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FIXED POINTS FOR WEAK CONTRACTION INVOLVING CUBIC TERMS OF DISTANCE FUNCTION

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Abstract: The purpose of this paper is to introduce a new weak contraction that involves cubic terms of distance function and using this weak contraction, we prove common fixed point theorems for compatible mappings and its variants.

Keywords: compatible mappings and its variants; common fixed point; weak contraction.

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1. INTRODUCTION AND PRELIMINARIES

The Banach Contraction Principle is popularly known as the Banach fixed point theorem which states that every contraction map on a complete metric space has a unique fixed point, i.e., Let (X, d) be a complete metric space. Let $T: X \to X$ be a map satisfying $d\big(T(x), T(y)\big) \leq k\big(d(x, y)\big), 0 \leq k < 1$, for all $x, y \in X$. Then T has a unique fixed point. This principle is widely used as a basic tool in solving existence problems in pure and applied sciences that

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include various areas such as numerical analysis, approximation theory, game theory, economics, physics, chemistry and computer science etc. It plays a very crucial role in nonlinear analysis. There has been significant interest of researchers to generalize this result for a pair/pairs of mappings satisfying various types of contractive/contraction conditions in various abstract metric spaces. Moreover, this fixed point can be explicitly obtained as a limit of repeated iteration of the mapping, initiating at any point of the underlying space. Obviously, every contraction is a continuous function but not conversely. Many mathematicians proved several fixed point theorems to explore some new contraction-type mappings in order to generalize the classical Banach Contraction Principle.

In 1969, Boyd and Wong [1] replaced the constant k in Banach contractive condition by an upper semi-continuous function as follows:

Let (X, d) be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be upper semi continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$. If $T : X \to X$ satisfies $d(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$, then it has a unique fixed point $x \in X$ and $\{T^n x\}$ converges to x for all $x \in X$.

Fixed point theorems basically involve sufficient conditions for the existence of fixed points. Therefore, the central concerns in fixed point theory are to find a minimal set of sufficient conditions which ensures the guarantee of fixed points or common fixed points. It was a turning point in the fixed point theory literature when the notion of commutativity mappings was used by Jungck [2] to obtain a common fixed point theorem for a pair of mappings.

A common fixed point result generally involves conditions on commutativity, continuity, and contraction along with a suitable condition on the containment of range of one mapping into the range of the other. One is always required to improve one or more of these conditions in order to prove a new common fixed point theorem.

The first ever attempt to relax the commutativity of mappings to a smaller subset of the domain of mappings was initiated by Sessa [11] who in 1982 gave the notion of weak commutativity. One can notice that the notion of weak commutativity is a point property, while the notion of compatibility is an iterate of sequence. Two self mappings f and g of a metric space (X, d) are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in X.

In 1986, Jungck [3] introduced the notion of compatible mappings as follows:

Definition 1.1. Two self maps f and g of a metric space (X, d) are called compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t ∈ X.

In 1993, Jungck *et al.* [5] introduced the concept of compatible mappings of type (*A*) as follows:

Definition 1.2. Two self mappings *f* and g on a metric space (X, d) are called compatible of type (A) if

$$\lim_{n\to\infty} d(ffx_n, gfx_n) = 0 \text{ and } \lim_{n\to\infty} d(ggx_n, fgx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1995, Pathak and Khan [7] introduced the notion of compatible mappings of type (*B*) as follows:

Definition 1.3. Two self mappings *f* and *g* on a metric space (*X*, *d*) are called compatible of type (B) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n)\right]$$

and

$$\lim_{n\to\infty} d(gfx_n, ggx_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, ggx_n)\right]$$

whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1998, Pathak *et al.* [9] introduced the notion of compatible mappings of type (C) as follows:

Definition 1.4. Two self mappings f and g on a metric space (X, d) are called compatible of type (C) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n) + \lim_{n\to\infty} d(ft, ggx_n)\right]$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \le \frac{1}{3}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, ggx_n) + \lim_{n\to\infty} d(gt, ffx_n)\right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t ∈ X.

In 1995, Pathak *et al.* [8] introduced the notion of compatible mappings of type (P) as follows:

Definition 1.5. Two self mappings f and g on a metric space (X, d) are called compatible of type (P) if

$$\lim_{n\to\infty} d(ffx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t ∈ X.

Now we highlight the properties of compatible mappings and its variants.

Proposition 1.6. [3] Let S and T be compatible mappings of a metric space (X, d) into itself. If $St = Tt$ for some t ∈ X, then $STt = SSt = TTt = TSt$.

Proposition 1.7. [3] Let S and T be compatible mappings of a metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ∈ X. Then

(i) $\lim_{n\to\infty} TSx_n = St$ if S is continuous at t.

(ii) $\lim_{n\to\infty} STx_n = Tt$ if T is continuous at t.

(iii) $STt = TSt$ and $St = Tt$ if S and T are continuous at t.

Proposition 1.8. [5] Let S and T be compatible mappings of type (A) of a metric space (X, d) into itself. If one of S and T is continuous, then S and T are compatible.

Proposition 1.9.[7] Let S and T be compatible mappings of type (B) of a metric space (X, d) into itself. If St = Tt for some t ∈ X, then $STt = SSt = TTt = TSt$.

Proposition 1.10. [7] Let S and T be compatible mappings of type (B) of a metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t ∈ X. Then

(i) $\lim_{n\to\infty} TTx_n = St$ if S is continuous at t;

(ii) $\lim_{n\to\infty} SSx_n = Tt$ if T is continuous at t;

(iii) $STt = TSt$ and $St = Tt$ if S and T are continuous at t.

Remark 1.11. In Proposition 1.9, if one assumes that S and T be compatible mappings of type (C) or of type (P) instead of type (B), the conclusion of the Proposition 1.9 remains true.

Remark 1.12. In Proposition 1.10, if one assumes that S and T be compatible mappings of type (C) or of type (P) instead of type (B), the conclusion of the proposition 1.10 remains true.

Let (X, d) be a metric space. A self map T on X is said to be weak contraction if for each x, y ∈ X, there exists a function ϕ: [0, ∞) → [0, ∞) such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \text{ with } \phi(t) > 0 \text{ and } \phi(0) = 0.$$

In this paper, we introduce a new weak contraction that involves cubic terms of distance function:

Let f, g, S and T be four mappings of a complete metric space (X, d) into itself satisfying the following condition:

$$(C1)\ d^3(fx, gy) \leq p\ max\{[d^2(Sx, fx)d(Ty, gy) + d(Sx, fx)d^2(Ty, gy)]/2,$$
$$d(Sx, fx)d(Sx, gy)d(Ty, fx), d(Sx, gy)d(Ty, fx)d(Ty, gy)\}$$
$$-\phi\big(m(Sx, Ty)\big),$$

for all $x, y \in X$, where

$$m(Sx, Ty) = \max\{d^2(Sx, Ty), d(Sx, fx)d(Ty, gy), d(Sx, gy)d(Ty, fx),$$
$$[d(Sx, fx)d(Sx, gy) + d(Ty, fx)d(Ty, gy)]/2\}$$

and p is a real number satisfying $0 < p < 1$. Further, $\phi: [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0\ for\ each\ t > 0$.

2. FIXED POINTS FOR COMPATIBLE MAPPINGS AND ITS VARIANTS

First we prove the following theorem for compatible mappings.

Theorem 2.1. Let f, g, S and T be four mappings of a complete metric space (X, d) into itself satisfying $(C1)$ and the following:

$(C2)$ $f(X) \subset T(X), g(X) \subset S(X)$

$(C3)$ One of $f, g, S\ and\ T$ is continuous.

Assume that the pairs (f, S) and (g, T) are compatible, then f, g, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and using $(C2)$, we can find $f(x_0) = T(x_1) = y_0$, for this x_1 we can find $x_2 \in X$ such that $g(x_1) = S(x_2) = y_1$. Proceeding in this fashion, we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n} = f(x_{2n}) = T(x_{2n+1}), y_{2n+1} = g(x_{2n+1}) = S(x_{2n+2}) \; for\ each\ n \geq 0. \tag{2.1}$$

Let $\beta_{2n} = d(y_{2n}, y_{2n+1})$. First, we prove that $\{\beta_{2n}\}$ is non increasing sequence and converges to zero.

