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THE HYPER ZAGREB INDEX OF SOME PRODUCT GRAPHS

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Abstract. In this paper some basic mathematical expressions for the Hyper Zagreb index of Product Graphs containing the symmetric difference, disjunction and tensor product have been computed.

Keywords:hyper-zagreb index; symmetric difference; disjunction; tensor product.

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1. Introduction

All graphs considered here are simple, connected and finite. Let $V(G), E(G), d_G(v)$ and $d_G(u,v)$ denote the vertex set, the edge set, the degree of a vertex and the distance between the vertices u and v of a graph G respectively. A graph with n vertices and m edges is called a (n,m) graph. First we present the definitions and notations which are required throughout this paper. The complement $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ in which 2 vertices are adjacent if they are not adjacent in G.

Clearly $d_{\overline{G}}(u) = n-1-d_G(u)$ where n is the number of vertices in G.

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A topological index is a numerical parameter mathematically derived from the graph structure. Topological indices and graph invariants based on the distances between vertices of a graph or vertex degrees are widely used for characterizing molecular graphs. The Wiener index is the first and most studied topological indices both from theoretical point of view and applications.

The Wiener index [12] is the first and most studied topological indices and it is defined as

$$W(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d_G(u,v).$$

Gutman and Trinajstic in 1972 [5] introduced the first and second Zagreb indices to study the structure-dependency of the total π-electron energy and are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

Ashrafi A.R in 2010 [1] defined the first and second Zagreb coindices as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)]$$

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v)$$

The forgotten index F [4] is defined as

$$F(G) = \sum_{uv \in E(G)} [d_G^2(u) + d_G^2(v)] = \sum_{v \in V(G)} (d_G(v))^3$$

G.H. Shirdel [10] defined the Hyper Zagreb index as

$$HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2$$

In this paper, the Hyper zagreb index of join, Cartesian product, composition and corona product of graphs are computed.The chemical properties of Hyper Zagreb indices were discussed in [2],[3] . Nilanjan De et al.[8] found the Hyper Zagreb index of bridge and chain graphs. In [9] K.Pattabiraman computed the Hyper Zagreb indices and its coindices of graphs. Here we continue the line of research by exploring the behaviour of the Hyper Zagreb index under several important graph operations.In [6] Liu.et.al. found the Hyper-Zagreb index of Cacti with perfect matchings.

In [11] the hyper Zagreb co index is defined as

$$\overline{HM}(G) = \sum_{uv \notin E(G)} (d_G(u) + d_G(v))^2$$

The tensor product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \times H) = \{(x,\ a)(y,\ b) : xy \in E(G)$ and $ab\ \in E(H)\}$.

The symmetric difference $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) =\{\ (x,\ a)(y,\ b) : xy \in E(G)$ or $ab \in E(H)$ but not both.$\}$

The disjunction $G \vee H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and

$$E(G \vee H) = \{(x,\ a)(y,\ b) : xy \in E(G) \text{ or } ab \in E(H)\}$$

In this paper, Hyper Zagreb index of symmetric difference, disjunction and tensor product of two graphs are computed.

We begin with following lemma which can be used for computing the above graph operations.

Lemma 1.1. [3, 5]

(a) $d_{G \times H}(x,y) = d_G(x) d_G(y)$, *where* $(x,\ y) \in E(G \times H)$

(b) $d_{G \oplus H}(x,\ y) = n_2 d_G(x) + n_1 d_H(y) - 2 d_G(x) d_H(y)$, *where* $(x,\ y) \in E(G \oplus H)$.

(c) $d_{G \vee H}(x,\ y) = n_2 d_G(x) + n_1 d_H(y) - d_G(x) d_H(y)$, *where* $(x,\ y) \in E(G \vee H)$.

