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ON SOME PROPERTIES OF $\mathscr{I}^{\mathscr{K}}$-CONVERGENCE

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Abstract. In this paper, we study $\mathscr{I}^{\mathscr{K}}$-convergent sequences and observe that various properties of usual convergence are exhibited by $\mathscr{I}^{\mathscr{K}}$-convergence in the set of real numbers $\mathbb{R}$. Subsequently, we prove the Sandwich Theorem for $\mathscr{I}^{\mathscr{K}}$-convergent sequences in $\mathbb{R}$. We also introduce $\mathscr{I}^{\mathscr{K}}$-convergence field and study its various properties.

Keywords: $\mathscr{I}^{\mathscr{K}}$-convergence; $\mathscr{I}^{\mathscr{K}}$-convergence field.

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1. Introduction

An ideal on a set $\mathscr{S}$ is a collection of subsets of $\mathscr{S}$ closed under finite unions and subset inclusion. Two basic ideals are Fin and $\mathscr{I}_0$ on $\mathbb{N}$ defined as Fin:= collection of all finite subsets of $\mathbb{N}$ and $\mathscr{I}_0$:= subsets of $\mathbb{N}$ with density 0. For a subset A of $\mathbb{N}$, $A \in \mathscr{I}_0$ if and only if

$$\limsup_{n\to\infty} \frac{|A\cap\{1,2,...,n\}|}{n} = 0.$$

An ideal $\mathscr{I}$ is a P-ideal if it is σ-directed modulo finite sets, i.e., for every sequence (A_n) of sets in $\mathscr{I}$ there exists $A \in \mathscr{I}$ such that $A_n \setminus A$ is finite for all $n \in \mathbb{N}$. For an ideal $\mathscr{I}$ in $P(\mathbb{N})$, we observe two additional subsets of $P(\mathbb{N})$ namely $\mathscr{I}^\star$, $\mathscr{I}^+$ where $\mathscr{I}^\star := \{A \subset \mathbb{N} : A^c \in \mathscr{I}\}$, the filter dual of $\mathscr{I}$ and $\mathscr{I}^+$:= collection of all subsets of $\mathbb{N}$ which does not belong to $\mathscr{I}$.

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The ideal convergence of a sequence of real numbers was introduced by Kostyrko et al. [8], as a generalisation of the existing notions of convergence. For an ideal $\mathscr{I}$, two modes of ideal convergence are denoted by $\mathscr{I}$-convergence and $\mathscr{I}^*$-convergence.

Definition 1.1. Let X be a topological space. Then a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathscr{I}$-convergent to ξ, denoted by $x_n \to_{\mathscr{I}} \xi$, if $\{n : x_n \notin U\} \in \mathscr{I}^+$, $\forall$ neighborhoods U of ξ.

Definition 1.2. Let X be a topological space. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathscr{I}^\star$-convergent to ξ if and only if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathscr{I}^\star$ (i.e. $\mathbb{N} \setminus M \in \mathscr{I}$)), such that $\underset{k\to\infty}{lim}(x_{m_k}) = \xi$.

It may be observed that these two definitions arose from two equivalent definitions of usual convergence. Kostyrko et al. [8] showed that $\mathscr{I}^\star$-convergence coincide $\mathscr{I}$-convergence for an admissible P-ideal $\mathscr{I}$, where admissible ideals contain elements in Fin.

In 2011, Macaj and Sleziak [5] defined the $\mathscr{I}^{\mathscr{K}}$-convergence of function in a topological space. Comparisions of $\mathscr{I}^{\mathscr{K}}$-convergence with ideal convergence [8] are studied by many authors [7, 6] in last decade. Some of the definitions and results of [2, 5] are listed below for further reference. Here X is a topological space and S is a set.

i) [5] A function $f : S \to X$ is called $\mathscr{I}^{\mathscr{K}}$-convergent to a point $x \in X$ if there exist $M \in \mathscr{I}^*$ such that the function $g : S \to X$ such that
$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases} \text{ is } \mathscr{K}\text{-convergent to } x.$$

ii) [5] A function $f : S \to X$ is called $\mathscr{I}^{\mathscr{K}^*}$-convergent to a point $x \in X$ if there exist $M \in \mathscr{I}^*$ such that the function $g : S \to X$ such that
$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases} \text{ is } \mathscr{K}^*\text{-convergent to } x.$$