Case I: If n is even, on putting $x = x_{2n}$ and $y = x_{2n+1}$ in (C1), we get

$$\begin{aligned} d^3(fx_{2n}, gx_{2n+1}) \leq p\ max\{&[d^2(Sx_{2n}, fx_{2n})d(Tx_{2n+1}, gx_{2n+1}) \\ &+d(Sx_{2n}, fx_{2n})d^2(Tx_{2n+1}, gx_{2n+1})]/2, \\ &d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fx_{2n}), \\ &d(Sx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fx_{2n})d(Tx_{2n+1}, gx_{2n+1})\} \\ &-\phi\big(m(Sx_{2n}, Tx_{2n+1})\big), \end{aligned}$$

where

$$\begin{aligned} m(Sx_{2n}, Tx_{2n+1}) = \max\{&d^2(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n})d(Tx_{2n+1}, gx_{2n+1}), \\ &d(Sx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fx_{2n}), \\ &[d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gx_{2n+1}) + d(Tx_{2n+1}, fx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}. \end{aligned}$$

Using (2.1), we have

$$\begin{aligned} d^3(y_{2n}, y_{2n+1}) \leq p\ max\{&[d^2(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}) \\ &+d(y_{2n-1}, y_{2n})d^2(y_{2n}, y_{2n+1})]/2, \\ &d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n}), \\ &d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})d(y_{2n}, y_{2n+1})\} \\ &-\phi\big(m(y_{2n-1}, y_{2n})\big), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} m(y_{2n-1}, y_{2n}) = \max\{&d^2(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}), \\ &d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n}), \end{aligned}$$

$$[d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})d(y_{2n}, y_{2n+1})]/2\}.$$

On using $\beta_{2n} = d(y_{2n}, y_{2n+1})$ in (2.2), we get

$$\beta_{2n}^3 \leq p \max\{[\beta_{2n-1}^2\beta_{2n} + \beta_{2n-1}\beta_{2n}^2]/2,0,0\} - \phi\big(m(y_{2n-1}, y_{2n})\big) \tag{2.3}$$

where $m(y_{2n-1}, y_{2n}) = \max\{\beta_{2n-1}^2, \beta_{2n-1}\beta_{2n}, 0, [\beta_{2n-1}d(y_{2n-1}, y_{2n+1}) + 0]/2\}$.

By using triangular inequality and property of ϕ, we get

$$d(y_{2n-1}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) = \beta_{2n-1} + \beta_{2n}$$

and $m(y_{2n-1}, y_{2n}) \leq \max\{\beta_{2n-1}^2, \beta_{2n-1}\beta_{2n}, 0, [\beta_{2n-1}(\beta_{2n-1} + \beta_{2n})]/2\}$.

If $\beta_{2n-1} < \beta_{2n}$, then (2.3) reduces to

$\beta_{2n}^3 \leq p\beta_{2n}^3 - \phi(\beta_{2n}^2)$, which is a contradiction, since $0< p < 1$.

Hence $\beta_{2n} \leq \beta_{2n-1}$.

In a similar way, if n is odd, then we can obtain $\beta_{2n+1} \leq \beta_{2n}$.

It follows that the sequence $\{\beta_{2n}\}$ is decreasing.

Let $\lim_{n\to\infty} \beta_{2n} = r$, for some $r \geq 0$.

Suppose $r > 0$; then from inequality (C1), we have

$$\begin{aligned} d^3(fx_{2n}, gx_{2n+1}) \leq p\, max\{&[d^2(Sx_{2n}, fx_{2n})d(Tx_{2n+1}, gx_{2n+1}) \\ &+d(Sx_{2n}, fx_{2n})d^2(Tx_{2n+1}, gx_{2n+1})]/2, \\ &d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fx_{2n}), \\ &d(Sx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fx_{2n})d(Tx_{2n+1}, gx_{2n+1})\} \\ &-\phi\big(m(Sx_{2n}, Tx_{2n+1})\big), \end{aligned}$$

where

$$\begin{aligned} m(Sx_{2n}, Tx_{2n+1}) = \max\{&d^2(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n})d(Tx_{2n+1}, gx_{2n+1}), \\ &d(Sx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fx_{2n}), \\ &[d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gx_{2n+1}) + d(Tx_{2n+1}, fx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}. \end{aligned}$$

Now by using (2.1), triangular inequality and property of ϕ and proceed limit as $n \to \infty$, we get

$$r^3 \leq pr^3 - \phi(r^2) < pr^3 \text{ , i.e., } p > 1,$$

which is a contradiction, therefore, we get $r = 0$. Thus

$$\lim_{n\to\infty} \beta_{2n} = \lim_{n\to\infty} d(y_{2n}, y_{2n-1}) = r = 0.$$

Now we show that $\{y_n\}$ is a Cauchy sequence.

If possible, let $\{y_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$, for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k with $n(k) > m(k) \geq k$, we have

$$d(y_{m(k)}, y_{n(k)}) \geq \epsilon,$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest positive integer with $n(k) > m(k)$ and satisfying $d(y_{m(k)}, y_{n(k)}) \geq \epsilon$, we have

$$d\left(y_{m(k)}, y_{n(k)-1}\right) < \epsilon.$$

Now, $\epsilon \leq d\left(y_{m(k)}, y_{n(k)}\right) \leq d\left(y_{m(k)}, y_{n(k)-1}\right) + d(y_{n(k)-1}, y_{n(k)})$.

Letting $k \to \infty$, we get $\lim_{k\to\infty} d\left(y_{m(k)}, y_{n(k)}\right) = \epsilon$

Now from the triangular inequality, we have,

$\left|d\left(y_{n(k)}, y_{m(k)+1}\right) - d\left(y_{m(k)}, y_{n(k)}\right)\right| \leq d\left(y_{m(k)}, y_{m(k)+1}\right)$.

Taking limits as $k \to \infty$ we have

$\lim_{k\to\infty} d\left(y_{n(k)}, y_{m(k)+1}\right) = \epsilon$.

Again, using triangular inequality, we have

$\left|d\left(y_{m(k)}, y_{n(k)+1}\right) - d\left(y_{m(k)}, y_{n(k)}\right)\right| \leq d\left(y_{n(k)}, y_{n(k)+1}\right)$.

Proceeding limits as $k \to \infty$ we get

$\lim_{k\to\infty} d\left(y_{m(k)}, y_{n(k)+1}\right) = \epsilon$.

Similarly, we have

$\left|d\left(y_{m(k)+1}, y_{n(k)+1}\right) - d\left(y_{m(k)}, y_{n(k)}\right)\right| \leq d\left(y_{m(k)}, y_{m(k)+1}\right) + d\left(y_{n(k)}, y_{n(k)+1}\right)$.

Taking limit as $k \to \infty$ in the above inequality, we have

$\lim_{k\to\infty} d\left(y_{n(k)+1}, y_{m(k)+1}\right) = \epsilon$.

On putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (C1), we get

$$d^3\left(fx_{m(k)}, gx_{n(k)}\right) \leq p\, max\{[d^2\left(Sx_{m(k)}, fx_{m(k)}\right)d\left(Tx_{n(k)}, gx_{n(k)}\right)$$

$$+d(Sx_{m(k)}, fx_{m(k)})d^2(Tx_{n(k)}, gx_{n(k)})]/2,$$
$$d(Sx_{m(k)}, fx_{m(k)})d(Sx_{m(k)}, gx_{n(k)})d(Tx_{n(k)}, fx_{m(k)}),$$
$$d(Sx_{m(k)}, gx_{n(k)})d(Tx_{n(k)}, fx_{m(k)})d(Tx_{n(k)}, gx_{n(k)})\}$$
$$-\phi\left(m(Sx_{m(k)}, Tx_{n(k)})\right),$$

where

$$m(Sx_{m(k)}, Tx_{n(k)}) = \max\{d^2(Sx_{m(k)}, Tx_{n(k)}), d(Sx_{m(k)}, fx_{m(k)})d(Tx_{n(k)}, gx_{n(k)}),$$
$$d(Sx_{m(k)}, gx_{n(k)})d(Tx_{n(k)}, fx_{m(k)}),$$
$$[d(Sx_{m(k)}, fx_{m(k)})d(Sx_{m(k)}, gx_{n(k)}) + d(Tx_{n(k)}, fx_{m(k)})d(Tx_{n(k)}, gx_{n(k)})]/2\}.$$

i.e.,

$$d^3(y_{m(k)}, y_{n(k)}) \leq p\max\{[d^2(y_{m(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)})$$
$$+d(y_{m(k)-1}, y_{m(k)})d^2(y_{n(k)-1}, y_{n(k)})]/2$$
$$d(y_{m(k)-1}, y_{m(k)})d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)}),$$
$$d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)})\}$$
$$-\phi(m(Sx_{m(k)}, Tx_{n(k)})),$$

where

$$m(Sx_{m(k)}, Tx_{n(k)}) = \max\{d^2(y_{m(k)-1}, y_{n(k)-1}), d(y_{m(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}),$$
$$d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)}),$$
$$[d(y_{m(k)-1}, y_{m(k)})d(y_{m(k)-1}, y_{n(k)}) + d(y_{n(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)})]/2\}.$$

Letting $k \to \infty$, we get

$$\epsilon^3 \leq pmax\left\{\frac{1}{2}[0+0], 0,0\right\} - \phi(\epsilon^2)$$
$$= -\phi(\epsilon^2),$$ a contradiction.

