a 2-distance forcing set for the path  $P_t^1$ . We observe that after getting the derived coloring of the path  $P_t^1$ , the color of the vertex  $u_1$  is forced to black. Similarly, the set  $A_2$  forms a 2-distance forcing set of the path  $P_t^2$  rooted at the vertex  $u_2$ . Then the black vertex  $p_2^1$  of the path  $P_t^1$  forces the vertex  $u_n$  of the cycle  $C_n$  to black. Again, the set  $A_3$  generates a 2-distance forcing set of the path  $P_t^3$  rooted at the vertex  $u_3$ . Proceeding like this, consider the path  $P_t^{n-2}$  rooted at the vertex  $u_{n-2}$ . We can observe that the set  $A_{n-2}$  will form a 2-distance forcing set of the path  $P_t^{n-2}$ . Now the black vertex  $p_2^{n-2}$  of the path  $P_t^{n-2}$  forces the vertex  $u_{n-1}$  to black. Then the black vertex  $u_{n-2}$  forces the vertex  $p_2^{n-1}$  of the path  $P_t^{n-1}$  to black, the black vertex  $u_1$  forces the vertex  $p_2^n$  of the path  $P_t^n$  to black. Now we can see that all remaining white vertices of the paths  $P_t^{n-1}$  and  $P_t^n$  will be colored to black. Thus the set  $Z_{2d}$  generates a 2-distance forcing set of  $G$ . Cardinality of the set  $Z_{2d}$  is  $2n - 4$ . Therefore,  $Z_{2d}(G) \leq 2n - 4$ , as we desired.  $\square$

We strongly believe that the above bound is sharp.

**Theorem 13.** *Let  $G$  be the rooted product  $C_n \odot C_m$  of a cycle  $C_n$  and the rooted cycle  $C_m$ ,  $m \geq 5, n \geq 4$ . Then,  $Z_{2d}(G) \leq 3n$ , where  $n$  is the number of vertices of the cycle  $C_n$  and we strongly believe that this bound is sharp.*

*Proof.* Denote the vertices of the cycle  $C_n$  in  $G$  by  $u_1, u_2, \dots, u_n$  and the  $n$  copies of the cycle  $C_m$  in  $G$  by  $C_m^1, C_m^2, \dots, C_m^n$ , where

$$C_m^1 = v_1^1, v_2^1, \dots, v_m^1, v_1^1$$

$$C_m^2 = v_1^2, v_2^2, \dots, v_m^2, v_1^2$$

...

...

...

$$C_m^n = v_1^n, v_2^n, \dots, v_m^n, v_1^n$$

Let the vertex  $v_1^1$  of the cycle  $C_m^1$  be rooted at the vertex  $u_1$ , the vertex  $v_1^2$  of the cycle  $C_m^2$  be rooted at the vertex  $u_2, \dots$ , the vertex  $v_1^n$  of the cycle  $C_m^n$  be rooted at the vertex  $u_n$ . That is,  $u_1 = v_1^1, u_2 = v_1^2, \dots, u_n = v_1^n$ . Suppose that  $A_1 = \{u_1, v_2^1, v_3^1, v_4^1\}$ ,  $A_2 = \{u_2, v_2^2, v_3^2, v_4^2\}$ ,  $A_3 = \{v_2^3, v_3^3, v_4^3\}$ ,  $A_4 =$

$\{v_2^4, v_3^4, v_4^4\}, A_5 = \{v_2^5, v_3^5, v_4^5\}, \dots, A_{n-2} = \{v_2^{n-2}, v_3^{n-2}, v_4^{n-2}\}, A_{n-1} = \{v_2^{n-1}, v_3^{n-1}\}, A_n = \{v_2^n, v_3^n\}$ . Also let  $Z_{2d} = A_1 \cup A_2 \cup \dots \cup A_n$ . Assign black color to all vertices of the set  $Z_{2d}$ . We claim that the set  $Z_{2d}$  forms a 2-distance forcing set of the graph  $G$ .

