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SOME FIXED POINT THEOREMS FOR α-ADMISSIBLE MAPPINGS IN S_b-METRIC SPACES

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Abstract. In this paper, we introduce the concept of generalized $S_b-\beta-\psi$ contractive type mappings in S_b-metric space and prove some fixed point theorems. Fixed point theorems on metric spaces endowed with a partial order are also discussed.

Keywords: α-admissible mapping; S_b-metric space; β-admissible mapping; generalized $S_b-\beta-\psi$ contractive mapping.

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1. Introduction

Fixed point theory has a wide application in all fields of quantitative science. Therefore, it is quite natural to consider various generalizations of metric space in order to address the needs in various fields of quantitative science. So, we consider different generalization of metric space. There is a lot of extension of the notions of metric space. Among which one of the most important generalization is the concept of S_b-metric space introduced by Souayah and Mlaiki [13] in 2016. For more, we refer ([2], [3], [4], [5], [11] , [12], [13]) and references therein. Also one of the important generalization is the result obtained by Samet et al. [10]. They introduced

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the concept of α-admissible mapping and defined the notion of $\alpha-\psi$ contractive mapping. The results of Samet et al. [10] had been generalized in various directions (see [7], [8], [9]).

Before starting our main work, we recall some well known definitions, properties and lemmas which will be used in this paper.

Samet et al. [10] defined α-admissible as follows.

Definition 1.1. [10] Let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ and $\alpha:\mathbb{U}\times\mathbb{U}\to[0,\infty)$. Then $\mathscr{A}$ is said to be α-admissible if $x,y\in\mathbb{U}$,

$$\alpha(x,y)\geq 1\Rightarrow\alpha(\mathscr{A}x,\mathscr{A}y)\geq 1.$$

Example 1.1. Let $\mathbb{U}=[0,\infty)$. Define $\mathscr{A}\colon\mathbb{U}\to\mathbb{U}$ and $\alpha\colon\mathbb{U}\times\mathbb{U}\to[0,\infty)$ by $\mathscr{A}x=\dfrac{x^2}{4}$ for all $x\in\mathbb{U}$ and

$$\alpha(x,y)=\begin{cases}1, & \text{if } x,y\in[0,3];\\ 0, & \text{otherwise.}\end{cases}$$

Obviously, $\mathscr{A}$ is $\alpha-$admissible mapping.

Berinde [6] defined (c)-comparison function as follows.

Let Ψ be a family of functions $\psi:[0,\infty)\to[0,\infty)$ satisfying the following conditions

(i) ψ is non-decreasing,

(ii) there exists $k_0\in\mathbb{N}$ and $a\in(0,1)$ and a convergent series of non-negative terms $\sum_{k=1}^{\infty}v_k$ such that

$$\psi^{k+1}(t)\leq a\psi^k(t)+v_k$$

for $k\geq k_0$ and any $t\in\mathbb{R}^+$, where $\mathbb{R}^+=[0,\infty)$.

Lemma 1.1. [6] If $\psi\in\Psi$, then the following hold:

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n\to\infty$ for all $t\in\mathbb{R}^+$,

(ii) $\psi(t)<t$ for any $t\in\mathbb{R}^+$,

(iii) ψ is continuous at 0,

(iv) the series $\sum_{k=1}^{\infty}\psi^k(t)$ converges for any $t\in\mathbb{R}^+$.

Karapinar and Samet [7] introduced the following contractive condition.

Definition 1.2. [7] Let $(\mathbb{U}, d)$ be a metric space and $\mathscr{A} : \mathbb{U} \to \mathbb{U}$ be a given mapping. We say that $\mathscr{A}$ is a generalized $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : \mathbb{U} \times \mathbb{U} \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(\mathscr{A}x, \mathscr{A}y) \leq \psi(M(x,y))$$

for all $x, y \in \mathbb{U}$, where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,\mathscr{A}x) + d(y,\mathscr{A}y)}{2}, \frac{d(x,\mathscr{A}y) + d(y,\mathscr{A}x)}{2}\right\}. \tag{1}$$

Example 1.2. Let $\mathbb{U} = [0,1]$ be endowed with the metric $d(x,y) = |x-y|$ for all $x,y \in \mathbb{U}$. Let $\psi(t) = 1 + \frac{t}{2}$, $\forall\, t \geq 0$.

Define the mapping $\mathscr{A} : \mathbb{U} \to \mathbb{U}$ by

$$\mathscr{A}x = \begin{cases} \frac{1}{2}, \text{ if } x \in [0,1), \\ 0, \text{ if } x = 1 \end{cases}$$

and $\alpha : \mathbb{U} \times \mathbb{U} \to [0, \infty)$ by $\alpha(x,y) = 1$ for all $x, y \in \mathbb{U}$. Then, $\mathscr{A}$ is a generalized $\alpha - \psi$ contractive mapping.

Obviously, $(\mathbb{U}, d)$ is a complete metric space. In this case, $\mathscr{A}$ is not continuous.