Remark 1.2. [7] For a graph G, let $A(G) = \{(x,y) \in V(G) \times V(G) \,|\, x$ and y are adjacent in $G\}$ and let $B(G) = \{(x,y) \in V(G) \times V(G) \,|\, x$ and y are not adjacent in $G\}$. For each $x \in V(G)$, $(x,x) \in B(G)$. Clearly, $A(G) \cup B(G) = V(G) \times V(G)$. The summation $\sum_{(x,y) \in A(G)}$ runs over the ordered pairs of $A(G)$. For simplicity, we write the summation $\sum_{(x,y) \in A(G)}$ as $\sum_{xy \in G}$. Similarly, we write the summation $\sum_{(x,y) \in B(G)}$ as $\sum_{xy \notin G}$. Also the summation $\sum_{xy \in E(G)}$ runs over the edges of G.

Let G be a simple graph with n vertices. Then

Lemma 1.3. [5]

(a) $\sum_{xy\in G} 1 = 2e(G)$

(b) $\sum_{xy\in G} d_G(x) = M_1(G)$.

(c) $\sum_{xy\in G} d_G(x)d_G(y) = 2M_2(G)$.

(d) $\sum_{xy\in G} d_G^2(x) = F(G)$

(e) $\sum_{xy\in G} (d_G(x)+d_G(y))^2 = 2HM(G)$

Here we find the following lemma to compute the graph operations.

Lemma 1.4. $\sum_{xy\notin G} d_G(x) = 2(n-1)e(\overline{G}) - M_1(\overline{G})$

Proof:

$$\begin{aligned}\sum_{xy\notin G} d_G(x) &= \sum_{xy\in\overline{G}} (n-1-d_{\overline{G}}(x))\\ &= (n-1)(2e(\overline{G})) - M_1(\overline{G})\end{aligned}$$

□

Lemma 1.5. $\sum_{xy\notin G} d_G^2(x) = 2(n-1)^2 e(\overline{G}) + F(\overline{G}) - 2(n-1)M_1(\overline{G})$

Proof:

$$\begin{aligned}\sum_{xy\notin G} d_G^2(x) &= \sum_{xy\in\overline{G}} (n-1-d_{\overline{G}}(x))^2\\ &= \sum_{xy\in\overline{G}} ((n-1)^2 + d_{\overline{G}}^2(x) - 2(n-1)d_{\overline{G}}(x))\\ &= (n-1)^2(2e(\overline{G})) + F(\overline{G}) - 2(n-1)M_1(\overline{G})\end{aligned}$$

□

Lemma 1.6. $\sum_{xy\notin G} d_G(x)d_G(y) = 2[(n-1)^2 e(\overline{G}) - (n-1)M_1(\overline{G}) + M_2(\overline{G})]$

Proof:

$$\sum_{xy\notin G} d_G(x)d_G(y) = \sum_{xy\in\overline{G}} (n-1-d_{\overline{G}}(x))(n-1-d_{\overline{G}}(y))$$

$$= \sum_{xy \in \overline{G}} [(n-1)^2 - (n-1)(d_{\overline{G}}(x) + d_{\overline{G}}(y)) + d_{\overline{G}}(x) d_{\overline{G}}(y)]$$

$$= 2[(n-1)^2 e(\overline{G}) - (n-1) M_1(\overline{G}) + M_2(\overline{G})]$$

$\square$

2. The Hyper Zagreb Index of Tensor Product of Graphs

Theorem 2.1. *Let G and H be two connected graphs with n_1 and n_2 vertices and m_1 and m_2 edges respectively. Then*

$$HM(G \times H) = 2F(G)F(H) + 8M_2(G)M_2(H) \ .$$

Proof:

$$\begin{aligned}
2HM(G \times H) &= \sum_{(x,a)(y,b) \in G \times H} (d_{G \times H}(x,\ a) + d_{G \times H}(y,\ b))^2 \\
&= \sum_{xy \in G} \sum_{ab \in H} [(d_{G \times H}(x,\ a) + d_{G \times H}(y,\ b))^2 + (d_{G \times H}(x,\ b) + d_{G \times H}(y,\ a))^2] \\
&= \sum_{xy \in G} \sum_{ab \in H} (d_G(x) d_H(a) + d_G(y) d_H(b))^2 \\
&\quad + \sum_{xy \in G} \sum_{ab \in H} (d_G(x) d_H(b) + d_G(y) d_H(a))^2 \\
&= T_1 + T_2 \\
T_1 &= \sum_{xy \in G} \sum_{ab \in H} (d_G(x) d_H(a) + d_G(y) d_H(b))^2 \\
&= \sum_{xy \in G} \sum_{ab \in H} (d_G^2(x) d_H^2(a) + d_G^2(y) d_H^2(b) + 2 d_G(x) d_H(a) d_G(y) d_H(b)) \\
&= 2F(G)F(H) + 8M_2(G)M_2(H) \\
T_2 &= \sum_{xy \in G} \sum_{ab \in H} (d_G(x) d_H(b) + d_G(y) d_H(a))^2 \\
&= \sum_{xy \in G} \sum_{ab \in H} (d_G^2(x) d_H^2(b) + d_G^2(y) d_H^2(a) + 2 d_G(x) d_H(b) d_G(y) d_H(a)) \\
&= 2F(G)F(H) + 8M_2(G)M_2(H)
\end{aligned}$$

Adding T_1 and T_2 we have

$$HM(G\times H) = 2F(G)F(H) + 8M_2(G)M_2(H)$$

$\square$

3. THE HYPER ZAGREB INDEX OF SYMMETRIC DIFFERENCE OF GRAPHS

Here we compute the hyper Zagreb index of symmetric difference of two graphs.

Theorem 3.1. *Let G and H be two connected graphs with n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then*

$$\begin{aligned}
HM(G\oplus H) = {} & 2n_2^3 HM(G) + 8n_1^2 m_1 M_1(H) + 8M_1(H)HM(G) + 16n_1n_2m_2M_1(G) \\
& - 16n_2m_2HM(G) - 16n_1M_1(G)M_1(H) + 2n_1^3 HM(H) + 8n_2^2 m_2 M_1(G) \\
& + 8M_1(G)HM(H) + 16n_1n_2m_1M_1(H) - 16n_2M_1(G)M_1(H) \\
& - 16m_1n_1HM(H) + 8n_2^2\overline{m}_2 HM(G) + 8m_1n_1^2\overline{HM}(H) \\
& + 16F(G)\left(2(n_2-1)^2\overline{m}_2 + F(\overline{H}) - 2(n_2-1)M_1(\overline{H})\right) \\
& + 8n_1n_2M_1(G)\left(4\overline{m}_2(n_2-1) - 2M_1(\overline{H})\right) \\
& - 16n_2\left(F(G)+2M_2(G)\right)\left(2\overline{m}_2(n_2-1) - M_1(\overline{H})\right) \\
& - 16n_1M_1(G)(4(n_2-1)^2\overline{m}_2 - 4(n_2-1)M_1(\overline{H}) + 2M_2(\overline{H}) + F(\overline{H})) \\
& + 64M_2(G)((n_2-1)^2\overline{m_2} - (n_2-1)M_1(\overline{H}) + M_2(\overline{H})) \\
+ {} & 8m_2n_2^2\overline{HM}(G) + 8\overline{m}_1n_1^2 HM(H) + 16F(H)[2(n_1-1)^2\overline{m_1} + F(\overline{G}) \\
& - 2(n_1-1)M_1(\overline{G})] + 16n_1n_2(2(n_1-1)\overline{m}_1 - M_1(\overline{G}))M_1(H) \\
& - 16n_2M_1(H)(4(n_1-1)^2\overline{m}_1 - 4(n_1-1)M_1(\overline{G}) + 2M_2(\overline{G}) + F(\overline{G})) \\
& - 16n_1(F(H)+2M_2(H))(2(n_1-1)\overline{m}_1 - M_1(\overline{G})) \\
& + 64M_2(H)((n_1-1)^2\overline{m}_1 - (n_1-1)M_1(\overline{G}) + M_2(\overline{G}))
\end{aligned}$$

