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SPECTRUM AND ENERGIES OF A GRAPH

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Abstract. In this paper, we compute spectrum, Laplacian spectrum, signless Laplacian spectrum and their corresponding energies of commuting conjugacy class graph of the non-abelian p-group of order p^n whose order of the centre is p^{n-2}, p is prime and $n \geq 3$. We derive some consequences along with the fact that commuting conjugacy class graph of the above group is super integral. We also compare various energies and determine whether commuting conjugacy class graph of the group is hyperenergetic, L-hyperenergetic or Q-hyperenergetic. We compute the genus of the graph. Finally we conclude the paper by noting the fact that the graph is neither toroidal nor double-toroidal nor triple-toroidal.

Keywords: commuting conjugacy class graph; spectrum; energy; finite group.

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1. INTRODUCTION

Let G be any group and $V(G) = \{a^G : a \in G \setminus Z(G)\}$, where a^G is the conjugacy class of a in G and $Z(G)$ is the center of G. We consider the graph $\Lambda(G)$, called commuting conjugacy class graph of G, with vertex set $V(G)$ and two distinct vertices a^G and b^G are adjacent if there exists some elements $a' \in x^G$ and $b' \in y^G$ such that a' and b' commute. Extending the notion of commuting graph of a group pioneered by Brauer and Fowler [11], Herzog et al. [10] were

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introduced commuting conjugacy class graph of groups in the first decade of this millennium. The second and third paper on this topic got published in the years 2016 and 2020 authered by Mohammadian et al. [1] and Salahshour et al. [9] respectively, where Mohammadian et al. characterize finite groups such that $\Lambda(G)$ is triangle-free and Salahshour et al. describe the structure of $\Lambda(G)$ considering G to be dihedral groups (D_{2n} for $n \geq 3$), generalized quaternion groups (Q_{4m} for $m \geq 2$), semidihedral groups (SD_{8n} for $n \geq 2$), the groups $V_{8n} = \langle a, b : a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle$ (for $n \geq 2$), $U_{(n,m)} = \langle a, b : a^{2n} = b^m = 1, a^{-1}ba = b^{-1} \rangle$ (for $m \geq 2$ and $n \geq 2$) and $G(p,m,n) = \langle a, b : a^{p^m} = b^{p^n} = [a,b]^p = 1, [a,[a,b]] = [b,[a,b]] = 1 \rangle$ (for any prime p, $m \geq 1$ and $n \geq 1$).

If $\Lambda = m_1 K_{n_1}$ then it is noteworthy that

$$\text{Spec}(\Lambda) = \left\{ (-1)^{m_1(n_1-1)}, (n_1-1)^{m_1} \right\} \tag{1.1}$$

$$\text{L-spec}(\Lambda) = \left\{ 0^{m_1}, n_1^{m_1(n_1-1)} \right\} \text{ and} \tag{1.2}$$

$$\text{Q-spec}(\Lambda) = \left\{ (2n_1-2)^{m_1}, (n_1-2)^{m_1(n_1-1)} \right\}, \tag{1.3}$$

where $\text{Spec}(\Lambda), \text{L-spec}(\Lambda)$ and $\text{Q-spec}(\Lambda)$ denote the spectrum, Laplacian spectrum and signless Laplacian spectrum of Λ. Recall that $\text{Spec}(\Lambda), \text{L-spec}(\Lambda)$ and $\text{Q-spec}(\Lambda)$ contain eigenvalues with the multiplicities (written as exponents) of $A(\Lambda)$, $L(\Lambda) := D(\Lambda) - A(\Lambda)$ and $Q(\Lambda) := D(\Lambda) + A(\Lambda)$ respectively, where $A(\Lambda)$ and $D(\Lambda)$ are adjacency and degree matrix of Λ respectively. Also, energy ($\mathscr{E}(\Lambda)$), Laplacian energy ($\mathscr{LE}(\Lambda)$) and signless Laplacian energy ($\mathscr{LE}^+(\Lambda)$) are defined as follows:

$$\mathscr{E}(\Lambda) = \sum_{x \in \text{Spec}(\Lambda)} |x|, \tag{1.4}$$

$$\mathscr{LE}(\Lambda) = \sum_{x \in \text{L-spec}(\Lambda)} \left| x - \frac{2|e(\Lambda)|}{|V(\Lambda)|} \right|, \tag{1.5}$$

$$\mathscr{LE}^+(\Lambda) = \sum_{x \in \text{Q-spec}(\Lambda)} \left| x - \frac{2|e(\Lambda)|}{|V(\Lambda)|} \right|, \tag{1.6}$$

where $V(\Lambda)$ is the set of vertices and $e(\Lambda)$ is the set of edges of Λ respectively.