If $x \in [0,1)$ and $y = 1$, we have

$$\alpha(x,y)d(\mathscr{A}x,\mathscr{A}y) = d(\frac{1}{2},0) = \frac{1}{2}d(y,\mathscr{A}y) \leq \psi\big(M(x,y)\big)$$

If $x = 1$ and $y \in [0,1)$, we have

$$\alpha(x,y)d(\mathscr{A}x,\mathscr{A}y) = d(\mathscr{A}x,\mathscr{A}y) = \frac{1}{2}d(x,\mathscr{A}x) \leq \psi\big(M(x,y)\big)$$

The other cases are trivial. So, $\mathscr{A} : \mathbb{U} \to \mathbb{U}$ is a generalized $\alpha - \psi$ contractive mapping.

Definition 1.3. [13] Let $\mathbb{U}$ be a non-empty set and let $s \geq 1$ be a given number. A function $S_b : \mathbb{U}^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $x, y, z, t \in \mathbb{U}$, the following conditions hold:

(i) $S_b(x,y,z) = 0$ if and only if $x = y = z$;

(ii) $S_b(x,x,y) = S_b(y,y,x)$ for all $x, y \in \mathbb{U}$;

(iii) $S_b(x,y,z) \leq s[S_b(x,x,t)+S_b(y,y,t)+S_b(z,z,t)]$.

The pair $(\mathbb{U},S_b)$ is called an S_b-metric space.

Rohen et al.[14] also defined the S_b-metric space as follows.

Definition 1.4. [14] Let $\mathbb{U}$ be a non-empty set and let $b \geq 1$ be a given number. A function $S:\mathbb{U}^3 \to [0,\infty)$ is said to be S_b-metric if and only if for all $x,y,z,t \in \mathbb{U}$, the following conditions hold:

(i) $S(x,y,z) = 0$ if and only if $x = y = z$,

(ii) $S(x,y,z) \leq b[S(x,x,t)+S(y,y,t)+S(z,z,t)]$.

The pair $(\mathbb{U},S)$ is called an S_b-metric space.

Definition 1.5. [14] An S_b-metric is said to be symmetric if

$$S(x,x,y) = S(y,y,x), \forall x,y \in \mathbb{U}.$$

Lemma 1.2. [11] In an S_b-metric space, we have

$$\begin{aligned} S(x,x,y) &\leq bS(y,y,x) \\ \text{and } S(y,y,x) &\leq bS(x,x,y) \end{aligned}$$

where $b \geq 1$ is a real number.

Lemma 1.3. [11] In an S_b- metric space, we have

$$S(x,x,z) \leq 2bS(x,x,y)+b^2S(y,y,z)$$

Definition 1.6. [13] Let $(\mathbb{U},S)$ be an S_b-metric space. A sequence $\{x_n\}$ in $\mathbb{U}$ is said to be

(i) S_b-Cauchy sequence if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $m,n \geq n_0$.

(ii) S_b-convergent to a point $x \in \mathbb{U}$ if, for each $\varepsilon > 0$, there exists integer n_0 such that $S(x_n,x_n,x) < \varepsilon$ or $S(x,x,x_n) < \varepsilon$ for all $n \geq n_0$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(iii) $(\mathbb{U},S_b)$ is said to be a complete S_b-metric space if every Cauchy sequence $\{x_n\}$ converges to a point $x \in \mathbb{U}$ such that

$$\lim_{n,m\to\infty} S_b(x_n,x_n,x_m) = \lim_{n\to\infty} S_b(x_n,x_n,x) = S_b(x,x,x)$$

Definition 1.7. [11] A mapping $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ is said to be S_b-continuous if $\{\mathscr{A}x\}$ is S_b-convergent to $\mathscr{A}x$, where $\{x_n\}$ is an S_b-convergent sequence converging to x.

The concept of β-admissible mapping is introduced by Alghamdi and Karapinar [1].

Definition 1.8. [1] Let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ and $\beta:\mathbb{U}\times\mathbb{U}\times\mathbb{U}\to[0,\infty)$, then $\mathscr{A}$ is said to be β-admissible if $x,y,z\in\mathbb{U}$,

$$\beta(x,y,z)\geq 1 \Rightarrow \beta(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z)\geq 1.$$

Example 1.3. Let $\mathbb{U}=[0,1]$. Define $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ and $\beta:\mathbb{U}\times\mathbb{U}\times\mathbb{U}\to[0,\infty)$ by $\mathscr{A}x=\dfrac{x^2}{4}+\dfrac{1}{2}$, $\forall\, x\in\mathbb{U}$ and $\beta(x,y,z)=1$ for all $x,y,z\in\mathbb{U}$. Then, $\mathscr{A}$ is $\beta-$admissible mapping.

We extend the concept of $\alpha-$admissible for n^{th} order and define as follows.