For, consider the set  $A_1$ . Clearly  $A_1$  generates a 2-distance forcing set of the cycle  $C_m^1$  rooted at the vertex  $u_1$ . Similarly, the set  $A_2$  generates a 2-distance forcing set of the cycle  $C_m^2$  rooted at the vertex  $u_2$ . Now since  $u_n$  is the only 2-distance white neighbor of the vertex  $v_2^1$  of the cycle  $C_m^1$ , the black vertex  $v_2^1$  forces the vertex  $u_n$  to black. In a similar way, the black vertex  $v_2^2$  of the cycle  $C_m^2$  forces the vertex  $u_3$  to black. Again, consider the set  $A_3$ . Clearly the set  $A_3$  together with the black vertex  $u_3$  generates a 2-distance forcing set of the cycle  $C_m^3$  rooted at the vertex  $u_3$ . Proceeding like this, consider the cycle  $C_m^{(n-2)}$  rooted at the vertex  $u_{n-2}$ . Note that at this stage the vertex  $u_{n-2}$  is forced to black by the vertex  $v_2^{n-3}$  of the cycle  $C_m^{(n-3)}$ . Now we can observe that the set  $A_{n-2}$  together with the vertex  $u_{n-2}$  generates a 2-distance forcing set of the cycle  $C_m^{(n-2)}$  rooted at the vertex  $u_{n-2}$ . In the cycle  $C_m^{n-1}$ , clearly the vertex  $u_{n-1}$  is forced to black by the vertex  $v_2^{n-2}$  of the cycle  $C_m^{n-2}$ . Then the vertex  $u_{n-2}$  forces the vertex  $v_m^{n-1}$  of the cycle  $C_m^{n-1}$  to black, the vertex  $u_1$  forces the vertex  $v_m^n$  of the cycle  $C_m^n$  to black. Now the set  $A_{n-1}$  together with the black vertices  $u_{n-1}$  and  $v_m^{n-1}$  forms a 2-distance forcing set for the cycle  $C_m^{(n-1)}$ . Similarly, the set  $A_n$  together with the black vertices  $u_n$  and  $v_m^n$  generates a 2-distance forcing set of the cycle  $C_m^n$ . Thus the set  $Z_{2d}$  generates a 2-distance forcing set of the entire graph  $G$ . Cardinality of the set  $Z_{2d}$  is  $4 + 4 + 3(n - 4) + 2 + 2 = 3n$ .

Hence  $Z_{2d}(G) \leq 3n$ . This completes our proof. □

#### 4. 2-DISTANCE FORCING NUMBER OF SQUARE OF GRAPH

Here we consider the square of path and the cycle.

**Theorem 14.** *Let  $G$  denotes the square graph of the path  $P_n$  of order  $n \geq 5$ . Then  $Z_{2d}(G) = 4$ .*

*Proof.* In  $G$ ,  $\delta_{2d}(G) = 5$ . So by theorem-1

$$(9) \quad Z_{2d}(G) \geq 4.$$



To establish the reverse inequality, we proceed as follows.

Let  $Z_{2d} = \{u_1, u_2, u_3, u_4\}$  be a set of black vertices. We observe that  $u_5$  is the only 2-distance white neighbor of the vertex  $u_1$ . So the black vertex  $u_1 \rightarrow u_5$  to black. Now consider the black vertex  $u_2$ . The 2-distance neighbors of the vertex  $u_2$  are  $u_1, u_3, u_4, u_5$  and  $u_6$ , out of which  $u_6$  is the only white vertex. Therefore, the vertex  $u_2 \rightarrow u_6$  to black. Again consider the black vertex  $u_3$ . The 2-distance neighbors of  $u_3$  are  $u_1, u_2, u_4, u_5, u_6$  and  $u_7$  of which  $u_7$  is the only white vertex. Hence the vertex  $u_3 \rightarrow u_7$  to black. Continue this process, we will obtain a forcing sequence  $u_1 \rightarrow u_5, u_2 \rightarrow u_6, u_3 \rightarrow u_7, \dots, u_{n-5} \rightarrow u_{n-1}, u_{n-4} \rightarrow u_n$ . Thus all vertices of  $G$  will be colored to black using the set  $Z_{2d}$ . Hence the set  $Z_{2d} = \{u_1, u_2, u_3, u_4\}$  forms a 2-distance forcing set of  $G$ . Here the cardinality of the set  $Z_{2d}$  is 4. Therefore,