In 2008, Gutman et al. [5] posed the following conjecture comparing $\mathscr{E}(\Lambda)$ and $\mathscr{LE}(\Lambda)$.

Conjecture 1.1. $\mathscr{E}(\Lambda) \leq \mathscr{LE}(\Lambda)$ for any graph Λ.

However, in the same year, Stevanović et al. [2] disproved the above conjecture. In 2009, Liu and Lin [7] also disproved Conjecture 1.1 by providing some counter examples.

2. Spectrum, Energies & Genus

We first find various spectra and energies of commuting conjugacy class graph of a group, and then we compute the genus of the group and find whether the group is planar, toroidal, double-toroidal, triple-toroidal.

Theorem 2.1. [8, Corollary 3.2] *If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, p is prime and $n \geq 3$, then $\Lambda(G) = (p+1)K_{p^{n-3}(p-1)}$.*

Theorem 2.2. *If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, p is prime and $n \geq 3$, then*

(1) $\operatorname{Spec}(\Lambda(G)) = \left\{(-1)^{(p+1)(p^{n-3}(p-1)-1)}, (p^{n-3}(p-1)-1)^{p+1}\right\}$.
(2) $\text{L-spec}(\Lambda(G)) = \left\{0^{p+1}, (p^{n-3}(p-1))^{(p+1)(p^{n-3}(p-1)-1)}\right\}$.
(3) $\text{Q-spec}(\Lambda(G)) = \left\{(2p^{n-3}(p-1)-2)^{p+1}, (p^{n-3}(p-1)-2)^{(p+1)(p^{n-3}(p-1)-1)}\right\}$.
(4) $\mathscr{E}(\Lambda(G)) = \mathscr{LE}(\Lambda(G)) = \mathscr{LE}^{+}(\Lambda(G)) = 2(p+1)(p^{n-3}(p-1)-1)$

Proof. By Theorem 2.1 we have,

$$\Lambda(G) = (p+1)K_{p^{n-3}(p-1)}.$$

Let $m_1 = p+1$ and $n_1 = p^{n-3}(p-1)$. Then, by (1.1)-(1.3), it follows that

$$\operatorname{Spec}(\Lambda(G)) = \{(-1)^{(p+1)(p^{n-3}(p-1)-1)}, (p^{n-3}(p-1)-1)^{p+1}\},$$

$$\text{L-spec}(\Lambda(G)) = \{0^{p+1}, (p^{n-3}(p-1))^{(p+1)(p^{n-3}(p-1)-1)}\}$$

and

$$\text{Q-spec}(\Lambda(G)) = \{(2p^{n-3}(p-1)-2)^{p+1}, (p^{n-3}(p-1)-2)^{(p+1)(p^{n-3}(p-1)-1)}\}.$$

Now, by (1.4), we get,

$$\begin{aligned}\mathscr{E}(\Lambda(G)) &= (p+1)(p^{n-3}(p-1)-1)\times 1+(p+1)\times(p^{n-3}(p-1)-1)\\ &= 2(p+1)(p^{n-3}(p-1)-1).\end{aligned}$$

We have, $|V(\Lambda(G))| = m_1n_1 = (p+1)\times p^{n-3}(p-1) = p^{n-3}(p^2-1)$ and

$$\begin{aligned}|e(\Lambda(G))| &= \frac{m_1n_1(n_1-1)}{2}\\ &= \frac{1}{2}p^{n-3}(p^2-1)(p^{n-3}(p-1)-1).\end{aligned}$$

Therefore,

$$\frac{2|e(\Lambda(G))|}{|V(\Lambda(G))|} = p^{n-3}(p-1)-1.$$

Also,

$$\begin{aligned}\left|0-\frac{2|e(\Lambda(G))|}{|V(\Lambda(G))|}\right| &= \frac{2|e(\Lambda(G))|}{|V(\Lambda(G))|}\\ &= p^{n-3}(p-1)-1,\end{aligned}$$

and

$$\begin{aligned}\left|p^{n-3}(p-1)-\frac{2|e(\Lambda(G))|}{|V(\Lambda(G))|}\right| &= \left|p^{n-3}(p-1)-(p^{n-3}(p-1)-1)\right|\\ &= 1.\end{aligned}$$

Now, by (1.5), we have,

$$\begin{aligned}\mathscr{LE}(\Lambda(G)) &= (p+1)\times(p^{n-3}(p-1)-1)+(p+1)(p^{n-3}(p-1)-1)\times 1\\ &= 2(p+1)(p^{n-3}(p-1)-1).\end{aligned}$$