Definition 1.9. Let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a self mapping on a non-empty set $\mathbb{U}$ and $\alpha:\mathbb{U}^n\to[0,\infty)$ be a mapping. Then we say $\mathscr{A}$ is an α-admissible of order n if $x_1,x_2,...,x_n\in\mathbb{U}$,

$$\alpha(x_1,x_2,...,x_n)\geq 1 \Rightarrow \alpha(\mathscr{A}x_1,\mathscr{A}x_2,...,\mathscr{A}x_n)\geq 1.$$

Remark 1.1. When $n=2$ and $n=3$, $\mathscr{A}$ is an $\alpha-$admissible and $\beta-$admissible respectively.

2. Main Results

Now we introduce the concept of generalized $S_b-\beta-\psi$ contractive mappings by generalising the concept of $\alpha-\psi$ contractive mapping in the setting of S_b-metric space.

Definition 2.1. Let $(\mathbb{U},S)$ be an S_b-metric space and $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a given mapping. We say that $\mathscr{A}$ is a $S_b-\beta-\psi$ contractive mapping if there exist two functions $\beta:\mathbb{U}\times\mathbb{U}\times\mathbb{U}\to[0,\infty)$ and $\psi\in\Psi$ such that for all $x,y,z\in\mathbb{U}$, we have

$$\beta(x,y,z)S(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z)\leq\psi(S(x,y,z))$$

Example 2.1. Let $\mathbb{U}=[0,\infty)$. Let $(\mathbb{U},S)$ be an S_b-metric space with $S_b(x,y,z)=d(x,y)+d(x,z)$, d is an ordinary metric on $\mathbb{U}$.
Define $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ by $\mathscr{A}x=x(x+9)$ for all $x\in\mathbb{U}$. We define $\beta:\mathbb{U}\times\mathbb{U}\times\mathbb{U}\to[0,\infty)$ by

$$\beta(x,y,z) = \begin{cases} 1, \text{ if } (x,y,z) = (0,0,0), \\ 0, \text{ otherwise.} \end{cases}$$

One can easily verify that

$$\beta(x,y,z)S(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z) \leq \frac{1}{4}S(x,y,z),\ \forall x,y,z \in \mathbb{U}.$$

Then, $\mathscr{A}$ is an $S_b-\beta-\psi$ contractive mapping with $\psi(t) = \dfrac{1}{4}t$ for all $t \geq 0$.

Definition 2.2. Let $(\mathbb{U},S)$ be an S_b-metric space and let $\mathscr{A} : \mathbb{U} \to \mathbb{U}$ be a given mapping. We say that $\mathscr{A}$ is a generalized $S_b-\beta-\psi$ contractive mapping of type I if there exist two functions $\beta : \mathbb{U}\times\mathbb{U}\times\mathbb{U} \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x,y,z \in \mathbb{U}$, we have

$$\beta(x,y,z)S(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z) \leq \psi(\Delta(x,y,z)) \tag{2}$$

where

$$\begin{aligned}\Delta(x,y,z) &= \max\big\{S(x,y,z),S(x,x,\mathscr{A}x),S(y,y,\mathscr{A}y),S(z,z,\mathscr{A}z),\\ &\quad \frac{1}{6b^2}(S(x,x,\mathscr{A}y)+S(y,y,\mathscr{A}z)+S(z,z,\mathscr{A}x))\big\}\end{aligned}$$

Definition 2.3. Let $(\mathbb{U},S)$ be an S_b-metric space and let $\mathscr{A} : \mathbb{U} \to \mathbb{U}$ be a given mapping. We say that $\mathscr{A}$ is a generalized $S_b-\beta-\psi$ contractive mapping of type II if there exist two functions $\beta : \mathbb{U}\times\mathbb{U}\times\mathbb{U} \to [0,\infty)$ and $\psi \in \Psi$ such that for all $x,y \in \mathbb{U}$, we have

$$\beta(x,x,y)S(\mathscr{A}x,\mathscr{A}x,\mathscr{A}y) \leq \psi(\Delta(x,x,y)) \tag{3}$$

where

$$\begin{aligned}\Delta(x,x,y) &= \max\big\{S(x,x,y),S(x,x,\mathscr{A}x),S(y,y,\mathscr{A}y),\\ &\quad \frac{1}{6b^2}\big(S(x,x,\mathscr{A}x)+S(x,x,\mathscr{A}y)+S(y,y,\mathscr{A}x)\big)\big\}\end{aligned}$$

Theorem 2.1. Let $(\mathbb{U},S)$ be a complete S_b-metric space. Suppose that $\mathscr{A} : \mathbb{U} \to \mathbb{U}$ is a generalized $S_b-\beta-\psi$ contractive mapping of type I and satisfy the following conditions:

(i) $\mathscr{A}$ is β-admissible,

(ii) there exists $u_0 \in \mathbb{U}$ such that $\beta(u_0,u_0,\mathscr{A}u_0) \geq 1$,

(iii) $\mathscr{A}$ is S_b-continuous.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$.