Thus $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{fx_{2n}\}, \{Sx_{2n}\}, \{gx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ of the sequence $\{y_n\}$ also converges to z.

Now suppose that S is continuous. Then $\{SSx_{2n}\}, \{Sfx_{2n}\}$ converges to Sz as $n \to \infty$. Since (f, S) are compatible on X, it follows from Proposition 1.7 that $\{fSx_{2n}\}$ converges to Sz as $n \to \infty$.

We claim that $z = Sz$. For this put $x = Sx_{2n}$ and $y = x_{2n+1}$ in condition (C1), we have

$$d^3(fSx_{2n}, gx_{2n+1}) \leq p \max\{[d^2(SSx_{2n}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1}) + d(SSx_{2n}, fSx_{2n})d^2(Tx_{2n+1}, gx_{2n+1})]/2, d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n}), d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})\} - \phi(m(SSx_{2n}, Tx_{2n+1})),$$

where

$$m(SSx_{2n}, Tx_{2n+1}) = \max\{d^2(SSx_{2n}, Tx_{2n+1}), d(SSx_{2n}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1}), d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n}), [d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gx_{2n+1}) + d(Tx_{2n+1}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(Sz, z) \leq p \max\{[d^2(Sz, Sz)d(z, z) + d(Sz, Sz)d^2(z, z)]/2, d(Sz, Sz)d(Sz, z)d(z, Sz), d(Sz, z)d(z, Sz)d(z, z)\} - \phi\big(m(Sz, z)\big),$$

where

$$m(Sz, z) = \max\{d^2(Sz, z), d(Sz, Sz)d(z, z), d(Sz, z)d(z, Sz), [d(Sz, Sz)d(Sz, z) + d(z, Sz)d(z, z)]/2\} = d^2(Sz, z).$$

Hence, we have

$$d^3(Sz, z) \leq p \max\{0,0,0\} - \phi(d^2(Sz, z)).$$

Thus, we get $d(Sz, z) = 0$ and hence $Sz = z$.

Next, we will show that $fz = z$. For this put x = z and $y = x_{2n+1}$ in (C1),

$$d^3(fz, gx_{2n+1}) \leq p \max\{[d^2(Sz, fz)d(Tx_{2n+1}, gx_{2n+1}) + d(Sz, fz)d^2(Tx_{2n+1}, gx_{2n+1})]/2,$$

$$d(Sz,fz)d(Sz,gx_{2n+1})d(Tx_{2n+1},fz),$$
$$d(Sz,gx_{2n+1})d(Tx_{2n+1},fz)d(Tx_{2n+1},gx_{2n+1})\}$$
$$-\phi(m(Sz,Tx_{2n+1})),$$

where

$$m(Sz,Tx_{2n+1}) = max\{d^2(Sz,Tx_{2n+1}), d(Sz,fz)d(Tx_{2n+1},gx_{2n+1}),$$
$$d(Sz,gx_{2n+1})d(Tx_{2n+1},fz),$$
$$[d(Sz,fz)d(Sz,gx_{2n+1}) + d(Tx_{2n+1},fz)d(Tx_{2n+1},gx_{2n+1})]/2\}.$$

Letting $n \to \infty$, we have

$$d^3(fz,z) \leq p\, max \begin{Bmatrix} [d^2(Sz,fz)d(z,z) + d(Sz,fz)d^2(z,z)]/2, \\ d(Sz,fz)d(Sz,z)d(z,fz), \\ d(Sz,z)d(z,fz)d(z,z) \end{Bmatrix}$$
$$-\phi\big(m(Sz,z)\big),$$

where

$$m(Sz,z) = \max\{d^2(Sz,z), d(Sz,fz)d(z,z), d(Sz,z)d(z,fz),$$
$$[d(Sz,fz)d(Sz,z) + d(z,fz)d(z,z)]/2\} = 0.$$

Hence, we get

$$d^3(fz,z) \leq p \max\{[0+0]/2,0,0\} - \phi(0).$$

Thus, we get, $d^3(fz,z) = 0$. This implies that $fz = z$. Since $f(X) \subset T(X)$ and hence there exists point $u \in X$ such that $z = fz = Tu$.

We claim that $z = gu$. For this, we put x = z and y = u in (C1), we get

$$d^3(fz,gu) \leq p\, max \begin{Bmatrix} [d^2(Sz,fz)d(Tu,gu) + d(Sz,fz)d^2(Tu,gu)]/2, \\ d(Sz,fz)d(Sz,gu)d(Tu,fz), \\ d(Sz,gu)d(Tu,fz)d(Tu,gu) \end{Bmatrix}$$
$$-\phi\big(m(Sz,Tu)\big),$$

where

$$m(Sz,Tu) = max \begin{Bmatrix} d^2(Sz,Tu), d(Sz,fz)d(Tu,gu), \\ d(Sz,gu)d(Tu,fz), \\ [d(Sz,fz)d(Sz,gu) + d(Tu,fz)d(Tu,gu)]/2 \end{Bmatrix}$$
$$= \max\{d^2(z,z), d(z,z)d(z,gu), d(z,gu)d(z,z),$$

$$[d(z,z)d(z,gu)+d(z,z)d(z,gu)]/2\}=0.$$

Hence, we have

$$d^3(z,gu)\leq p\,max\left\{\begin{matrix}[d^2(z,z)d(z,gu)+d(z,z)d^2(z,gu)]/2,\\ d(z,z)d(z,gu)d(z,z),d(z,gu)d(z,z)d(z,gu)\end{matrix}\right\}-\phi(0),$$

which implies that $d^3(z,gu)=0$ and hence $z=gu$. Since (g,T) is compatible in X and $Tu=gu=z$, by proposition 1.6, we have $Tgu=gTu$ and hence $Tz=Tgu=gTu=gz$. Also, from (C1) we have

$$\begin{aligned}d^3(z,Tz)=d^3(fz,gz)\leq p\max\{&[d^2(Sz,fz)d(Tz,gz)+d(Sz,fz)d^2(Tz,gz)]/2\\ &d(Sz,fz)d(Sz,gz)d(Tz,fz),d(Sz,gz)d(Tz,fz)d(Tz,gz)\}\\ &-\phi(m(Sz,Tz),\end{aligned}$$

where

$$\begin{aligned}m(Sz,Tz)=\max\{&d^2(Sz,Tz),d(Sz,fz)d(Tz,gz),d(Sz,gz)d(Tz,fz),\\ &[d(Sz,fz)d(Sz,gz)+d(Tz,fz)d(Tz,gz)]/2\}\\ =&\,d^2(z,Tz).\end{aligned}$$

Hence, we get

$$d^3(z,Tz)\leq p\max\{[0+0]/2,0,0\}-\phi(d^2(z,Tz)),$$

which implies that $z=Tz$. Hence, $z=Tz=gz=Sz=fz$. Therefore, z is a common fixed point of f,g,S and T.

Similarly, we can also complete the proof when T is continuous.

Next suppose that f is continuous. Then $\{ffx_{2n}\}$ and $\{fSx_{2n}\}$ converges to fz as $n\to\infty$. Since the mappings f and S are compatible on X, it follows from Proposition 1.7 that $\{Sfx_{2n}\}$ converges to fz as $n\to\infty$.

Now we claim that $z=fz$. For this we put $x=fx_{2n}$ and $y=x_{2n+1}$ in (C1), we get

$$\begin{aligned}&d^3(ffx_{2n},gx_{2n+1})\\ &\leq p\max\{[d^2(Sfx_{2n},ffx_{2n})d(Tx_{2n+1},gx_{2n+1})\\ &+d(Sfx_{2n},ffx_{2n})d^2(Tx_{2n+1},gx_{2n+1})]/2,\\ &d(Sfx_{2n},ffx_{2n})d(Sfx_{2n},gx_{2n+1})d(Tx_{2n+1},ffx_{2n}),\end{aligned}$$

$$d(Sfx_{2n}, gx_{2n+1})d(Tx_{2n+1}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1})\}$$
$$-\phi\big(m(Sfx_{2n}, Tx_{2n+1})\big),$$

where

$$m(Sfx_{2n}, Tx_{2n+1}) = \max\{d^2(Sfx_{2n}, Tx_{2n+1}), d(Sfx_{2n}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1}),$$
$$d(Sfx_{2n}, gx_{2n+1})d(Tx_{2n+1}, ffx_{2n}),$$
$$[d(Sfx_{2n}, ffx_{2n})d(Sfx_{2n}, gx_{2n+1})$$
$$+d(Tx_{2n+1}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(fz, z) \leq p \max\{[0+0]/2, 0, 0\} - \phi(m(fz, z)),$$

where $m(fz, z) = d^2(fz, z)$.

This implies $d^3(fz, z) \leq -\phi(d^2(fz, z))$.