Lemma 1.3. [5, Lemma 2.1] *If $\mathscr{I}$ and $\mathscr{K}$ are two ideals on $\mathbb{N}$ and $f : S \to X$ is a function such that $\mathscr{K} - \lim f = x$, then $\mathscr{I}^{\mathscr{K}} - \lim f = x$.*

Proposition 1.4. [5, Lemma 2.1] *Let $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{K}_1$ and $\mathscr{K}_2$ be ideals on a set S such that $\mathscr{I}_1 \subseteq \mathscr{I}_2$ and $\mathscr{K}_1 \subseteq \mathscr{K}_2$ and X be a topological space. Then for any function $f : S \to X$ we have*

$$\mathscr{I}^{\mathscr{K}_1} - \lim x_n = x \implies \mathscr{I}^{\mathscr{K}_2} - \lim y_n = x.$$

$$\mathscr{I}_1^{\mathscr{K}} - \lim x_n = x \implies \mathscr{I}_2^{\mathscr{K}} - \lim y_n = x.$$

A sequence $\{x_n\} \in X$ is said to be $\mathscr{I}$-bounded for an ideal $\mathscr{I}$, if there exists $M > 0$ such that $\{k \in \mathbb{N} : x_k > M\} \in \mathscr{I}$.

Result 1.5. [5, Result 3.3] If a sequence is $\mathscr{I}$-convergent, then it is $\mathscr{I}$-bounded.

Theorem 1.6. [1, Theorem 4.1] *If a series $\sum x_n$ is $\mathscr{I}$-convergent, then there exists a subset $P = \{n_1, n_2, \ldots\}$ such that $P \in \mathscr{I}$ and $\sum x_{n_i}$ is convergent.*

Throughout this paper we deal with the ideals $\mathscr{I}$ containing Fin and $S \notin \mathscr{I}$.

2. $\mathscr{I}^{\mathscr{K}}$-CONVERGENT SEQUENCES

There are certain properties of $\mathscr{I}^{\mathscr{K}}$-convergent sequences that can be shown straightway from usual convergence setup. Following results are obvious, we prefer to skip some of the proofs.

Theorem 2.1. *If a sequence is $\mathscr{I}^{\mathscr{K}}$-convergent then it is $\mathscr{I} \cup \mathscr{K}$-bounded, provided $\mathscr{I} \cup \mathscr{K}$ is an ideal.*

Proof. Let a sequence $x = \{x_n\}$ is $\mathscr{I}^{\mathscr{K}}$-convergent. Subsequently, we can observe that x is $\mathscr{I} \cup \mathscr{K}$-convergent. That means by Theorem 1.5, x is $\mathscr{I} \cup \mathscr{K}$-bounded. ☐

Result 2.2. Let $\mathscr{I}$ and $\mathscr{K}$ be two ideals on $\mathbb{N}$. $\{x_n\}$, $\{y_n\}$ be two sequences such that $x_n \leq y_n$ for all $n \in \mathscr{K}$. Then

(1) $\mathscr{I}^{\mathscr{K}} - \lim x_n = \infty \implies \mathscr{I}^{\mathscr{K}} - \lim y_n = \infty$.

(2) $\mathscr{I}^{\mathscr{K}} - \lim y_n = -\infty \implies \mathscr{I}^{\mathscr{K}} - \lim x_n = -\infty$.

Result 2.3. Let $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{K}_1$, $\mathscr{K}_2$ be ideals on $\mathbb{N}$ such that $\mathscr{I}_1 \subseteq \mathscr{I}_2$ and $\mathscr{K}_1 \subseteq \mathscr{K}_2$. Also $\{x_n\}$, $\{y_n\}$ be two sequences such that $x_n \leq y_n$ for all $n \in \mathscr{K}$. Then

(1) $\mathscr{I}^{\mathscr{K}_1} - \lim x_n = \infty \implies \mathscr{I}^{\mathscr{K}_2} - \lim y_n = \infty$.

(2) $\mathscr{I}^{\mathscr{K}_1} - \lim y_n = -\infty \implies \mathscr{I}^{\mathscr{K}_2} - \lim x_n = -\infty$.

(3) $\mathscr{I}_1^{\mathscr{K}} - \lim x_n = \infty \implies \mathscr{I}_2^{\mathscr{K}} - \lim y_n = \infty$.

(4) $\mathscr{I}_1{}^{\mathscr{K}} - \lim y_n = -\infty \implies \mathscr{I}_2{}^{\mathscr{K}} - \lim x_n = -\infty$.