Proof. Let $u_0 \in \mathbb{U}$ be the element for which $\beta(u_0, u_0, \mathscr{A}u_0) \geq 1$. We define the sequence $\{u_n\}$ in $\mathbb{U}$ as

$$u_{n+1} = \mathscr{A}u_n \text{ for all } n \geq 0.$$

Suppose that $u_n \neq u_{n+1}$ for all $n \geq 0$. Otherwise, for some $k \in \mathbb{N}$ we would have $u_k = u_{k+1} = \mathscr{A}u_k$, that is, $u = u_k$ would be a fixed point of $\mathscr{A}$ and the proof would be completed.

Since $\mathscr{A}$ is β-admissible, we have

$$\beta(u_0, u_0, u_1) = \beta(u_0, u_0, \mathscr{A}u_0) \geq 1 \Rightarrow \beta(\mathscr{A}u_0, \mathscr{A}u_0, \mathscr{A}u_1) = \beta(u_1, u_1, u_2) \geq 1.$$

or, in general

$$\beta(u_n, u_n, u_{n+1}) \geq 1, \tag{4}$$

for all $n = 0, 1, ...$

From (2) and (4), for all $n \geq 1$, we have

$$\begin{aligned} S(u_n, u_n, u_{n+1}) &= S(\mathscr{A}u_{n-1}, \mathscr{A}u_{n-1}, \mathscr{A}u_n) \\ &\leq \beta(u_{n-1}, u_{n-1}, u_n) S(\mathscr{A}u_{n-1}, \mathscr{A}u_{n-1}, \mathscr{A}u_n) \\ &\leq \psi(\Delta(u_{n-1}, u_{n-1}, u_n)) \end{aligned}$$

where

$$\begin{aligned} \Delta(u_{n-1}, u_{n-1}, u_n) &= \max\big\{S(u_{n-1}, u_{n-1}, u_n), S(u_{n-1}, u_{n-1}, \mathscr{A}u_{n-1}), \\ &\quad S(u_{n-1}, u_{n-1}, \mathscr{A}u_{n-1}), S(u_n, u_n, \mathscr{A}u_n), \\ &\quad \frac{1}{6b^2}\big(S(u_{n-1}, u_{n-1}, \mathscr{A}u_{n-1}) + S(u_{n-1}, u_{n-1}, \mathscr{A}u_n) \\ &\quad + S(u_n, u_n, \mathscr{A}u_{n-1})\big)\big\} \\ &= \max\{S(u_{n-1}, u_{n-1}, u_n), S(u_{n-1}, u_{n-1}, u_n), S(u_n, u_n, u_{n+1}) \\ &\quad \frac{1}{6b^2}(S(u_{n-1}, u_{n-1}, u_n), S(u_{n-1}, u_{n-1}, u_{n+1}), S(u_n, u_n, u_n))\} \end{aligned}$$

$$\begin{aligned}
&\leq \max\{S(u_{n-1},u_{n-1},u_n),S(u_n,x_n,u_{n+1}),\\
&\quad \frac{1}{6b^2}(S(u_{n-1},u_{n-1},u_n)+2bS(u_{n-1},u_{n-1},u_n)+b^2S(u_n,u_n,u_{n+1}))\}\\
&< \max\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n+1}),\\
&\quad \frac{1}{6b^2}(3b^2S(u_{n-1},u_{n-1},u_n)+b^2S(u_n,u_n,u_{n+1}))\}\\
&\leq \max\left\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n+1}),\right.\\
&\quad \left.\frac{1}{4}\big(3S(u_{n-1},u_{n-1},u_n)+S(u_n,u_n,u_{n+1})\big)\right\}\\
&= \max\left\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n+1})\right\}
\end{aligned}$$

Thus, we have

$$S(u_n,u_n,u_{n+1}) \leq \psi\Big(\max\left\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n+1})\right\}\Big)$$

We may consider the following two cases:

Case I : If $\max\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n+1})\} = S(u_n,u_n,u_{n+1})$ for some n, then

$$\begin{aligned}
S(u_n,u_n,u_{n+1}) &\leq \psi(S(u_n,u_n,u_{n+1}))\\
&< S(u_n,u_n,u_{n+1})
\end{aligned}$$

which is not possible.

Case II : If $\max\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n+1})\} = S(u_{n-1},u_{n-1},u_n)$, then

$$S(u_n,u_n,u_{n+1}) \leq \psi(S(u_{n-1},u_{n-1},u_n))$$

for all $n \geq 1$. Since ψ is non-decreasing by induction, we get

(5) $$S(u_n,u_n,u_{n+1}) \leq \psi^n(S(u_0,u_0,u_1) = h^n S_{b_0}$$

for all $n \geq 1$.