$$(10) \quad Z_{2d}(G) \leq 4.$$

From the inequalities (9) and (10), the proof follows. □

**Theorem 15.** *If  $G$  is the square graph of the cycle  $C_n, n \geq 9$ . Then,  $Z_{2d}(G) = 8$ .*

*Proof.* The square graph of the cycle  $C_n$  is a 4-regular graph. Denote the vertices of the cycle  $C_n$  in  $G$  by  $u_1, u_2, \dots, u_n$ . Each vertex  $u_i, (i=1,2,3,\dots,n)$  has eight vertices in its 2-distance neighborhood by the definition of the square graph of a cycle. The 2-distance vertices of  $u_1, u_2, u_3, u_4, \dots, u_{n-1}, u_n$  are displayed in the following table.

Vertex	2-distance vertices
$u_1$	$u_2, u_3, u_4, u_5, u_n, u_{n-1}, u_{n-2}, u_{n-3}$
$u_2$	$u_1, u_3, u_4, u_5, u_6, u_n, u_{n-1}, u_{n-2}$
$u_3$	$u_1, u_2, u_4, u_5, u_6, u_7, u_n, u_{n-1}$
$u_4$	$u_1, u_2, u_3, u_5, u_6, u_7, u_8, u_n$
...	.....
...	.....
...	.....
$u_{n-1}$	$u_1, u_2, u_3, u_n, u_{n-2}, u_{n-3}, u_{n-4}, u_{n-5}$
$u_n$	$u_1, u_2, u_3, u_4, u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}$

Since  $\delta_{2d}(G) = 9$ , by Theorem- 1, we have

$$(11) \quad Z_{2d}(G) \geq 8.$$

To prove the reverse inequality, we proceed as follows.

Start the color change rule by coloring the vertex  $u_1$  as black. Since the vertex  $u_1$  has eight 2-distance white neighbors, to proceed further, we have to color at least seven of these white vertices to black. Let them be  $u_{n-3}, u_{n-2}, u_{n-1}, u_n, u_2, u_4$  and  $u_5$ . Then  $u_1 \rightarrow u_3$  to black. Now consider the black vertex  $u_2$ . The vertices  $u_1, u_3, u_4, u_5, u_6, u_n, u_{n-1}, u_{n-2}$  are the 2-distance vertices of the vertex  $u_2$ . Out of these vertices,  $u_1, u_n, u_{n-1}, u_{n-2}, u_3, u_4, u_5$  are already black. So, the vertex  $u_2 \rightarrow u_6$  to black. Then the vertex  $u_3 \rightarrow u_7$  to black, since the other 2-distance vertices  $u_1, u_2, u_n, u_{n-1}, u_4, u_5$  and  $u_6$  of the vertex  $u_3$  are already black. Similarly, apply the same argument for the black vertex  $u_4$  to force the vertex  $u_8$  to black. Proceeding like this, the black vertex  $u_{n-8}$  forces the vertex  $u_{n-4}$  to black. We note that the vertices  $u_{n-3}, u_{n-2}, u_{n-1}, u_n$  are already coloured black. Hence with the black vertices  $u_1, u_2, u_4, u_5, u_n, u_{n-1}, u_{n-2}, u_{n-3}$ , we can force all vertices of the graph  $G$  to black. So, the set  $Z_{2d} = \{u_1, u_2, u_4, u_5, u_n, u_{n-1}, u_{n-2}, u_{n-3}\}$  of black vertices is a 2-distance forcing set of the graph  $G$ . Cardinality of the set  $Z_{2d}$  is 8. There fore,

$$(12) \quad Z_{2d}(G) \leq 8.$$

This completes our proof. □

### 5. 2-DISTANCE FORCING NUMBER OF SPLITTING GRAPH

In this section, we consider the splitting graph of the path. Determining the 2-distance forcing number of the splitting graph of other families of graphs is left as an exercise.