Thus, we get $d(fz, z) = 0$, which implies that $fz = z$. Since $f(X) \subset T(X)$, there exists a point $v \in X$ such that $z = fz = Tv$.

We claim that $z = gv$. For this, we put $x = fx_{2n}$ and $y = v$ in (C1), we get

$$d^3(ffx_{2n}, gv)$$
$$\leq p \max\{[d^2(Sfx_{2n}, ffx_{2n})d(Tv, gv)$$
$$+ d(Sfx_{2n}, ffx_{2n})d^2(Tv, gv)]/2,$$
$$d(Sfx_{2n}, ffx_{2n})d(Sfx_{2n}, gv)d(Tv, ffx_{2n}),$$
$$d(Sfx_{2n}, gv)d(Tv, ffx_{2n})d(Tv, gv)\}$$
$$-\phi\big(m(Sfx_{2n}, Tv)\big),$$

where

$$m(Sfx_{2n}, Tv) = \max\{d^2(Sfx_{2n}, Tv), d(Sfx_{2n}, ffx_{2n})d(Tv, gv),$$
$$d(Sfx_{2n}, gv)d(Tv, ffx_{2n}),$$
$$[d(Sfx_{2n}, ffx_{2n})d(Sfx_{2n}, gv) + d(Tv, ffx_{2n})d(Tv, gv)]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(z, gv) \leq p \max\{[d^2(z, z)d(z, gv) + d(z, z)d^2(z, gv)]/2,$$
$$d(z, z)d(z, gv)d(z, z), d(z, gv)d(z, z)d(z, gv)\} - \phi(m(z, Tv)),$$

where

$$m(z,Tv) = \max\{d^2(z,z), d(z,z)d(z,gv), d(z,gv)d(z,z),$$
$$[d(z,z)d(z,gv) + d(z,z)d(z,gv)]/2\} = 0.$$

which implies that $d^3(z,gv) = 0$ and hence $z = gv$. Since (g,T) is compatible on X and $Tv = gv = z$, by proposition 1.6, we have $Tgv = gTv$ and hence $Tz = Tgv = gTv = gz$.

We claim that $z = gz$. For this we put $x = x_{2n}$ and y = z in (C1).

$$d^3(fx_{2n}, gz) \leq p \max\{[d^2(Sx_{2n}, fx_{2n})d(Tz,gz) + d(Sx_{2n}, fx_{2n})d^2(Tz,gz)]/2,$$
$$d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gz)d(Tz, fx_{2n}),$$
$$d(Sx_{2n}, gz)d(Tz, fx_{2n})d(Tz,gz)\}$$
$$-\phi\big(m(Sx_{2n}, Tz)\big),$$

where

$$m(Sx_{2n}, Tz) = \max\{d^2(Sx_{2n}, Tz), d(Sx_{2n}, fx_{2n})d(Tz,gz),$$
$$d(Sx_{2n}, gz)d(Tz, fx_{2n}),$$
$$[d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gz) + d(Tz, fx_{2n})d(Tz,gz)]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(z,gz) \leq p \max\{[0+0]/2, 0, 0\} - \phi(m(z,gz)).$$

where $m(z,Tz) = d^2(z,gz)$.

This implies $d^3(z,gz) \leq -\phi(d^2(z,gz))$.

This gives $z = gz$. Since $g(X) \subset S(X)$, there exists a point $w \in X$ such that $z = gz = Sw$.

We claim that $z = fw$. For this, we put x = w and y = z in (C1), we get

$$d^3(fw,gz) \leq p \max\{[d^2(Sw,fw)d(Tz,gz) + d(Sw,fw)d^2(Tz,gz)]/2,$$
$$d(Sw,fw)d(Sw,gz)d(Tz,fw), d(Sw,gz)d(Tz,fw)d(Tz,gz)\}$$
$$-\phi\big(m(Sw,Tz)\big),$$

where

$$m(Sw,Tz) = \max\{d^2(Sw,Tz), d(Sw,fw)d(Tz,gz), d(Sw,gz)d(Tz,fw),$$
$$[d(Sw,fw)d(Sw,gz) + d(Tz,fw)d(Tz,gz)]/2\}$$
$$= \max\{d^2(z,z), d(z,fw)d(z,z), d(z,z)d(z,fw),$$

$$[d(z, fw)d(z, z) + d(z, fw)d(z, z)]/2\}$$

$$= 0.$$

Hence we have,

$$d^3(fw, z) \leq p \max\{[d^2(z, fw)d(z, z) + d(z, fw)d^2(z, z)]/2,$$
$$d(z, fw)d(z, z)d(z, fw), d(z, z)d(z, fw)d(z, z)\} - \phi(0).$$

which implies that $fw = z$. Since (f, S) is compatible on X, $fw = Sw = z$, by Proposition 1.6, we have $fSw = Sfw$ and hence $Sz = Sfw = fSw = fz$. That is, $z = Sz = fz = Tz = gz$. Therefore, z is a fixed point of f, g, S and T.

Similarly, we can complete the proof when g is continuous.

Finally, in order to prove uniqueness, suppose that z and w $(z \neq w)$ are two common fixed points of f, g, S and T.

Put x = z and y = w in (C1), we have

$$d^3(z, w) = d^3(fz, gw) \leq p \max\{0,0,0\} - \phi(m(Sz, Tw))$$
$$= -\phi\left(d^2(z, w)\right).$$

Thus, we have $d(z, w) = 0$ and hence z = w. Therefore, f, g, S and T have a unique common fixed point in X. This completes the proof.

Now we give the following theorem for compatible mappings of type (A).

Theorem 2.2. Let f, g, S and T be four mappings of a complete metric space (X, d) into itself satisfying the conditions (C1)-(C3).

Assume that the pairs (f, S) and (g, T) are compatible of type (A), then f, g, S and T have a unique common fixed point in X.

Proof. Suppose that S is continuous on X. Since f and S are compatible of type (A). From Proposition 1.8, f and S are compatible and so the result easily follows from Theorem 2.1.

Similarly, if T is continuous and the pair (g, T) is compatible of type (A), then g and T are compatible and so the result easily follows from Theorem 2.1.

Also, we can get the same results when f or g is continuous. This completes the proof.

Next, we prove the following result for a weak contractive mapping satisfying a compatibility of type (B).

Theorem 2.3. Let f, g, S and T be four mappings of a complete metric space (X, d) into itself satisfying the conditions (C1)-(C3). Assume that the pairs (f, S) and (g, T) are compatible of type (B). Then f, g, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point and using $(C2)$ we can find $f(x_0) = T(x_1) = y_0$, for this x_1 we can find $x_2 \in X$ such that $g(x_1) = S(x_2) = y_1$. Proceeding in this way, we can construct a sequence $\{y_n\}$ such that

$$y_{2n} = f(x_{2n}) = T(x_{2n+1}), y_{2n+1} = g(x_{2n+1}) = S(x_{2n+2})\ for\ each\ n \geq 0.$$

From the proof of Theorem 2.1, $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{fx_{2n}\}$, $\{Sx_{2n}\}$, $\{gx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ also converge to the same point z.

Now suppose that f is continuous. Then $\{ffx_{2n}\}$ and $\{fSx_{2n}\}$ converges to fz as $n \to \infty$. Since the mappings f and S are compatible of type (B), it follows from the Proposition 1.10 that $\{SSx_{2n}\}$ converges to fz as $n \to \infty$.

Now we claim that $z = fz$. For this put $x = Sx_{2n}$ and $y = x_{2n+1}$ in (C1), we get

$$\begin{aligned}
& d^3(fSx_{2n}, gx_{2n+1}) \\
& \leq p \max\{[d^2(SSx_{2n}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1}) \\
& + d(SSx_{2n}, fSx_{2n})d^2(Tx_{2n+1}, gx_{2n+1})]/2, \\
& \quad d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n}), \\
& \quad d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})\} \\
& \quad -\phi\big(m(SSx_{2n}, Tx_{2n+1})\big),
\end{aligned}$$

where

$$\begin{aligned}
m(SSx_{2n}, Tx_{2n+1}) = \max\{& d^2(SSx_{2n}, Tx_{2n+1}), d(SSx_{2n}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1}), \\
& d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n}), \\
& [d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gx_{2n+1}) \\
& + d(Tx_{2n+1}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$d^3(fz,z) \le p \max\{[d^2(fz,fz)d(z,z) + d(fz,fz)d^2(z,z)]/2,$$
$$d(fz,fz)d(fz,z)d(z,fz), d(fz,z)d(z,fz)d(z,z)\} - \phi(m(fz,z)).$$

where

$$m(fz,z) = \max\{d^2(fz,z), d(fz,fz)d(z,z), d(fz,z)d(z,fz),$$
$$[d(fz,fz)d(fz,z) + d(z,fz)d(z,z)]/2\} = d^2(fz,z).$$

Hence, we get

$$d^3(fz,z) \le p \max\{[0+0]/2, 0, 0\} - \phi(d^2(fz,z)).$$

Thus, we get $d(fz,z) = 0$, which implies that $fz = z$. Since $f(X) \subset T(X)$, there exists a point $u \in X$ such that z = fz = Tu.