Using Lemma 1.3, we have

$$\begin{aligned}
S(u_n,u_n,u_m) &\leq 2bS(u_n,u_n,u_{n+1})+b^2S(u_{n+1},u_{n+1},u_m)\\
&\leq 2bS(u_n,u_n,u_{n+1})+b^2\{2bS(u_{n+1},u_{n+1},u_{n+2})+b^2S(u_{n+2},u_{n+2},u_m)\}\\
&\leq 2bS(u_n,u_n,u_{n+1})+2b^3S(u_{n+1},u_{n+1},u_{n+2})\\
&\quad +b^4\{2bS(u_{n+2},u_{n+2},u_{n+3})+b^2S(u_{n+3},u_{n+3},u_m)\}\\
&\leq 2bS(u_n,u_n,u_{n+1})+2b^3S(u_{n+1},u_{n+1},u_{n+2})+2b^5S(u_{n+2},u_{n+2},u_{n+3})\\
&\quad +2b^7S(u_{n+3},u_{n+3},u_{n+4})+\ \dots\ +2b^{m-n}S(u_{m-1},u_{m-1},u_m)\\
&\quad \dots\ +S(u_{m-2},u_{m-2},u_{m-1})\}+S(u_{m-1},u_{m-1},u_m)\\
&< 2bh^n(1+b^2h+b^4h^2+\ \dots\ +b^{m-n}h^{m-1})S_{b_0}\\
&< 2bh^n(1+b^2h+b^4h^2+\ \dots\ )S_{b_0}\\
&= \frac{2bh^n}{1-b^2h}S_{b_0}\to 0 \text{ as } m,n\to\infty.
\end{aligned}$$

This implies that $\{u_n\}$ is an S_b-Cauchy sequence in the S_b-metric space $(\mathbb{U},S)$. Since $(\mathbb{U},S)$ is complete, there exists $u\in\mathbb{U}$ such that $\{u_n\}$ is S_b-convergent to u. Since $\mathscr{A}$ is S_b-continuous, it follows that $\{\mathscr{A}u_n\}$ is S_b-convergent to $\mathscr{A}u$. By the uniqueness of the limit, we get $u=\mathscr{A}u$, that is u is a fixed point of $\mathscr{A}$. □

Corollary 2.1. Let $(\mathbb{U},S)$ be a complete S_b-metric space. Suppose that $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ is an $S_b-\beta-\psi$ contractive mapping and satisfy the following conditions:

(i) $\mathscr{A}$ is β-admissible,

(ii) there exists $u_0\in\mathbb{U}$ such that $\beta(u_0,u_0,\mathscr{A}u_0)\geq 1$,

(iii) $\mathscr{A}$ is S_b-continuous.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$.

Example 2.2. In the above Example 2.1, we observe that $\mathscr{A}$ is an $S_b-\beta-\psi$ contractive mapping.

For $x=y=z=0$, we have $\beta(x,y,z)=\beta(0,0,0)=1$ and $\beta(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z)=\beta(0,0,0)=1$. So, $\mathscr{A}$ is $\beta-$admissible. And, there exists a point $u_0=0\in\mathbb{U}$ such that $\beta(u_0,u_0,\mathscr{A}u_0)=$

$\beta(0,0,\mathscr{A}0)=\beta(0,0,0)=1$. Also, $\mathscr{A}$ is S_b-continuous. All the conditions of Corollary 2.1 are satisfied. Hence, $\mathscr{A}$ has a fixed point. Here, the fixed point of $\mathscr{A}$ is 0.

Corollary 2.2. Let $(\mathbb{U},S)$ be a complete S_b-metric space. Suppose that $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ is a generalized $S_b-\beta-\psi$ contractive mapping of type II and satisfy the following conditions :

(i) $\mathscr{A}$ is β-admissible,

(ii) there exists $u_0\in\mathbb{U}$ such that $\beta(u_0,u_0,\mathscr{A}u_0)\geq 1$,

(iii) $\mathscr{A}$ is S_b-continuous.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$.

By omitting the continuity of $\mathscr{A}$, we state the following theorem.

Theorem 2.2. Let $(\mathbb{U},S)$ be a complete S_b-metric space. Suppose that $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ is a generalized $S_b-\beta-\psi$ contractive mapping of type I such that ψ is continuous and satisfy the following conditions:

(i) $\mathscr{A}$ is β-admissible,

(ii) there exists $u_0\in\mathbb{U}$ such that $\beta(u_0,u_0,\mathscr{A}u_0)\geq 1$,

(iii) if $\{u_n\}$ is a sequence in $\mathbb{U}$ such that $\beta(u_n,u_n,u_{n+1})\geq 1$ for all n and $\{u_n\}$ is an S_b-convergent to $u\in\mathbb{U}$, then $\beta(u_n,u_n,u)\geq 1$ for all n.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$.

Proof. Taking $u_0\in\mathbb{U}$ as the element satisfying the condition (ii), we construct the sequence $\{u_n\}$ as usual, that is,

$$u_{n+1}=\mathscr{A}u_n,\ \text{for all } n\geq 0.$$

This sequence $\{u_n\}$ is an S_b-Cauchy sequence in the complete S_b-metric space $(\mathbb{U},S)$ which can be shown exactly as in the proof of Theorem 2.1, that is, the sequence $\{x_n\}$ is S_b-convergent to $u\in\mathbb{U}$.