**Definition** [15]. The splitting graph of a graph  $G$ , denoted by  $S(G)$ , is the graph obtained by taking a vertex  $v'$  corresponding to each vertex  $v \in G$  and join  $v'$  to all the vertices of  $G$  adjacent to  $v$ .

**Theorem 16.** *Let  $G$  be the splitting graph of the path  $P_n$ ,  $n \geq 4$ . Then  $Z_{2d}(G) = n$ .*

*Proof.* Denote the vertices of the path  $P_n$  in  $G$  by  $u_1, u_2, \dots, u_n$ . Represent the corresponding vertices of  $u_1, u_2, \dots, u_n$  in  $G$  by  $u_1^1, u_2^1, \dots, u_n^1$ . In this graph, the vertices  $u_1^1$  and  $u_n^1$  have the least number of 2-distance vertices and the number of such 2-distance vertices is 4.

We form a 2-distance forcing set of the graph  $G$  as follows.

Begin with the vertex  $u_1^1$ . Assign black color to  $u_1^1$ . The vertex  $u_1^1$  has four 2-distance neighbors  $u_1, u_2, u_3$  and  $u_3^1$ . To begin the color change rule, color at least three of these 2-distance neighbors to black. Let the black vertices be  $u_1, u_2$  and  $u_3^1$ . Then the black vertex  $u_1^1$  forces the vertex  $u_3$  to black. Consequently, the black vertex  $u_1$  forces  $u_2^1$  to black. Now the black vertex  $u_2$  has only two 2-distance white neighbors  $u_4$  and  $u_4^1$ . Assign black color to the vertex  $u_4$  to force the vertex  $u_4^1$  to black by the vertex  $u_2$ . Similarly, the black vertex  $u_3$  has two 2-distance white vertices  $u_5$  and  $u_5^1$ . Color the vertex  $u_5$  to black. Then the black vertex  $u_3$  forces the vertex  $u_5^1$  to black. Now consider the vertex  $u_4$ . The vertices  $u_6$  and  $u_6^1$  are the 2-distance white neighbors of  $u_4$ . By assigning black color to the vertex  $u_6$ , we can force the vertex  $u_6^1$  to black by the vertex  $u_4$ . Continue this process step by step, consider the black vertex  $u_{n-1}^1$  (we observe that the vertex  $u_{n-1}^1$  is forced to black by the vertex  $u_{n-3}$ ). Now the black vertex  $u_{n-1}^1$  has only one 2-distance white neighbor  $u_n$ . So,  $u_{n-1}^1$  forces  $u_n$  to black. In turn, the black vertex  $u_{n-1}$  forces the vertex  $u_n^1$  to black. Thus the set  $Z_{2d} = \{u_1^1, u_3^1, u_1, u_2, u_4, u_5, \dots, u_{n-1}\}$  forms a 2-distance forcing set of  $G$ . Clearly, the cardinality of the set  $Z_{2d}$  is  $2 + n - 2 = n$ . There fore,

$$(13) \quad Z_{2d}(G) \leq n.$$

To prove the converse, we claim that no set with  $(n - 1)$  black vertices generates a 2-distance forcing set of  $G$ . For, we consider the following cases.

**Case 1.** Let  $Z_{2d} = \{u_1, u_2, \dots, u_{n-1}\}$  be a set of  $(n - 1)$  black vertices. Then we can observe that further forcing is not possible because each black vertex has at least three 2-distance white neighbors.

**Case 2.** Assume that  $Z_{2d} = \{u_1^1, u_2^1, \dots, u_{n-1}^1\}$  be a set of black vertices. Here also the color

change rule is not possible since each black vertex has at least three 2-distance white neighbors.