Proof:

$$\begin{aligned}
2HM(G\oplus H) &= \sum_{xy\in G}\sum_{a\in V(H)}(d_{G\oplus H}(x,\ a)+d_{G\oplus H}(y,\ a))^2 \\
&+\sum_{ab\in H}\sum_{x\in V(G)}(d_{G\oplus H}(x,\ a)+d_{G\oplus H}(x,\ b))^2 \\
&+\sum_{xy\in G}\sum_{ab\notin H}\{(d_{G\oplus H}(x,\ a)+d_{G\oplus H}(y,\ b))^2+(d_{G\oplus H}(x,\ b)+d_{G\oplus H}(y,\ a))^2\} \\
&+\sum_{xy\notin G}\sum_{ab\in H}\{(d_{G\oplus H}(x,\ a)+d_{G\oplus H}(y,\ b))^2+(d_{G\oplus H}(x,\ b)+d_{G\oplus H}(y,\ a))^2\} \\
&= S_1+S_2+S_3+S_4
\end{aligned}$$

where S_1, S_2, S_3 and S_4 denote the sums of the above terms in order.

$$\begin{aligned}
S_1 &= \sum_{xy\in G}\sum_{a\in V(H)}(d_{G\oplus H}(x,\ a)+d_{G\oplus H}(y,\ a))^2 \\
&= \sum_{xy\in G}\sum_{a\in V(H)}[(n_2d_G(x)+n_1d_H(a)-2d_G(x)d_H(a)) \\
&\quad +(n_2d_G(y)+n_1d_H(a)-2d_G(y)d_H(a))]^2 \\
&= \sum_{xy\in G}\sum_{a\in V(H)}[n_2(d_G(x)+d_G(y))+2n_1d_H(a)-2d_H(a)(d_G(x)+d_G(y))]^2 \\
S_1 &= 2n_2^3HM(G)+8n_1^2m_1M_1(H)+8M_1(H)HM(G)+16n_1n_2m_2M_1(G) \\
&\quad -16n_2m_2HM(G)-16n_1M_1(G)M_1(H)
\end{aligned}$$

Now,

$$\begin{aligned}
S_2 &= \sum_{x\in V(G)}\sum_{ab\in H}(d_{G\oplus H}(x,\ a)+d_{G\oplus H}(x,\ b))^2 \\
&= \sum_{x\in V(G)}\sum_{ab\in H}(n_2d_G(x)+n_1d_H(a)-2d_G(x)d_H(a)+n_2d_G(x)+n_1d_H(b)-2d_G(x)d_H(b))^2 \\
&= \sum_{x\in V(G)}\sum_{ab\in H}(2n_2d_G(x)+n_1(d_H(a)+d_H(b))-2d_G(x)(d_H(a)+d_H(b))^2 \\
S_2 &= 8n_2^2M_1(G)m_2+2n_1^3HM(H)+8M_1(G)HM(H)+16n_1n_2m_1M_1(H) \\
&\quad -16n_2M_1(G)M_1(H)-16n_1m_1HM(H)
\end{aligned}$$