From (iii), we have

$$\beta(u_n,u_n,u)\geq 1 \tag{6}$$

for all $n \geq 0$.

Using (6), we have

$$\begin{aligned} S(u_{n+1},u_{n+1},\mathscr{A}u) &= S(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u) \\ &\leq \beta(u_n,u_n,u)S(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u) \\ &\leq \psi(\Delta(u_n,u_n,u)) \end{aligned}$$

where

$$\begin{aligned} \Delta(u_n,u_n,u) &= \max\Big\{S(u_n,u_n,u),S(u_n,u_n,\mathscr{A}u_n),S(u,u,\mathscr{A}u), \\ &\quad \frac{1}{6b^2}\big(S(u_n,u_n,\mathscr{A}u_n)+S(u_n,u_n,\mathscr{A}u)+S(u,u,\mathscr{A}u_n)\big)\Big\} \\ &= \max\Big\{S(u_n,u_n,u),S(u_n,u_n,u_{n+1}),S(u,u,\mathscr{A}u) \\ &\quad \frac{1}{6b^2}\big(S(u_n,u_n,u_{n+1})+S(u_n,u_n,\mathscr{A}u)+S(u,u,u_{n+1})\big)\Big\} \end{aligned}$$

Letting $n \to \infty$ in the above inequality, that is,

$$S(u_{n+1},u_{n+1},\mathscr{A}u) \leq \psi(\Delta(u_n,u_n,u))$$

it follows that

$$S(u,u,\mathscr{A}u) \leq \psi(S(u,u,\mathscr{A}u))$$

which is not possible.

Thus, $S(u,u,\mathscr{A}u) = 0$ and hence $u = \mathscr{A}u$. □

Corollary 2.3. Let $(\mathbb{U},S)$ be a complete S_b-metric space. Suppose that $\mathscr{A} : \mathbb{U} \to \mathbb{U}$ is a generalized $S_b - \beta - \psi$ contractive mapping of type II such that ψ is continuous and satisfy the following conditions:

(i) $\mathscr{A}$ is β-admissible,

(ii) there exists $u_0 \in \mathbb{U}$ such that $\beta(u_0,u_0,\mathscr{A}u_0) \geq 1$,

(iii) if $\{u_n\}$ is a sequence in $\mathbb{U}$ such that $\beta(u_n,u_n,u_{n+1}) \geq 1$ for all n and $\{u_n\}$ is an S_b-convergent to $u \in \mathbb{U}$, then $\beta(u_n,u_n,u) \geq 1$ for all n.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$.

Theorem 2.3. Let all the conditions of Theorem 2.1 (respectively Theorem 2.2, Corollary 2.2, Corollary 2.3) hold. Furthermore, assume that for $u \in Fix(\mathscr{A})$, $\beta(u,u,z) \geq 1$ for all $z \in \mathbb{U}$. Then, the fixed point of the mapping $\mathscr{A}$ is unique.

Proof. Let $v,w \in Fix(\mathscr{A})$ be two fixed points of $\mathscr{A}$. By the hypothesis, we have $\beta(v,v,w) \geq 1$. Since v and w are fixed points of $\mathscr{A}$, we have, $\beta(\mathscr{A}v,\mathscr{A}v,\mathscr{A}w) = \beta(v,v,w)$.

Consequently, we have

$$\begin{aligned} S(v,v,w) &= S(\mathscr{A}v,\mathscr{A}v,\mathscr{A}w) \\ &\leq \beta(v,v,w)S(\mathscr{A}v,\mathscr{A}v,\mathscr{A}w) \\ &\leq \psi(\Delta(v,v,w)) \end{aligned}$$

where

$$\begin{aligned} \Delta(v,v,w) &= \max\big\{S(v,v,w),S(v,v,\mathscr{A}v),S(w,w,\mathscr{A}w), \\ &\quad \frac{1}{6b^2}\big(S(v,v,\mathscr{A}v)+S(v,v,\mathscr{A}w)+S(w,w,\mathscr{A}v)\big)\big\} \\ &= \max\big\{S(v,v,w),\frac{1}{6b^2}\big(S(v,v,w)+S(w,w,v)\big)\big\} \\ &= S(v,v,w) \end{aligned}$$

Thus, we get that

$$\begin{aligned} S(v,v,w) &\leq \psi(\Delta(v,v,w)) \\ &\leq \psi(S(v,v,w)) \\ &< S(v,v,w). \end{aligned}$$

which is not possible.

Hence, $v = w$, that is, the fixed point of $\mathscr{A}$ is unique. □

Corollary 2.4. Let $(\mathbb{U},S)$ be a complete S_b-metric space and $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a given mapping. Suppose that there exists a continuous function $\psi \in \Psi$ such that

$$S(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z) \leq \psi(\Delta(x,y,z))$$

for all $x,y,z \in \mathbb{U}$. Then $\mathscr{A}$ has a unique fixed point.