**Case 3.** Let the set  $Z_{2d}$  be the vertex set of a path  $P_{n-1}$ . Assume that all the vertices of  $P_{n-1}$  are black. Then we consider the following sub cases.

**Sub case 3.1** Assume that  $n$  is odd.

Let  $P_{n-1}$  be the path  $u_1u_2^1u_3u_4^1\dots u_{n-3}^1u_{n-2}u_{n-1}^1$ . Then further forcing is not possible, since each black vertex has at least two 2-distance white neighbors.

**Sub case 3.2.** Consider the path  $u_1^1u_2u_3^1u_4\dots u_{n-3}u_{n-2}^1u_{n-1}$ . Here also derived coloring is not possible because each black vertex has at least two white vertices in its 2-distance neighborhood.

**Sub case 3.3** Suppose that  $n$  is even.

Consider the paths  $u_1u_2^1u_3u_4^1\dots u_{n-3}u_{n-2}^1u_{n-1}$  and  $u_1^1u_2u_3^1u_4\dots u_{n-3}u_{n-2}u_{n-1}^1$ . In this case also color change rule is not possible because each black vertex has at least two 2-distance white neighbors.

It is worth mentioning that a set of  $(n - 1)$  black vertices generated in any way other than mentioned above cannot form a 2-distance forcing set of  $G$ . So from the above cases, we can observe that

$$(14) \quad Z_{2d}(G) \geq n.$$

Therefore from (13) and (14), we get  $Z_{2d}(G) = n$ .

□

## 6. 2-DISTANCE FORCING NUMBER OF CARTESIAN PRODUCT OF GRAPHS

In this section, we compute the 2-distance forcing number of Cartesian product of graphs like the ladder graph  $P_n \square P_2$ , the grid graph  $P_n \square P_m$  and the circular ladder graph  $C_n \square K_2$ . We start with the ladder graph  $P_n \square P_2$ .

**Theorem 17.** *Let  $G$  be the ladder graph  $P_n \square P_2$ ,  $n \geq 3$ . Then, the 2-distance forcing number of  $G$  is four. That is,  $Z_{2d}(G) = 4$ .*

*Proof.* Let  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  be the vertices of the graph  $G = P_n \square P_2$ . Note that  $\delta_{2d}(G) = 5$  in  $G$ . Since  $Z_{2d}(G) \geq \delta_{2d}(G) - 1$  by Theorem- 1, we have

$$(15) \quad Z_{2d}(G) \geq 4.$$

—

Now it suffices to prove the reverse inequality. For, consider the set  $Z_{2d} = \{u_1, u_2, v_1, v_2\}$ . Assign the vertices of the set  $Z_{2d}$  the black color. Take the black vertex  $v_1$ . Since  $v_3$  is the only 2-distance white neighbor of  $v_1$ , the black vertex  $v_1 \rightarrow v_3$  to black. Now clearly  $u_1 \rightarrow u_3$  to black. Repeatedly apply the color change rule, we get the forcing sequence  $v_2 \rightarrow v_4, u_2 \rightarrow u_4, v_3 \rightarrow v_5, u_3 \rightarrow u_5, v_{n-3} \rightarrow v_{n-1}, u_{n-3} \rightarrow u_{n-1}, v_{n-2} \rightarrow v_n, u_{n-2} \rightarrow u_n$ . Thus all vertices of  $G$  will be colored black. Therefore, the set  $Z_{2d} = \{u_1, u_2, v_1, v_2\}$  generates a 2-distance forcing set of the graph  $G$ . Cardinality of the set  $Z_{2d}$  is 4. Hence

$$(16) \quad Z_{2d}(G) \leq 4.$$

From (15) and (16), the result follows. □

**Theorem 18.** *Let  $G$  be the Cartesian product  $P_n \square P_m$  of the path  $P_n$  and the path  $P_m$ , ( $n \geq m; n, m \geq 5$ ). Then,  $Z_{2d}(G) = 2m$ .*

*Proof.* Denote the vertices of the graph  $G$  by

$$p_1^1, p_2^1, \dots, p_n^1$$

$$p_1^2, p_2^2, \dots, p_n^2$$

$$p_1^3, p_2^3, \dots, p_n^3$$

...