Proof. Using Proposition 1.4, above results are immediate. □

Theorem 2.4. *Let $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$ be real sequences such that $x_n \leq y_n \leq z_n$ for all $n \in K$, where $K \in \mathscr{K}^*$. If $\mathscr{I}^{\mathscr{K}} - \lim x = L = \mathscr{I}^{\mathscr{K}} - \lim z$ then $\mathscr{I}^{\mathscr{K}} - \lim y = L$.*

Proof. For a given $\varepsilon > 0$, Then, for $x = \{x_n\}$, $z = \{z_n\}$ there exist M_1, $M_2 \in \mathscr{I}^*$ such that the sets

$$B_x = \{n \in M_1 : |x_n - L| \geq \varepsilon\},$$

$$B_z = \{n \in M_2 : |y_n - L| \geq \varepsilon\}$$

belong to $\mathscr{K}$. Then, for the set $M = M_1 \cap M_2 \in \mathscr{I}^*$, we have the sets

$$B_x{}' = \{n \in M : |x_n - L| \geq \varepsilon\},$$

$$B_z{}' = \{n \in M : |y_n - L| \geq \varepsilon\}$$

belong to $\mathscr{K}$. Therefore, for $M \in \mathscr{I}^*$, we have $B_y{}' \subseteq (B_x{}' \cup B_z{}') \cap K$ and hence the set

$$B_y{}' = \{n \in M : |z_n - L| \geq \varepsilon\}$$

is in $\mathscr{K}$. It follows that $\{y_n\}$ is $\mathscr{I}^{\mathscr{K}}$-convergent to L. □

Following results are immediate, so we prefer to omit the proofs.

Result 2.5. Let $x_n \geq \alpha$ for all $n \in K(\subseteq \mathbb{N})$ with $K \in \mathscr{K}$. If $\mathscr{I}^{\mathscr{K}} - \lim x_n = L$, then $L \geq \alpha$.

Result 2.6. Let $x_n \leq y_n$, for all $n \in I(\in \mathscr{I})$.

(1) If $\mathscr{I}^{\mathscr{K}} - \lim x_n$ and $\mathscr{I}^{\mathscr{K}} - \lim y_n$ exist then $\mathscr{I}^{\mathscr{K}} - \lim x_n \leq \mathscr{I}^{\mathscr{K}} - \lim y_n$.
(2) If $\mathscr{I}^{\mathscr{K}} - \lim y_n \leq B$, then $\mathscr{I}^{\mathscr{K}} - \lim x_n \leq B$.

Result 2.7. Let $x_n > 0$ for all $n \in K(K \in \mathscr{K})$ and $x_n \neq 0$ for all $n \in \mathbb{N}$, then $\mathscr{I}^{\mathscr{K}} - \lim x_n = \infty$ if and only if $\mathscr{I}^{\mathscr{K}} - \lim x_n{}^{-1} = 0$.

Result 2.8. If $\mathscr{I}^{\mathscr{K}} - \lim x_n = L$, then $\mathscr{I}^{\mathscr{K}} - \lim |x_n| = |L|$ but the converse is not true.

3. $\mathscr{I}^{\mathscr{K}}$-Convergent Series

In this section, we introduce the notion of $\mathscr{I}^{\mathscr{K}}$-convergence for series of real or complex numbers which unifies and generalize different notions of convergence of series.

Definition 3.1. A series $\sum_{k=1}^{\infty} x_k$ is said to be $\mathscr{I}^{\mathscr{K}}$-convergent if the sequence of its partial sums (s_n), where $s_n = x_1 + x_2 + ... + x_n$ is $\mathscr{I}^{\mathscr{K}}$-convergent.

Theorem 3.2. *If a series $\sum x_n$ is $\mathscr{I}^{\mathscr{K}}$-convergent, then there exists a subset $P = \{n_1, n_2, ...\}$ such that $P \in \mathscr{I} \cup \mathscr{K}$ and $\sum x_{n_i}$ is convergent provided $\mathscr{I} \cup \mathscr{K}$ is an ideal.*

Proof. We observe that if a series $\sum x_n$ is $\mathscr{I}^{\mathscr{K}}$-convergent, then it follows that $\sum x_n$ is $\mathscr{I} \cup \mathscr{K}$-convergent to the same limit. Then by Theorem 1.6, we have a set $P = \{n_1, n_2, ...\} \in \mathscr{I} \cup \mathscr{K}$ such that $\sum x_{n_i}$ is convergent. □

Result 3.3. The series $\sum z_n$ with complex terms is $\mathscr{I}^{\mathscr{K}}$-convergent if and only if the real part and the imerginary part is $\mathscr{I}^{\mathscr{K}}$-convergent.