Again, we have,

$$\begin{aligned}\left|(2p^{n-3}(p-1)-2)-\frac{2|e(\Lambda(G))|}{|V(\Lambda(G))|}\right| &= \left|(2p^{n-3}(p-1)-2)-(p^{n-3}(p-1)-1)\right|\\ &= p^{n-3}(p-1)-1\end{aligned}$$

and

$$\left|(p^{n-3}(p-1)-2)-\frac{2|e(\Lambda(G))|}{|V(\Lambda(G))|}\right| = \left|(p^{n-3}(p-1)-2)-(p^{n-3}(p-1)-1)\right|$$
$$= 1.$$

By (1.6), we have

$$\begin{aligned}\mathscr{LE}^+(\Lambda(G)) &= (p+1)\times(p^{n-3}(p-1)-1)+(p+1)(p^{n-3}(p-1)-1)\times 1\\ &= 2(p+1)(p^{n-3}(p-1)-1),\end{aligned}$$

This completes the proof. □

By Theorem 2.2, it follows that $\mathrm{Spec}(\Lambda), \text{L-spec}(\Lambda)$ and $\text{Q-spec}(\Lambda)$ contain only integers for the group G given in Theorem 2.1. Therefore, the commuting conjugacy class graph of G is super integral. In Theorem 2.2, various energies of commuting conjugacy class graphs of G are compared. It is also observed that $\mathscr{E}(\Lambda) \leq \mathscr{LE}(\Lambda)$. Thus, it follows that E-LE Conjecture of Gutman et al. [5] holds for $\Lambda = \Lambda(G)$.

A graph Λ having n vertices is called hyperenergetic, L-hyperenergetic or Q-hyperenergetic according as $\mathscr{E}(K_n) < \mathscr{E}(\Lambda), \mathscr{LE}(K_n) < \mathscr{LE}(\Lambda)$ or $\mathscr{LE}^+(K_n) < \mathscr{LE}^+(\Lambda)$. Also, Λ is called borderenergetic, L-borderenergetic and Q-borderenergetic if $\mathscr{E}(K_n) = \mathscr{E}(\Lambda), \mathscr{LE}(K_n) = \mathscr{LE}(\Lambda)$ and $\mathscr{LE}^+(K_n) = \mathscr{LE}^+(\Lambda)$ respectively. We consider commuting conjugacy class graph $\Lambda(G)$ for the groups considered in Theorem 2.1 and determine whether they are hyperenergetic, L-hyperenergetic or Q-hyperenergetic. We shall also determine whether they are borderenergetic, L-borderenergetic or Q-borderenergetic.

Lemma 2.3. *If K_n is the complete graph of order n then*

$$\mathscr{E}(K_n) = \mathscr{LE}(K_n) = \mathscr{LE}^+(K_n) = 2(n-1).$$

Theorem 2.4. *If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, p is prime and $n \geq 3$, then $\Lambda(G)$ is neither hyperenergetic nor L-hyperenergetic nor Q-hyperenergetic. Also $\Lambda(G)$ is neither borderenergetic nor L-borderenergetic nor Q-borderenergetic.*

Proof. By Theorem 2.1 we have

$$\Lambda(G) = (p+1)K_{p^{n-3}(p-1)}.$$

Therefore, $|V(\Lambda(G))| = p^{n-3}(p^2-1)$ and hence by the Lemma 2.3

$$\mathscr{E}(K_{|V(\Lambda(G))|}) = \mathscr{L}\mathscr{E}^+(K_{|V(\Lambda(G))|}) = \mathscr{L}\mathscr{E}(K_{|V(\Lambda(G))|}) = 2(p^{n-3}(p^2-1)-1).$$

Now,

$$\mathscr{E}(K_n) - \mathscr{E}(\Lambda(G)) = 2(p^{n-3}(p^2-1)-1) - 2(p+1)(p^{n-3}(p-1)-1) = 2p > 0.$$

Similarly,

$$\mathscr{L}\mathscr{E}(K_n) - \mathscr{L}\mathscr{E}(\Lambda(G)) = 2p > 0$$

and

$$\mathscr{L}\mathscr{E}^+(K_n) - \mathscr{L}\mathscr{E}^+(\Lambda(G)) = 2p > 0.$$

Therefore,

$$\mathscr{E}(K_n) > \mathscr{E}(\Lambda(G)),$$

$$\mathscr{L}\mathscr{E}(K_n) > \mathscr{L}\mathscr{E}(\Lambda(G))$$

and

$$\mathscr{L}\mathscr{E}^+(K_n) > \mathscr{L}\mathscr{E}^+(\Lambda(G)).$$

Hence the result is true. □

The smallest non-negative integer l is called the *genus* of a graph Λ if Λ can be embedded on the surface obtained by attaching l handles to a sphere. We write $\gamma(\Lambda)$ to denote the genus of Λ. It is easy to observe that, if Λ_o is a sub-graph of Λ then $\gamma(\Lambda_o) \leq \gamma(\Lambda)$. Let K_n be the complete graph on n vertices and mK_n the disjoint union of m copies of K_n. then, by [13, Theorem 6-38], we have

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \text{ if } n \geq 3. \tag{2.1}$$

A graph Λ is called *planar, toroidal, double-toroidal* and *triple-toroidal* if $\gamma(\Lambda) = 0, 1, 2$ and 3 respectively.