We claim that z = gu. For this, we put x = Sx_{2n} and y = u in (C1), we get

$$d^3(fSx_{2n}, gu) \le p \max\{[d^2(SSx_{2n}, fSx_{2n})d(Tu,gu)$$
$$+d(SSx_{2n}, fSx_{2n})d^2(Tu,gu)]/2,$$
$$d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gu)d(Tu, fSx_{2n}),$$
$$d(SSx_{2n}, gu)d(Tu, fSx_{2n})d(Tu,gu)\}$$
$$-\phi\big(m(SSx_{2n}, Tu)\big),$$

where

$$m(SSx_{2n}, Tu) = \max\{d^2(SSx_{2n}, Tu), d(SSx_{2n}, fSx_{2n})d(Tu,gu),$$
$$d(SSx_{2n}, gu)d(Tu, fSx_{2n}), [d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gu)$$
$$+d(Tu, fSx_{2n})d(Tu,gu)]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(fz,gu) = d^3(z,gu) \le p \max\{[d^2(z,z)d(z,gu) + d(z,z)d^2(z,gu)]/2,$$
$$d(z,z)d(z,gu)d(z,z), d(z,gu)d(z,z)d(z,gu)\} - \phi(m(z,Tu)),$$

where

$$m(z,Tu) = \max\{d^2(z,z), d(z,z)d(z,gu), d(z,gu)d(z,z),$$
$$[d(z,z)d(z,gu) + d(z,z)d(z,gu)]/2\} = 0.$$

Hence, we get

$$d^3(z, gu) \leq p \max\{[0+0]/2, 0, 0\} - \phi(0).$$

This implies that $d^3(z, gu) = 0$ and hence $z = gu = fz$. Since the pair (g, T) is compatible of type (B) and $gu = Tu = z$, by Proposition 1.9, we have $Tg\,u = gTu$ and hence $Tz = Tgu = gTu = gz$.

Now we claim that $z = gz$. For this we put $x = x_{2n}$ and $y = z$ in $(C1)$

$$\begin{aligned} d^3(fx_{2n}, gz) \leq p \max\{&[d^2(Sx_{2n}, fx_{2n})d(Tz, gz) \\ &+ d(Sx_{2n}, fx_{2n})d^2(Tz, gz)]/2, \\ &d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gz)d(Tz, fx_{2n}), \\ &d(Sx_{2n}, gz)d(Tz, fx_{2n})d(Tz, gz)\} \\ &-\phi\big(m(Sx_{2n}, Tz)\big), \end{aligned}$$

where

$$\begin{aligned} m(Sx_{2n}, Tz) = \max\{&d^2(Sx_{2n}, Tz), d(Sx_{2n}, fx_{2n})d(Tz, gz), \\ &d(Sx_{2n}, gz)d(Tz, fx_{2n}), [d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gz) \\ &+ d(Tz, fx_{2n})d(Tz, gz)]/2\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$d^3(z, gz) \leq p \max\{[0+0]/2, 0, 0\} - \phi(d^2(z, gz)).$$

This implies that $gz = z$. Since $g(X) \subset S(X)$ and hence there exists a point $v \in X$ such that $z = gz = Sv$.

We claim that $z = fv$. For this we put $x = v$ and $y = z$ in $(C1)$ we get

$$\begin{aligned} d^3(fv, gz) \leq p \max\{&[d^2(Sv, fv)d(Tz, gz) + d(Sv, fv)d^2(Tz, gz)]/2, \\ &d(Sv, fv)d(Sv, gz)d(Tz, fv), d(Sv, gz)d(Tz, fv)d(Tz, gz)\} \\ &-\phi\big(m(Sv, Tz)\big), \end{aligned}$$

where

$$\begin{aligned} m(Sv, Tz) &= \max\{d^2(Sv, Tz), d(Sv, fv)d(Tz, gz), d(Sv, gz)d(Tz, fv), \\ &\quad [d(Sv, fv)d(Sv, gz) + d(Tz, fv)d(Tz, gz)]/2\} \\ &= \max\{d^2(z, z), d(z, fv)d(z, z), d(z, z)d(z, fv), \\ &\quad [d(z, fv)d(z, z) + d(z, fv)d(z, z)]/2\} = 0. \end{aligned}$$

Hence, we get

$$d^3(fv,z) \le p \max\{[d^2(z,fv)d(z,z)+d(z,fv)d^2(z,z)]/2,$$
$$d(z,fv)d(z,z)d(z,fv), d(z,z)d(z,fv)d(z,z)\} - \phi(0).$$

This implies that $d^3(fv,z)=0$ and hence fv = z. Since the pair (f, S) is compatible of type (B) and $fv = Sv = z$, by Proposition 1.9, we have $fz = Sfv = fSv = Sz$. Hence $z = Sz = fz = Tz = gz$. Therefore, z is a common fixed point of f, g, S and T.

Now suppose that S is continuous. Then $\{SSx_{2n}\}$ and $\{Sfx_{2n}\}$ converges to Sz as $n \to \infty$. Since the mappings f and S are compatible of type (B), it follows from the Proposition 1.10 that $\{ffx_{2n}\}$ converges to Sz as $n \to \infty$.

Now we prove that $z = Sz$. For this we put $x = fx_{2n}$ and $y = x_{2n+1}$ in ($C1$) we get

$$d^3(ffx_{2n}, gx_{2n+1})$$
$$\le p \max\{[d^2(Sfx_{2n}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1})$$
$$+ d(Sfx_{2n}, ffx_{2n})d^2(Tx_{2n+1}, gx_{2n+1})]/2,$$
$$d(Sfx_{2n}, ffx_{2n})d(Sfx_{2n}, gx_{2n+1})d(Tx_{2n+1}, ffx_{2n}),$$
$$d(Sfx_{2n}, gx_{2n+1})d(Tx_{2n+1}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1})\}$$
$$-\phi\big(m(Sfx_{2n}, Tx_{2n+1})\big),$$

where

$$m(Sfx_{2n}, Tx_{2n+1}) = \max\{d^2(Sfx_{2n}, Tx_{2n+1}), d(Sfx_{2n}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1}),$$
$$d(Sfx_{2n}, gx_{2n+1})d(Tx_{2n+1}, ffx_{2n}),$$
$$[d(Sfx_{2n}, ffx_{2n})d(Sfx_{2n}, gx_{2n+1})$$
$$+d(Tx_{2n+1}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(Sz,z) \le p \max\{[d^2(Sz,Sz)d(z,z) + d(Sz,Sz)d^2(z,z)]/2,$$
$$d(Sz,Sz)d(Sz,z)d(z,Sz), d(Sz,z)d(z,Sz)d(z,z)\}$$
$$-\phi\big(m(Sz,z)\big),$$

where

$$m(Sz,z) = \max\{d^2(Sz,z), d(Sz,Sz)d(z,z), d(Sz,z)d(z,Sz),$$

$$[d(Sz,Sz)d(Sz,z)+d(z,Sz)d(z,z)]/2\} = d^2(Sz,z).$$

Hence, we get

$$d^3(Sz,z) \leq p \max\{[0+0]/2,0,0\} - \phi(d^2(Sz,z)).$$

Thus, we get $d^2(Sz,z)=0$, which implies that $Sz=z$.

Next we claim that $fz=z$. For this put $x = z$ and $y = x_{2n+1}$in $(C1)$ and taking $n \to \infty$, we get

$$d^3(fz,z) \leq p \max\{[d^2(Sz,fz)d(z,z)+d(Sz,fz)d^2(z,z)]/2,$$
$$d(Sz,fz)d(Sz,z)d(z,fz), d(Sz,z)d(z,fz)d(z,z)\}$$
$$-\phi\big(m(Sz,z)\big),$$

where

$$m(Sz,z) = \max\{d^2(Sz,z), d(Sz,fz)d(z,z), d(Sz,z)d(z,fz),$$
$$[d(Sz,fz)d(Sz,z)+d(z,fz)d(z,z)]/2\} = 0.$$

Hence, we get

$$d^3(fz,z) \leq p \max\{[0+0]/2,0,0\} - \phi(0).$$

Thus $d(fz,z)=0$. This implies that $fz = z$. Since $f(X) \subset T(X)$ and hence there exists a point $w \in X$ such that $z = fz = Tw$.