Corollary 2.5. Let $(\mathbb{U},S)$ be a complete S_b-metric space and $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a given mapping. Suppose that there exists a continuous function $\psi \in \Psi$ such that

$$S(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z) \leq \psi(S(x,y,z))$$

for all $x,y,z \in \mathbb{U}$. Then $\mathscr{A}$ has a unique fixed point.

Corollary 2.6. Let $(\mathbb{U},S)$ be a complete S_b-metric space and $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a given mapping. Suppose that there exists $\lambda \in [0,1)$ such that

$$\begin{aligned} S(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z) \quad &\leq \quad \lambda \max\{S(x,y,z), S(x,x,\mathscr{A}x), S(y,y,\mathscr{A}y), \\ &\qquad S(z,z,\mathscr{A}z), \frac{1}{6b^2}(S(x,x,\mathscr{A}y)+S(y,y,\mathscr{A}z)+S(z,z,\mathscr{A}x))\} \end{aligned}$$

for all $x,y,z \in \mathbb{U}$. Then $\mathscr{A}$ has a unique fixed point.

Corollary 2.7. Let $(\mathbb{U},S)$ be a complete S_b-metric space and let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a given mapping. Suppose that there exist non-negative real numbers p,q,r,s,t with $p+q+r+s+t<1$ such that

$$\begin{aligned} S(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z) \quad &\leq \quad pS(x,y,z)+qS(x,x,\mathscr{A}x)+rS(y,y,\mathscr{A}y) \\ &\qquad +s(S(z,z,\mathscr{A}z))+\frac{t}{6b^2}(S(x,x,\mathscr{A}y)+S(y,y,\mathscr{A}z)+S(z,z,\mathscr{A}x)) \end{aligned}$$

for all $x,y,z \in \mathbb{U}$. Then $\mathscr{A}$ has a unique fixed point.

Corollary 2.8. Let $(\mathbb{U},S)$ be a complete S_b-metric space and let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a given mapping. Suppose that there exists $\lambda \in [0,1)$ such that

$$S(\mathscr{A}x,\mathscr{A}y,\mathscr{A}z) \leq \lambda S(x,y,z)$$

for all $x,y,z \in \mathbb{U}$. Then $\mathscr{A}$ has a unique fixed point.

3. CONSEQUENCES

We state fixed point theorems on metric spaces endowed with a partial order.

Definition 3.1. Let $(\mathbb{U},\preceq)$ be a partially ordered set and $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a given mapping. We say that $\mathscr{A}$ is non-decreasing with respect to $\preceq$ if $x,y \in \mathbb{U}$, $x \preceq y \Rightarrow \mathscr{A}x \preceq \mathscr{A}y$.

Definition 3.2. Let $(\mathbb{U},\preceq)$ be a partially ordered set. A sequence $\{u_n\}\subset\mathbb{U}$ is said to be non-decreasing with respect to $\preceq$ if $u_n\preceq u_{n+1}$ for all n.

Definition 3.3. Let $(\mathbb{U},\preceq)$ be a partially ordered set and S be an S_b-metric space on $\mathbb{U}$. We say $(\mathbb{U},\preceq,S)$ an S_b-regular if for every non-decreasing sequence $\{u_n\}\subset\mathbb{U}$ such that $u_n\to u\in\mathbb{U}$ as $n\to\infty$, $u_n\preceq u$ for all n.

Theorem 3.1. Let $(\mathbb{U},\preceq)$ be a partially ordered set and S be an S_b-metric on $\mathbb{U}$ such that $(\mathbb{U},S)$ is a complete S_b-metric space. Let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a non-decreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi\in\Psi$ such that

$$S(\mathscr{A}x,\mathscr{A}x,\mathscr{A}y)\preceq\psi(\Delta(x,x,y)) \tag{7}$$

for all $x,y\in\mathbb{U}$ with $x\preceq y$. Suppose also that the following conditions hold :

(i) there exists $u_0\in\mathbb{U}$ such that $u_0\preceq\mathscr{A}u_0$,

(ii) $\mathscr{A}$ is S_b-continuous or $(\mathbb{U},\preceq,S)$ is S_b-regular and ψ is continuous.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$. Moreover, if for $u\in Fix(\mathscr{A})$, $u\preceq z$ for all $z\in\mathbb{U}$, one has the uniqueness of the fixed point.

Proof. Define the mapping $\beta:\mathbb{U}\times\mathbb{U}\times\mathbb{U}\to[0,\infty)$ by

$$\beta(x,x,y)=\begin{cases}1, & \text{if } x\preceq y\\ \frac{1}{10}, & \text{otherwise.}\end{cases} \tag{8}$$

From (7), for all $x,y\in\mathbb{U}$, we have

$$\beta(x,x,y)S(\mathscr{A}x,\mathscr{A}x,\mathscr{A}y)\leq\psi(\Delta(x,x,y))$$

It follows that $\mathscr{A}$ is a generalized $S_b-\beta-\psi$ contractive mapping of type II. From the condition (i), we have

$$\beta(u_0,u_0,\mathscr{A}u_0)\geq 1$$

Since $\mathscr{A}$ is a non-decreasing mapping with respect to $\preceq$, we have, for all $x,y\in\mathbb{U}$,

$$x\preceq y\Rightarrow\mathscr{A}x\preceq\mathscr{A}y\Rightarrow\beta(\mathscr{A}x,\mathscr{A}x,\mathscr{A}y)\geq 1.$$

It follows that

$$\beta(x,x,y) \geq 1 \implies \beta(\mathscr{A}x,\mathscr{A}x,\mathscr{A}y) \geq 1.$$

Thus $\mathscr{A}$ is β-admissible. Moreover, if $\mathscr{A}$ is S_b-continuous, by Corollary 2.2, $\mathscr{A}$ has a fixed point.