Lemma 2.5. [6, Corollary 2] *If Λ is the disjoint union of K_m and K_n, then $\gamma(\Lambda) = \gamma(K_m) + \gamma(K_n)$.*

Theorem 2.6. *If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, p is prime and $n \geq 3$, then $\Lambda(G)$ is planar if $p = 2, n = 3,4,5; p = 3, n = 3; p = 5, n = 3$.*

Proof. By Theorem 2.1 we have

$$\Lambda(G) = (p+1)K_{p^{n-3}(p-1)}.$$

For $p = 2$ we have $\Lambda(G) = 3K_{2^{n-3}}$, hence for $n = 3,4,5$ we get $\Lambda(G)$ as $3K_1, 3K_2, 3K_4$ respectively. Therefore $\Lambda(G)$ is planar. Again for $p = 3$ we have $\Lambda(G) = 4K_{2.3^{n-3}}$, hence for $n = 3$ we get $\Lambda(G)$ as $4K_2$. Therefore $\Lambda(G)$ is planar. Also for $p = 5$ we have $\Lambda(G) = 6K_{4.5^{n-3}}$, hence for $n = 3$ we get $\Lambda(G)$ as $6K_4$. Therefore $\Lambda(G)$ is planar. □

Theorem 2.7. *If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, p is prime and $n \geq 3$, then $\Lambda(G)$ is neither toroidal nor double-toroidal nor triple-toroidal.*

Proof. By Theorem 2.6, we have, $\Lambda(G)$ is planar if $p = 2, n = 3,4,5; p = 3, n = 3; p = 5, n = 3$. Now we consider the following cases.

Case 1. If $p = 2; n \geq 6$ then $\Lambda(G) = 3K_{2^{n-3}}$. Therefore $\Lambda(G)$ has a sub-graph $3K_8$. Since by (2.1) and Lemma 2.5, $\gamma(3K_8) = 6$, therefore $\gamma(\Lambda(G)) \geq 6$. Hence $\Lambda(G)$ is neither toroidal nor double-toroidal nor triple-toroidal.

Case 2. If $p = 3; n \geq 4$ then $\Lambda(G) = 4K_{2.3^{n-3}}$. Therefore $\Lambda(G)$ has a sub-graph $4K_6$. Since by (2.1) and Lemma 2.5, $\gamma(3K_8) = 4$, therefore $\gamma(\Lambda(G)) \geq 4$. Hence $\Lambda(G)$ is neither toroidal nor double-toroidal nor triple-toroidal.

Case 3. If $p = 5; n \geq 4$ then $\Lambda(G) = 6K_{4.5^{n-3}}$. Therefore $\Lambda(G)$ has a sub-graph $6K_{20}$. Since by (2.1) and Lemma 2.5, $\gamma(6K_{20}) = 138$, therefore $\gamma(\Lambda(G)) \geq 138$. Hence $\Lambda(G)$ is neither toroidal nor double-toroidal nor triple-toroidal.

Case 4. If $p \geq 7; n \geq 3$ then $\Lambda(G) = (p+1)K_{p^{n-3}(p-1)}$. Therefore $\Lambda(G)$ has a sub-graph $8K_6$. Since by (2.1) and Lemma 2.5, $\gamma(8K_6) = 8$, therefore $\gamma(\Lambda(G)) \geq 8$. Hence $\Lambda(G)$ is neither toroidal nor double-toroidal nor triple-toroidal. □

Theorem 2.8. *If G is a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, p is prime and $n \geq 3$, then*

$$\gamma(\Lambda(G)) = \begin{cases} 0, & \text{if } p=2, n=3,4,5; p=3, n=3; \\ & p=5, n=3 \\ (p+1)\left\lceil \frac{(p^{n-3}(p-1)-3)(p^{n-3}(p-1)-4)}{12} \right\rceil, & \text{otherwise.} \end{cases}$$

The proof of Theorem 2.8 can be done by using (2.1) and Theorem 2.6.

Finally, in Theorem 2.4, various energies of $\Lambda(G)$ and $K_{|V(G)|}$ are compared and obtained that $\Lambda(G)$ is neither hyperenergetic nor L-hyperenergetic nor Q-hyperenergetic. Also $\Lambda(G)$ is neither borderenergetic nor L-borderenergetic nor Q-borderenergetic. We find the condition for the graph to be planar. We conclude this paper by proving the fact that the graph is neither toroidal nor double-toroidal nor triple-toroidal.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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