We claim that $z=gw$. For this we put $x = z$ and $y = w$ in $(C1)$ we get

$$d^3(fz,gw) \leq p \max\{[d^2(Sz,fz)d(Tw,gw)+d(Sz,fz)d^2(Tw,gw)]/2,$$
$$d(Sz,fz)d(Sz,gw)d(Tw,fz), d(Sz,gw)d(Tw,fz)d(Tw,gw)\}$$
$$-\phi\big(m(Sz,Tw)\big),$$

where

$$m(Sz,Tw) = \max\{d^2(Sz,Tw), d(Sz,fz)d(Tw,gw), d(Sz,gw)d(Tw,fz),$$
$$[d(Sz,fz)d(Sz,gw)+d(Tw,fz)d(Tw,gw)]/2\}$$
$$= \max\{d^2(z,z), d(z,z)d(z,gw), d(z,gw)d(z,z),$$
$$[d(z,z)d(z,gw)+d(z,z)d(z,gw)]/2\} = 0.$$

Hence, we get

$$d^3(z,gw) \le p \max\{[d^2(z,z)d(z,gw) + d(z,z)d^2(z,gw)]/2,$$
$$d(z,z)d(z,gw)d(z,z), d(z,gw)d(z,z)d(z,gw)\} - \phi(0),$$

which implies that $d^3(z,gw) = 0$ and hence $z = gw$. Since the pair (g,T) is compatible of type (B) and $Tw = gw = z$, by Proposition 1.9, we have $Tgw = gTw$ and hence $Tz = Tgw = gTw = gz$. Also, from condition (C1) we have

$$d^3(fz,gz) \le p \max\{[d^2(Sz,fz)d(Tz,gz) + d(Sz,fz)d^2(Tz,gz)]/2,$$
$$d(Sz,fz)d(Sz,gz)d(Tz,fz), d(Sz,gz)d(Tz,fz)d(Tz,gz)\}$$
$$-\phi\big(m(Sz,Tz)\big),$$

where

$$m(Sz,Tz) = \max\{d^2(Sz,Tz), d(Sz,fz)d(Tz,gz), d(Sz,gz)d(Tz,fz),$$
$$[d(Sz,fz)d(Sz,gz) + d(Tz,fz)d(Tz,gz)]/2\}$$
$$= d^2(z,gz)$$

Hence, we get

$$d^3(z,gz) \le p \max\{[0+0]/2,0,0\} - \phi(d^2(z,gz)).$$

This implies that $z = gz$. Hence, $z = Tz = gz = Sz = fz$. Therefore, z is a common fixed point of f, g, S.

Similarly, we can also complete the proof when T or g is continuous.

Finally, in order to prove uniqueness, suppose that z and w ($z \neq w$) are two common fixed points of f, g, S and T.

Put $x = z$ and $y = w$ in $(C1)$.

$$d^3(z,w) = d^3(fz,gw) \le p \max\{0,0,0\} - \phi(m(Sz,Tw)).$$
$$= -\phi\big(d^2(z,w)\big).$$

Thus, we have $d(z,w) = 0$ and hence $z = w$. Therefore f, g, S and T have a unique common fixed point in X. This completes the proof.

Next we give the following theorem for compatible mappings of type (C).

Theorem 2.4. Let f, g, S and T be four mappings of a complete metric space (X, d) into itself satisfying the conditions (C1)-(C3). Assume that the pairs (f,S) and (g,T) are compatible of type (C). Then f, g, S and T have a unique common fixed point in X.

Proof. From the proof of Theorem 2.1, $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{fx_{2n}\}$, $\{Sx_{2n}\}$, $\{gx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ also converge to the same point z.

Now suppose that f is continuous. Then $\{ffx_{2n}\}$ and $\{fSx_{2n}\}$ converges to fz as $n \to \infty$. Since the mappings f and S are compatible of type (C), it follows from Remark 1.12 that $\{SSx_{2n}\}$ converges to fz as $n \to \infty$.

Now we claim that $z = fz$. For this we put x = Sx_{2n} and y = x_{2n+1}in (C1) we get

$$\begin{aligned} &d^3(fSx_{2n}, gx_{2n+1}) \\ &\leq p \max\{[d^2(SSx_{2n}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1}) \\ &+ d(SSx_{2n}, fSx_{2n})d^2(Tx_{2n+1}, gx_{2n+1})]/2, \\ &\quad d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n}), \\ &\quad d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})\} \\ &\quad -\phi\big(m(SSx_{2n}, Tx_{2n+1})\big), \end{aligned}$$

where

$$\begin{aligned} m(SSx_{2n}, Tx_{2n+1}) = \max\{&d^2(SSx_{2n}, Tx_{2n+1}), d(SSx_{2n}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1}), \\ &d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n}), \\ &[d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gx_{2n+1}) \\ &+d(Tx_{2n+1}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d^3(fz, z) \leq p \max\{&[d^2(fz, fz)d(z, z) + d(fz, fz)d^2(z, z)]/2, \\ &d(fz, fz)d(fz, z)d(z, fz), d(fz, z)d(z, fz)d(z, z)\} - \phi(m(fz, z)). \end{aligned}$$

where

$$\begin{aligned} m(fz, z) = \max\{&d^2(fz, z), d(fz, fz)d(z, z), d(fz, z)d(z, fz), \\ &[d(fz, fz)d(fz, z) + d(z, fz)d(z, z)]/2\} = d^2(fz, z). \end{aligned}$$

Hence, we get

$$d^3(fz, z) \leq p \max\{[0 + 0]/2, 0, 0\} - \phi(d^2(fz, z)).$$

Thus, we get $d(fz, z) = 0$, which implies that fz = z. Since $f(X) \subset T(X)$, there exists a point $u \in X$ such that z = fz = Tu.

We claim that z = gu. For this, we put x = Sx_{2n} and y = u in (C1), we get

$$d^3(fSx_{2n}, gu) \leq p \max\{[d^2(SSx_{2n}, fSx_{2n})d(Tu, gu) + d(SSx_{2n}, fSx_{2n})d^2(Tu, gu)]/2, d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gu)d(Tu, fSx_{2n}), d(SSx_{2n}, gu)d(Tu, fSx_{2n})d(Tu, gu)\} - \phi\big(m(SSx_{2n}, Tu)\big),$$

where

$$m(SSx_{2n}, Tu) = \max\{d^2(SSx_{2n}, Tu), d(SSx_{2n}, fSx_{2n})d(Tu, gu), d(SSx_{2n}, gu)d(Tu, fSx_{2n}), [d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gu) + d(Tu, fSx_{2n})d(Tu, gu)]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(fz, gu) = d^3(z, gu) \leq p \max\{[d^2(z,z)d(z,gu) + d(z,z)d^2(z,gu)]/2, d(z,z)d(z,gu)d(z,z), d(z,gu)d(z,z)d(z,gu)\} - \phi(m(z,Tu)),$$

where

$$m(z,Tu) = \max\{d^2(z,z), d(z,z)d(z,gu), d(z,gu)d(z,z), [d(z,z)d(z,gu) + d(z,z)d(z,gu)]/2\} = 0.$$

Hence, we get

$$d^3(z, gu) \leq p \max\{[0+0]/2, 0, 0\} - \phi(0).$$

This implies that $z = gu = fz$. Since the pair (g, T) is compatible of type (C) and Tu = $gu = z$, by remark 1.11, we have $Tgu = gTu$ and hence $Tz = Tgu = gTu = gz$.

Next, we claim that gz = z. For this we put x = x_{2n} and y = z in (C1)

$$d^3(fx_{2n}, gz) \leq p \max\{[d^2(Sx_{2n}, fx_{2n})d(Tz, gz) + d(Sx_{2n}, fx_{2n})d^2(Tz, gz)]/2, d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gz)d(Tz, fx_{2n}),$$

$$d(Sx_{2n},gz)d(Tz,fx_{2n})d(Tz,gz)\}$$
$$-\phi\big(m(Sx_{2n},Tz)\big),$$

where

$$m(Sx_{2n},Tz) = \max\{d^2(Sx_{2n},Tz), d(Sx_{2n},fx_{2n})d(Tz,gz),$$
$$d(Sx_{2n},gz)d(Tz,fx_{2n}), [d(Sx_{2n},fx_{2n})d(Sx_{2n},gz)$$
$$+d(Tz,fx_{2n})d(Tz,gz)]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(z,gz) \le p\max\{[0+0]/2,0,0\} - \phi(d^2(z,gz)).$$

This implies that $gz = z$. Since $g(X) \subset S(X)$ and hence there exists a point $v \in X$ such that $z = gz = Sv$.