...

...

$$p_1^m, p_2^m, \dots, p_n^m$$

We can easily infer that the vertices  $p_1^1, p_n^1, p_1^m$  and  $p_n^m$  have the least number of 2-distance vertices and the number of such vertices is 5. We generate a 2-distance forcing set of  $G$  as follows. Consider the set  $Z_{2d} = \{p_1^1, p_2^1, p_1^2, p_2^2, p_1^3, p_2^3, p_1^4, p_2^4, \dots, p_1^{m-1}, p_2^{m-1}, p_1^m, p_2^m\}$  of  $2m$  black vertices. Then clearly the black vertex  $p_1^1$  forces the vertex  $p_3^1$  to black, since  $p_3^1$  is the only 2-distance white neighbor of the vertex  $p_1^1$ . Similarly, the black vertex  $p_1^2$  forces the vertex  $p_3^2$  to black because  $p_3^2$  is the only 2-distance white neighbor of the vertex  $p_1^2$ . Continue like this, we can see that the vertex  $p_1^3$  forces  $p_3^3$  to black,  $p_1^4$  forces  $p_3^4$  to black, ...,  $p_1^m$  forces the vertex  $p_3^m$  to black. Again the black vertex  $p_2^1$  forces the vertex  $p_4^1$  to black,  $p_2^2$  forces the vertex  $p_4^2$  to black, ...,  $p_2^m$  forces the vertex  $p_4^m$  to black. Apply this process step by step, finally the black vertex  $p_{n-2}^1$  forces the vertex  $p_n^1$  to black,  $p_{n-2}^2$  forces the vertex  $p_n^2$  to black, ...,  $p_{n-2}^m$  forces the vertex  $p_n^m$  to black. Thus the set  $Z_{2d}$  generates a 2-distance forcing set of  $G$ . Clearly, the cardinality of the set  $Z_{2d}$  is  $2m$ . Hence,

$$(17) \quad Z_{2d}(G) \leq 2m$$

To establish the reverse inequality, we proceed as follows.

**Case 1.**

Omit the black vertex  $p_1^1$ (or  $p_1^m$ ) from the set  $Z_{2d}$ . Then after possible forcings, there exists only  $2 + \frac{m(m-1)}{2}$  black vertices in the graph  $G$ . So in this case, we cannot generate a 2-distance forcing set of  $G$ .

**Case 2.**

Delete the black vertex  $p_1^2$ (or  $p_1^{m-1}$ ) from the 2-distance forcing set  $Z_{2d}$ . In this case, after possible number of forcings, there exists only  $4 + \frac{(m-1)(m-2)}{2}$  black vertices in the graph  $G$ . Therefore, the set  $Z_{2d}$  will never form a 2-distance forcing set for  $G$ .

**Case 3.**

If we exclude the vertex  $P_2^1$  (or  $P_2^m$ ) from the set  $Z_{2d}$ , then after possible forcings there exists only  $\frac{m(m+1)}{2}$  black vertices in the graph  $G$ . Hence the set  $Z_{2d}$  will not form a 2-distance set for  $G$ .

**Case 4.**

If we omit the vertex  $P_2^2$  (or  $P_2^{m-1}$ ) from the set  $Z_{2d}$ , then clearly there are only  $2 + \frac{(m-1)m}{2}$  black vertices in the graph  $G$ .

**Case 5.**

Delete any one black vertex  $p_1^j$  ( $j=3,4,\dots,m-2$ ) from the set  $Z_{2d}$ . Then we can easily assert that the set  $Z_{2d}$  cannot form a 2-distance forcing set of  $G$ .

**Case 6.**

Remove the vertex  $P_2^j$ , ( $j = 3, 4, \dots, m - 2$ ) from the set  $Z_{2d}$ . Then we can observe that the derived coloring of  $G$  is not possible.