By (ii), for every non-decreasing sequence $\{u_n\} \subset \mathbb{U}$ such that $u_n \to u \in \mathbb{U}$ as $n \to \infty$, we have, $u_n \preceq u$ for all n. By the definition of $\beta-$admissible mapping, we have, $\beta(u_n,u_n,u_{n+1}) \geq 1 \implies \beta(u_n,u_n,u) \geq 1$ for all n. Thus, all the hypotheses of Corollary 2.3 are satisfied and hence there exists $u \in \mathbb{U}$ such that $\mathscr{A}u = u$.

To prove the uniqueness, since $u \in Fix(\mathscr{A})$, we have $u \preceq z$ for all $z \in \mathbb{U}$. By the definition of $\beta-$admissible mapping, $\beta(u,u,z) \geq 1$ for all $z \in \mathbb{U}$. Therefore, the hypotheses of Theorem 2.3 are satisfied and hence the uniqueness.

□

Corollary 3.1. Let $(\mathbb{U},\preceq)$ be a partially ordered set and S be an S_b-metric on $\mathbb{U}$ such that $(\mathbb{U},S)$ is a complete S_b-metric space. Let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a non-decreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that

$$S(\mathscr{A}x,\mathscr{A}x,\mathscr{A}y) \leq \psi(S(x,x,y))$$

for all $x,y \in \mathbb{U}$ with $x \preceq y$. Suppose also that the following conditions hold

(i) there exists $u_0 \in \mathbb{U}$ such that $u_0 \preceq \mathscr{A}u_0$,

(ii) $\mathscr{A}$ is S_b-continuous or $(\mathbb{U},\preceq,S)$ is S_b-regular.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$. Moreover, if for $u \in Fix(\mathscr{A})$, $u \preceq z$ for all $z \in \mathbb{U}$, one has the uniqueness of the fixed point.

Corollary 3.2. Let $(\mathbb{U},\preceq)$ be a partially ordered set and S be an S_b-metric space on $\mathbb{U}$ such that $(\mathbb{U},S)$ is a complete S_b-metric space. Let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a non-decreasing mapping with respect to $\preceq$. Suppose that there exist non-negative real numbers p,q,r and s with $p+q+r+s<1$

such that

$$\begin{aligned} S(\mathscr{A}x,\mathscr{A}x,\mathscr{A}y) &\leq pS(x,x,y)+qS(x,x,\mathscr{A}x)+rS(y,y,\mathscr{A}y) \\ &\quad +\frac{s}{6b^2}(S(x,x,\mathscr{A}x)+S(x,x,\mathscr{A}y)+S(y,y,\mathscr{A}x)) \end{aligned}$$

for all $x,y\in\mathbb{U}$ with $x\preceq y$. Suppose also that the following conditions hold:

(i) there exists $u_0\in\mathbb{U}$ such that $u_0\preceq\mathscr{A}u_0$,

(ii) $\mathscr{A}$ is S_b-continuous or $(\mathbb{U},\preceq,S)$ is S_b-regular.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$. Moreover, if for $u\in Fix(\mathscr{A})$, $u\preceq z$ for all $z\in\mathbb{U}$, one has the uniqueness of the fixed point.

Corollary 3.3. Let $(\mathbb{U},\preceq)$ be a partially ordered set and S be an S_b-metric space on $\mathbb{U}$ such that $(\mathbb{U},S)$ is a complete S_b-metric space. Let $\mathscr{A}:\mathbb{U}\to\mathbb{U}$ be a non-decreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda\in[0,1)$ such that

$$S(\mathscr{A}x,\mathscr{A}x,\mathscr{A}y)\leq\lambda S(x,x,y)$$

for all $x,y\in\mathbb{U}$ with $x\preceq y$. Suppose also that the following conditions hold :

(i) there exists $u_0\in\mathbb{U}$ such that $u_0\preceq\mathscr{A}u_0$,

(ii) $\mathscr{A}$ is S_b-continuous or $(\mathbb{U},\preceq,S)$ is S_b-regular.

Then $\mathscr{A}$ has a fixed point in $\mathbb{U}$. Moreover, if for $u\in Fix(\mathscr{A})$, $x\preceq z$ for all $z\in\mathbb{U}$, one has the uniqueness of the fixed point.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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