We claim that $z = fv$. For this we put $x = v$ and $y = z$ in ($C1$) we get

$$d^3(fv,gz) \le p\max\{[d^2(Sv,fv)d(Tz,gz) + d(Sv,fv)d^2(Tz,gz)]/2,$$
$$d(Sv,fv)d(Sv,gz)d(Tz,fv), d(Sv,gz)d(Tz,fv)d(Tz,gz)\}$$
$$-\phi\big(m(Sv,Tz)\big),$$

where

$$m(Sv,Tz) = \max\{d^2(Sv,Tz), d(Sv,fv)d(Tz,gz), d(Sv,gz)d(Tz,fv),$$
$$[d(Sv,fv)d(Sv,gz) + d(Tz,fv)d(Tz,gz)]/2\}.$$
$$=\max\{d^2(z,z), d(z,fv)d(gz,gz), d(z,z)d(z,fv),$$
$$[d(z,fv)d(z,z) + d(z,fv)d(gz,gz)]/2\} = 0.$$

Hence, we get

$$d^3(fv,z) \le p\max\{[d^2(z,fv)d(z,z) + d(z,fv)d^2(z,z)]/2,$$
$$d(z,fv)d(z,z)d(z,fv), d(z,z)d(z,fv)d(z,z)\} - \phi(0),$$

which implies that $d^3(fv,z) = 0$ and hence $fv = z$. Since the pair (f,S) is compatible of type (C) and $fv = Sv = z$, by Remark 1.11 that $fz = Sfv = fSv = Sz$. Hence $z = Sz = fz = Tz = gz$. Therefore, z is a common fixed point of f, g, S and T.

Now suppose that S is continuous. Then $\{SSx_{2n}\}$ and $\{Sfx_{2n}\}$ converges to Sz as $n \to \infty$. Since the mappings f and S are compatible of type (C), it follows from the Remark 1.12 that $\{ffx_{2n}\}$ converges to Sz as $n \to \infty$.

Now we prove that $z = Sz$. For this we put $x = fx_{2n}$ and $y = x_{2n+1}$ in $(C1)$ we get

$$\begin{aligned} d^3(ffx_{2n}, gx_{2n+1}) \\ &\leq p \max\{[d^2(Sfx_{2n}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1}) \\ &+ d(Sfx_{2n}, ffx_{2n})d^2(Tx_{2n+1}, gx_{2n+1})]/2, \\ &d(Sfx_{2n}, ffx_{2n})d(Sfx_{2n}, gx_{2n+1})d(Tx_{2n+1}, ffx_{2n}), \\ &d(Sfx_{2n}, gx_{2n+1})d(Tx_{2n+1}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1})\} \\ &-\phi\big(m(Sfx_{2n}, Tx_{2n+1})\big), \end{aligned}$$

where

$$\begin{aligned} m(Sfx_{2n}, Tx_{2n+1}) = \max\{&d^2(Sfx_{2n}, Tx_{2n+1}), d(Sfx_{2n}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1}), \\ &d(Sfx_{2n}, gx_{2n+1})d(Tx_{2n+1}, ffx_{2n}), \\ &[d(Sfx_{2n}, ffx_{2n})d(Sfx_{2n}, gx_{2n+1}) \\ &+d(Tx_{2n+1}, ffx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d^3(Sz, z) \leq p \max\{&[d^2(Sz, Sz)d(z, z) + d(Sz, Sz)d^2(z, z)]/2, \\ &d(Sz, Sz)d(Sz, z)d(z, Sz), d(Sz, z)d(z, Sz)d(z, z)\} \\ &-\phi\big(m(Sz, z)\big), \end{aligned}$$

where

$$\begin{aligned} m(Sz, z) = \max\{&d^2(Sz, z), d(Sz, Sz)d(z, z), d(Sz, z)d(z, Sz), \\ &[d(Sz, Sz)d(Sz, z) + d(z, Sz)d(z, z)]/2\} = d^2(Sz, z). \end{aligned}$$

Hence, we get

$$d^3(Sz, z) \leq p \max\{[0 + 0]/2, 0, 0\} - \phi(d^2(Sz, z)).$$

Thus, we get $d(Sz, z) = 0$, which implies that Sz = z.

Next, we claim that $fz = z$.

For this put $x = z$ and $y = x_{2n+1}$ in $(C1)$ and taking $n \to \infty$, we get

$$d^3(fz,z) \leq p \max\{[d^2(Sz,fz)d(z,z) + d(Sz,fz)d^2(z,z)]/2, \\ d(Sz,fz)d(Sz,z)d(z,fz), d(Sz,z)d(z,fz)d(z,z)\} \\ -\phi\big(m(Sz,z)\big).$$

where

$$m(Sz,z) = \max\{d^2(Sz,z), d(Sz,fz)d(z,z), d(Sz,z)d(z,fz), \\ [d(Sz,fz)d(Sz,z) + d(z,fz)d(z,z)]/2\} = 0.$$

Hence, we get

$$d^3(fz,z) \leq p \max\{[0+0]/2,0,0\} - \phi(0).$$

Thus $d^3(fz,z) = 0$. This implies that fz = z. Since $f(X) \subset T(X)$ and hence there exists a point $w \in X$ such that $z = fz = Tw$.

We claim that $z = gw$.

For this we put $x = z$ and $y = w$ in $(C1)$ we get

$$d^3(fz,gw) \leq p \max\{[d^2(Sz,fz)d(Tw,gw) + d(Sz,fz)d^2(Tw,gw)]/2, \\ d(Sz,fz)d(Sz,gw)d(Tw,fz), d(Sz,gw)d(Tw,fz)d(Tw,gw)\} \\ -\phi\big(m(Sz,Tw)\big),$$

where

$$m(Sz,Tw) = \max\{d^2(Sz,Tw), d(Sz,fz)d(Tw,gw), d(Sz,gw)d(Tw,fz), \\ [d(Sz,fz)d(Sz,gw) + d(Tw,fz)d(Tw,gw)]/2\} \\ = \max\{d^2(z,z), d(z,z)d(z,gw), d(z,gw)d(z,z), \\ [d(z,z)d(z,gw) + d(z,z)d(z,gw)]/2\} = 0.$$

Hence, we get

$$d^3(z,gw) \leq p \max\{[d^2(z,z)d(z,gw) + d(z,z)d^2(z,gw)]/2, \\ d(z,z)d(z,gw)d(z,z), d(z,gw)d(z,z)d(z,gw)\} - \phi(0).$$

which implies that $d^3(z,gw) = 0$ and hence $z = gw$. Since the pair (g,T) is compatible of type (C) and $Tw = gw = z$, by Remark 1.11, we have $Tgw = gTw$ and hence $Tz = Tgw = gTw = gz$. Also, from condition (C1) we have,

$$d^3(fz,gz) \leq p \max\{[d^2(Sz,fz)d(Tz,gz)+d(Sz,fz)d^2(Tz,gz)]/2,$$
$$d(Sz,fz)d(Sz,gz)d(Tz,fz), d(Sz,gz)d(Tz,fz)d(Tz,gz)\}$$
$$-\phi\big(m(Sz,Tz)\big),$$

where

$$m(Sz,Tz) = \max\{d^2(Sz,Tz), d(Sz,fz)d(Tz,gz), d(Sz,gz)d(Tz,fz),$$
$$[d(Sz,fz)d(Sz,gz)+d(Tz,fz)d(Tz,gz)]/2\}$$
$$= d^2(z,gz)$$

Hence, we get

$$d^3(z,gz) \leq p \max\{[0+0]/2,0,0\} - \phi(d^2(z,gz)).$$

This implies that $z = gz$. Hence, $z = Tz = gz = Sz = fz$. Therefore, z is a common fixed point of f, g, S and T.

Similarly, we can also complete the proof when g or T is continuous.

Uniqueness follows easily. This completes the proof.

Finally, we give the following theorem for compatible mappings of type (P).

Theorem 2.5. Let f, g, S and T be four mappings of a complete metric space (X, d) into itself satisfying the conditions (C1)-(C3). Assume that the pairs (f,S) and (g,T) are compatible of type (P). Then f, g, S and T have a unique common fixed point in X.

Proof. From the proof of Theorem 2.1, $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{fx_{2n}\}$, $\{Sx_{2n}\}$, $\{gx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ also converge to the same point z.

Now suppose that f is continuous. Then $\{ffx_{2n}\}$ and $\{fSx_{2n}\}$ converges to fz as $n \to \infty$. Since the mappings f and S are compatible of type (P), it follows from Remark 1.12 that $\{SSx_{2n}\}$ converges to fz as $n \to \infty$.

Now we claim that $z = fz$.