Hence from the above cases, we can conclude that a set with  $(2m - 1)$  black vertices will not form a 2-distance forcing set of  $G$ . Therefore,

$$(18) \quad Z_{2d}(G) \geq 2m.$$

Hence from (17) and (18),  $Z_{2d}(G) = 2m$ . □

**Theorem 19.** *Let  $G$  represents the circular ladder graph  $C_n \square K_2$ . Then,*

$$Z_{2d}(G) = \begin{cases} 5 & \text{if } n = 3 \\ 6 & \text{if } n = 4 \\ 7 & \text{if } n = 5 \\ 8 & \text{if } n \geq 6 \end{cases}$$

where  $n$  is the number of vertices of either cycle.

*Proof. Case 1.* Assume that  $n = 3$ . Let  $\{u_1, u_2, u_3, v_1, v_2, v_3\}$  be the vertex set of  $G$ , where  $u_1, u_2, u_3$  are the vertices of the inner cycle. In this case since the diameter of  $G$  is 2, the proof follows obviously from [8].

**Case 2.** Assume that  $n = 4$ . Represent the vertex set of  $G$  by

$\{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ , where  $u_1, u_2, u_3, u_4$  are the vertices of the inner cycle. Here,  $\Delta_{2d}(G) = \delta_{2d}(G) = 7$ . So from Theorem-1, we have

$$(19) \quad Z_{2d}(G) \geq 6.$$

On the other hand, consider the set  $Z_{2d} = \{v_1, v_2, v_3, v_4, u_1, u_2\}$  of black vertices. Then clearly the vertex  $v_1 \rightarrow u_4$  to black, since  $u_4$  is the only 2-distance white neighbor of the black vertex  $v_1$ . Consequently, the vertex  $u_3$  will be colored to black. Therefore, the set  $Z_{2d}$  forms a 2-distance forcing set of  $G$ . Cardinality of the set  $Z_{2d}$  is 6. Hence we have,

$$(20) \quad Z_{2d}(G) \leq 6.$$

Hence from (19) and (20), the result follows.

**Case 3.** Assume that  $n = 5$ . Denote the vertex set of  $G$  by

$\{u_1, u_2, \dots, u_5, v_1, v_2, \dots, v_5\}$ , where  $u_1, u_2, u_3, u_4, u_5$  are the vertices of the inner cycle. In  $G$ ,  $\Delta_{2d}(G) = \delta_{2d}(G) = 8$ . Therefore we have,

$$(21) \quad Z_{2d}(G) \geq 7.$$

Conversely, Let  $Z_{2d} = \{v_1, v_2, v_3, v_5, u_1, u_2, u_5\}$  be a set of seven black vertices. Then we can easily see that the black vertex  $v_1 \rightarrow v_4$  to black. Now, the vertex  $v_5 \rightarrow u_4$  to black. Consequently, the vertex  $u_3$  will be colored to black. Thus the set  $Z_{2d}$  generates a 2-distance forcing set of  $G$ . Cardinality of the set  $Z_{2d}$  is 7. Therefore,

$$(22) \quad Z_{2d}(G) \leq 7.$$



Hence from (21) and (22), the result follows.

**Case 4.** Assume that  $n \geq 6$ . Let  $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$  be the vertex set of the circular ladder graph  $G$ , where  $u_1, u_2, \dots, u_n$  are the vertices of the inner cycle. We construct a 2-distance forcing set of  $G$  as follows.

Clearly  $\delta_{2d}(G) = 8$  in  $G$ . Since  $Z_{2d}(G) \geq \delta_{2d}(G) - 1$  for a connected graph  $G$ , we have  $Z_{2d}(G) \geq 7$ . But it is obvious that with 7 black vertices we cannot form a 2-distance forcing set of the graph  $G$ , because with 7 black vertices, the maximum number of further forcing possible is only one. Hence we can conclude that

$$(23) \quad Z_{2d}(G) \geq 8.$$

To claim the reverse part, we proceed as follows.