Result 3.4. If $\sum x_n$ and $\sum y_n$ be two $\mathscr{I}^{\mathscr{K}}$-convergent series then for any complex numbers α and β, we have the series $\sum(\alpha x_n + \beta y_n)$ is $\mathscr{I}^{\mathscr{K}}$-convergent to $\alpha \sum x_n + \beta \sum y_n$.

4. $\mathscr{I}^{\mathscr{K}}$-Convergence Field

Definition 4.1. A convergence field of $\mathscr{I}^{\mathscr{K}}$-convergence is a set defined as

$$F(\mathscr{I}^{\mathscr{K}}) = \{x = (x_n) \in l_\infty : \text{there exist } \mathscr{I}^{\mathscr{K}} - \lim x \in \mathbb{R}\}.$$

l_∞ denote the space of all bounded complex valued sequences with $||.||_\infty$ norm.

Now define a function $g : F(\mathscr{I}^{\mathscr{K}}) \to \mathbb{R}$ such that

$$g(x) = \mathscr{I}^{\mathscr{K}} - \lim x, \text{ for all } x \in F(\mathscr{I}^{\mathscr{K}}).$$

Theorem 4.2. *The function $g : F(\mathscr{I}^{\mathscr{K}}) \to \mathbb{R}$ is Lipschitz function and hence uniformly continuous.*

Proof. Let $x,\, y \in F(\mathscr{I}^{\mathscr{K}})$ and $x \neq y$. That means $||x - y|| > 0$. So, there exist $M_1 \in \mathscr{I}^*$ such that

$$A_x = \{n \in M_1 : |x_n - g(x)| \geq ||x-y||\} \in \mathscr{K}$$

and also there exist $M_2 \in \mathscr{I}^*$ such that

$$A_y = \{n \in M_2 : |y_n - g(y)| \geq ||x-y||\} \in \mathscr{K}.$$

Then for $M_1 \cap M_2 = M \in \mathscr{I}^*$, the sets

$$A_x = \{n \in M : |x_n - g(x)| \geq ||x-y||\},$$

$$A_y = \{n \in M : |y_n - g(y)| \geq ||x-y||\}$$

belong to $\mathscr{K}$. Thus

$$A_x{}' = \{n \in M : |x_n - g(x)| < ||x-y||\},$$

$$A_y{}' = \{n \in M : |y_n - g(y)| < ||x-y||\}$$

belong to $\mathscr{K}^*$. So, $A = A_x{}' \cap A_y{}' \in \mathscr{K}^*$. Now taking n in A, we have

$$|g(x) - g(y)| \leq |g(x) - x_n| + |x_n - y_n| + |y_n - g(y)| \leq 3||x-y||.$$

This implies that g is a Lipchitz function. □

Theorem 4.3. *If $x, y \in F(\mathscr{I}^{\mathscr{K}})$ then $xy \in F(\mathscr{I}^{\mathscr{K}})$ and $g(xy) = g(x)g(y)$.*

Proof. Let $\varepsilon > 0$. Then there exist $M_1, M_2 \in \mathscr{I}^*$ such that the sets

$$B_x = \{n \in M_1 : |x_n - g(x)| < \varepsilon\},$$

$$B_y = \{n \in M_2 : |y_n - g(y)| < \varepsilon\}$$

belong to $\mathscr{K}^*$. Then, for $M = M_1 \cap M_2 \in \mathscr{I}^*$, the following sets

$$B_x = \{n \in M : |x_n - g(x)| < \varepsilon\},$$

$$B_y = \{n \in M : |y_n - g(y)| < \varepsilon\}$$

belong to $\mathscr{K}^*$. Now,

$$|x_n y_n - g(x)g(y)| = |x_n y_n - x_n g(y) + x_n g(y) - g(x)g(y)|$$
$$\leq |x_n||y_n - g(y)| + |g(y)||x_n - g(x)|.$$

As $F(\mathscr{I}^{\mathscr{K}}) \subseteq l_\infty$, there exist $N \in \mathbb{R}$ such that $|x_n| < N$ and $|g(y)| < N$. Thus, we get

$$|x_n y_n - g(x)g(y)| \leq N\varepsilon + N\varepsilon = 2N\varepsilon,$$

for all $n \in B_x \cap B_y \in \mathscr{K}^*$. Hence $xy \in F(\mathscr{I}^{\mathscr{K}})$ and $g(xy) = g(x)g(y)$. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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