For this put $x = Sx_{2n}$ and $y = x_{2n+1}$ in ($C1$) we get

$$d^3(fSx_{2n}, gx_{2n+1})$$
$$\leq p\max\{[d^2(SSx_{2n}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})$$
$$+ d(SSx_{2n}, fSx_{2n})d^2(Tx_{2n+1}, gx_{2n+1})]/2,$$
$$d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n}),$$
$$d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})\}$$
$$-\phi\big(m(SSx_{2n}, Tx_{2n+1})\big),$$

where

$$m(SSx_{2n}, Tx_{2n+1}) = \max\{d^2(SSx_{2n}, Tx_{2n+1}), d(SSx_{2n}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1}),$$
$$d(SSx_{2n}, gx_{2n+1})d(Tx_{2n+1}, fSx_{2n}),$$
$$[d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gx_{2n+1})$$
$$+d(Tx_{2n+1}, fSx_{2n})d(Tx_{2n+1}, gx_{2n+1})]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(fz, z) \leq p\max\{[d^2(fz, fz)d(z, z) + d(fz, fz)d^2(z, z)]/2,$$
$$d(fz, fz)d(fz, z)d(z, fz), d(fz, z)d(z, fz)d(z, z)\} - \phi(m(fz, z)).$$

where

$$m(fz, z) = \max\{d^2(fz, z), d(fz, fz)d(z, z), d(fz, z)d(z, fz),$$
$$[d(fz, fz)d(fz, z) + d(z, fz)d(z, z)]/2\} = d^2(fz, z).$$

Hence, we get

$$d^3(fz, z) \leq p\max\{[0 + 0]/2, 0, 0\} - \phi(d^2(fz, z)).$$

Thus we get $d(fz, z) = 0$, which implies that fz = z. Since $f(X) \subset T(X)$, there exists a point $u \in X$ such that z = fz = Tu.

We claim that z = gu. For this, we put x = Sx_{2n} and y = u in (C1), we get

$$d^3(fSx_{2n}, gu) \leq p\max\{[d^2(SSx_{2n}, fSx_{2n})d(Tu, gu)$$
$$+d(SSx_{2n}, fSx_{2n})d^2(Tu, gu)]/2,$$
$$d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gu)d(Tu, fSx_{2n}),$$

$$d(SSx_{2n}, gu)d(Tu, fSx_{2n})d(Tu, gu)\}$$
$$-\phi\big(m(SSx_{2n}, Tu)\big),$$

where

$$m(SSx_{2n}, Tu) = \max\{d^2(SSx_{2n}, Tu), d(SSx_{2n}, fSx_{2n})d(Tu, gu),$$
$$d(SSx_{2n}, gu)d(Tu, fSx_{2n}), [d(SSx_{2n}, fSx_{2n})d(SSx_{2n}, gu)$$
$$+d(Tu, fSx_{2n})d(Tu, gu)]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(z, gu) \leq p \max\{[d^2(z, z)d(z, gu) + d(z, z)d^2(z, gu)]/2,$$
$$d(z, z)d(z, gu)d(z, z), d(z, gu)d(z, z)d(z, gu)\} - \phi(m(z, Tu)),$$

where

$$m(z, Tu) = \max\{d^2(z, z), d(z, z)d(z, gu), d(z, gu)d(z, z),$$
$$[d(z, z)d(z, gu) + d(z, z)d(z, gu)]/2\} = 0.$$

Hence, we get

$$d^3(z, gu) \leq p \max\{[0 + 0]/2, 0, 0\} - \phi(0).$$

This implies that $z = gu = fz$. Since the pair g, T is compatible of type (P) and $gu = Tu = z$, by remark 1.11, we have $Tgu = gTu$ and hence $Tz = Tgu = gTu = gz$.

Now we claim that $gz = z$.

For this we put $x = x_{2n}$ and $y = z$ in $(C1)$

$$d^3(fx_{2n}, gz) \leq p \max\{[d^2(Sx_{2n}, fx_{2n})d(Tz, gz)$$
$$+d(Sx_{2n}, fx_{2n})d^2(Tz, gz)]/2,$$
$$d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gz)d(Tz, fx_{2n}),$$
$$d(Sx_{2n}, gz)d(Tz, fx_{2n})d(Tz, gz)\}$$
$$-\phi\big(m(Sx_{2n}, Tz)\big),$$

where

$$m(Sx_{2n}, Tz) = \max\{d^2(Sx_{2n}, Tz), d(Sx_{2n}, fx_{2n})d(Tz, gz),$$

$$d(Sx_{2n}, gz)d(Tz, fx_{2n}), [d(Sx_{2n}, fx_{2n})d(Sx_{2n}, gz)$$
$$+d(Tz, fx_{2n})d(Tz, gz)]/2\}.$$

Letting $n \to \infty$, we get

$$d^3(z, gz) \leq p \max\{[0+0]/2, 0, 0\} - \phi(d^2(z, gz)).$$

This implies that $gz = z$. Since $g(X) \subset S(X)$ and hence there exists a point $v \in X$ such that $z = gz = Sv$.

We claim that $z = fv$. For this we put $x = v$ and $y = z$ in $(C1)$ we get

$$d^3(fv, gz) \leq p \max\{[d^2(Sv, fv)d(Tz, gz) + d(Sv, fv)d^2(Tz, gz)]/2,$$
$$d(Sv, fv)d(Sv, gz)d(Tz, fv), d(Sv, gz)d(Tz, fv)d(Tz, gz)\}$$
$$-\phi\big(m(Sv, Tz)\big),$$

where

$$m(Sv, Tz) = \max\{d^2(Sv, Tz), d(Sv, fv)d(Tz, gz), d(Sv, gz)d(Tz, fv),$$
$$[d(Sv, fv)d(Sv, gz) + d(Tz, fv)d(Tz, gz)]/2\}.$$
$$= \max\{d^2(z, z), d(z, fv)d(gz, gz), d(z, z)d(z, fv),$$
$$[d(z, fv)d(z, z) + d(z, fv)d(gz, gz)]/2\} = 0.$$

Hence, we get

$$d^3(fv, z) \leq p \max\{[d^2(z, fv)d(z, z) + d(z, fv)d^2(z, z)]/2,$$
$$d(z, fv)d(z, z)d(z, fv), d(z, z)d(z, fv)d(z, z)\} - \phi(0).$$

This implies that $fv = z$. Since the pair f, S is compatible of type (P) and $fv = Sv = z$, by Remark 1.11, we have $ffv = SSv$ which implies that $fz = Sz$. Hence $z = Sz = fz = Tz = gz$. Therefore, z is a common fixed point of f, g, S and T.

Similarly, we can complete the proof when S or T or g is continuous. The uniqueness follows easily. This completes the proof.

Corollary 2.6. Let f and g be two mappings of a complete metric space (X, d) into itself satisfying the following condition:

$$d^3(fx, gy) \leq p\, max\{[d^2(x, fx)d(y, gy) + d(x, fx)d^2(y, gy)]/2,$$
$$d(x, fx)d(x, gy)d(y, fx), d(x, gy)d(y, fx)d(y, gy)\}$$
$$-\phi\big(m(x, y)\big),$$

for all $x, y \in X$, where

$$m(x, y) = \max\{d^2(x, y), d(x, fx)d(y, gy), d(x, gy)d(y, fx),$$
$$[d(x, fx)d(x, gy) + d(y, fx)d(y, gy)]/2\}$$

and p is a real number satisfying $0 < p < 1$. Further, $\phi: [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0$ $for\ each\ t > 0$. Then f and g have a unique common fixed point in X.

Proof. Taking $S = T = I$ (Identity map) in theorem 2.1, we get the required result.

Now we give an example in support of our main theorems.

Example 2.7. Let X = [0, 2] and (X, d) be a metric space defined by $d(x, y) = |x - y|$. We define f, g, S, T: X $\to$ X by

$$fx = \frac{5}{4}, \forall x \in [0, 2], \qquad gx = \begin{cases} \frac{7}{4}, x \in [0, 1] \\ \frac{5}{4}, x \in (1,2] \end{cases}$$

$$Sx = \begin{cases} 1, x \in [0, 1] \\ \frac{5}{4}, x \in (1,2), \\ \frac{7}{4}, x = 2 \end{cases} \qquad Tx = \begin{cases} \frac{1}{4}, x \in [0, 1] \\ \frac{5}{4}, x \in (1,2) \\ \frac{3}{2}, x = 2 \end{cases}$$

(i) Clearly, we get $f(X) \subset T(X) and\ g(X) \subset S(X)$.

(ii) f is continuous mapping in X and g, S and T are not continuous mappings in X.

(iii) the pairs (f, S) and (g, T) are compatible and they are compatible mappings of type (A), of type (B), of type (C) and of type (P).

Consider $\{x_n\} \in (1,2)$, we have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n = \frac{5}{4} = t \in X$$

Also, we have

$$\lim_{n\to\infty} d(Sfx_n, fSx_n) = 0,\ \lim_{n\to\infty} d(Tgx_n, gTx_n) = 0,$$

$$\lim_{n\to\infty} d(Sfx_n, ffx_n) = 0,\ \lim_{n\to\infty} d(fSx_n, SSx_n) = 0,$$

$$\lim_{n\to\infty} d(Tgx_n, ggx_n) = 0,\ \lim_{n\to\infty} d(gTx_n, TTx_n) = 0,$$

(iv) for $p = \frac{3}{4}$ and $\phi(t) = \frac{t}{4}$, condition (C2) is satisfied.

Therefore, all conditions of main theorems are satisfied and $\frac{5}{4}$ is a unique common fixed point of f, g, S and T.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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