Let  $Z_{2d} = \{u_1, u_2, u_n, u_{n-1}, v_1, v_2, v_n, v_{n-1}\}$  be the set of black vertices. Then the black vertex  $u_1 \rightarrow u_3$  to black, since  $u_3$  is the only 2-distance white vertex of  $u_1$ . Similarly the black vertex  $v_1 \rightarrow v_3$  to black, since  $v_3$  is the only 2-distance white vertex of  $v_1$ . Then clearly  $u_2 \rightarrow u_4$  to black,  $v_2 \rightarrow v_4$  to black and so on. Apply this process step by step, we get the forcing sequence  $u_3 \rightarrow u_5, v_3 \rightarrow v_5, u_4 \rightarrow u_6, v_4 \rightarrow v_6, \dots, u_{n-4} \rightarrow u_{n-2}, v_{n-4} \rightarrow v_{n-2}$ . Thus we will get a derived coloring of  $G$  using the set  $Z_{2d}$ . Therefore, the set  $Z_{2d}$  forms a 2-distance forcing set of  $G$ . Cardinality of the set  $Z_{2d}$  is 8. Hence

$$(24) \quad Z_{2d}(G) \leq 8$$

This completes our proof. □

**Theorem 20.** [8] *Let  $G$  be a connected graph of order  $n \geq 3$  with  $\text{diam}(G) = 2$ . Then  $Z_{2d}(G) = n - 1$ .*

**Corollary 21.** *Let  $G$  be the complete bipartite graph  $K_{m,n}$ ;  $m, n \geq 2$ . Then  $Z_{2d}(G) = m + n - 1$ .*

*Proof.* Since the complete bipartite graph  $K_{mn}$  is a graph with  $diam(K_{mn}) = 2$  and having more than two vertices, the proof follows by the Theorem 20. □

## 7. 2-DISTANCE FORCING NUMBER OF COMPLEMENT OF GRAPHS

In this section, we compute the 2-distance forcing number of complement of graphs like path and cycle.

**Theorem 22.** *Let  $G$  denotes the complement of the path  $P_n$ ,  $n \geq 5$ . Then  $Z_{2d}(G) = n - 1$ .*

*Proof.* Since the graph  $G$  is connected with  $n \geq 3$  and  $diam(G) = 2$ , the proof follows by Theorem 20. □

**Theorem 23.** *Let  $G$  represents the complement of the cycle  $C_n$ ,  $n \geq 5$ . Then  $Z_{2d}(G) = n - 1$ .*

*Proof.* Here the graph  $G$  is connected with  $n \geq 3$  having  $diam(G) = 2$ . Therefore, the proof follows by an immediate consequence of Theorem 20. □

## 8. CONCLUSIONS

In this article, we studied the notion of 2-distance forcing number of some graphs with large diameter. In Section 2, we determined the exact values of the 2-distance forcing number  $Z_{2d}(G)$  of graphs like the shadow graph  $D_2(P_n)$  of the path  $P_n$ , middle graph of the path  $P_n$ ,  $S^{th}$  Necklace graph  $N_s$ , the triangular snake graph, the n-sunlet graph and the graph  $F_p^k$ .

In Section 3, we focused our attention on computing the 2-distance forcing number of rooted product of some graphs. The 2-distance forcing number of rooted product of path with path, path with cycle, cycle with path and cycle with cycle were discussed. Finding the exact values of 2-distance forcing number of these graphs are still open.

Section 4 dealt with the 2-distance forcing number of square of some graphs. Square of graphs like path and cycle were considered for discussion.

In Section 5 , we found the 2-distance forcing number of the splitting graph of the path. Determining the value of this parameter for splitting graph of other classes of graphs is left as an exercise.

The 2-distance forcing number of Cartesian product of graphs like the ladder graph, the grid graph and the circular ladder graph were obtained in Section 6.

The final Section 7 dealt with the 2-distance forcing number of complement of path and cycle.

### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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