and

$$\begin{aligned}
S_3 &= \sum_{xy\in G}\sum_{ab\notin H}(d_{G\oplus H}(x,\, a)+d_{G\oplus H}(y,\, b))^2+(d_{G\oplus H}(x,\, b)+d_{G\oplus H}(y,\, a))^2\\
&= \sum_{xy\in G}\sum_{ab\notin H}(d_{G\oplus H}(x,\, a)+d_{G\oplus H}(y,\, b))^2+\sum_{xy\in G}\sum_{ab\notin H}(d_{G\oplus H}(x,\, b)+d_{G\oplus H}(y,\, a))^2\\
&= S_{3,1}+S_{3,2}\\
S_{3,1} &= \sum_{xy\in G}\sum_{ab\notin H}(d_{G\oplus H}(x,\, a)+d_{G\oplus H}(y,\, b))^2\\
&= \sum_{xy\in G}\sum_{ab\notin H}(n_2d_G(x)+n_1d_H(a)-2d_G(x)d_H(a))+(n_2d_G(y)+n_1d_H(b)\\
&\quad -2d_G(y)d_H(b))^2\\
&= \sum_{xy\in G}\sum_{ab\notin H}(n_2(d_G(x)+d_G(y))+n_1(d_H(a)+d_H(b))-2d_G(x)d_H(a)\\
&\quad -2d_G(y)d_H(b))^2\\
S_{3,1} &= 4n_2^2\overline{m}_2HM(G)+4m_1n_1^2\overline{HM}(H)+8F(G)[2(n_2-1)^2\overline{m}_2+F(\overline{H})\\
&\quad -2(n_2-1)M_1(\overline{H})]+4n_1n_2M_1(G)(4\overline{m}_2(n_2-1)-2M_1(\overline{H}))\\
&\quad -8n_2(F(G)+2M_2(G))(2\overline{m}_2(n_2-1)-M_1(\overline{H}))-8n_1M_1(G)(4(n_2-1)^2\overline{m}_2\\
&\quad -4(n_2-1)M_1(\overline{H})+2M_2(\overline{H})+F(\overline{H}))\\
&\quad +32M_2(G)((n_2-1)^2\overline{m}_2-(n_2-1)M_1(\overline{H})+M_2(\overline{H}))\\
S_{3,2} &= \sum_{xy\in G}\sum_{ab\notin H}(d_{G\oplus H}(x,\, b)+d_{G\oplus H}(y,\, a))^2\\
&= \sum_{xy\in G}\sum_{ab\notin H}(n_2d_G(x)+n_1d_H(b)-2d_G(x)d_H(b))+(n_2d_G(y)+n_1d_H(a)\\
&\quad -2d_G(y)d_H(a))^2\\
&= \sum_{xy\in G}\sum_{ab\notin H}(n_2(d_G(x)+d_G(y))+n_1(d_H(a)+d_H(b))-2d_G(x)d_H(b)\\
&\quad -2d_G(y)d_H(a))^2\\
S_{3,2} &= 4n_2^2\overline{m}_2HM(G)+4m_1n_1^2\overline{HM}(H)+8F(G)(2(n_2-1)^2\overline{m}_2+F(\overline{H})\\
&\quad -2(n_2-1)M_1(\overline{H}))+4n_1n_2M_1(G)(4\overline{m}_2(n_2-1)-2M_1(\overline{H}))
\end{aligned}$$

$$-8n_2(F(G)+2M_2(G))(2\overline{m}_2(n_2-1)-M_1\overline{H}))-8n_1M_1(G)(4(n_2-1)^2\overline{m}_2$$

$$-4(n_2-1)M_1(\overline{H})+2M_2(\overline{H})+F(\overline{H}))$$

$$+32M_2(G)((n_2-1)^2\overline{m}_2-(n_2-1)M_1(\overline{H})+M_2(\overline{H}))$$

$$S_3 = S_{3,1}+S_{3,2}$$

$$= 8n_2^2\overline{m}_2HM(G)+8m_1n_1^2\overline{HM}(H)+16F(G)(2(n_2-1)^2\overline{m}_2+F(\overline{H})$$

$$-2(n_2-1)M_1(\overline{H}))+8n_1n_2M_1(G)(4\overline{m}_2(n_2-1)-2M_1(\overline{H}))$$

$$-16n_2(F(G)+2M_2(G))(2\overline{m}_2(n_2-1)-M_1(\overline{H}))-16n_1M_1(G)(4(n_2-1)^2\overline{m}_2$$

$$-4(n_2-1)M_1(\overline{H})+2M_2(\overline{H})+F(\overline{H}))$$

$$+64M_2(G)((n_2-1)^2\overline{m}_2-(n_2-1)M_1(\overline{H})+M_2(\overline{H}))$$

Similarly,

$$S_4 = \sum_{xy\notin G}\sum_{ab\in H}(d_{G\oplus H}(x,\ a)+d_{G\oplus H}(y,\ b))^2+(d_{G\oplus H}(x,\ b)+d_{G\oplus H}(y,\ a))^2$$

$$= S_{4,1}+S_{4,2}$$

$$S_{4,1} = \sum_{xy\notin G}\sum_{ab\in H}(d_{G\oplus H}(x,\ a)+d_{G\oplus H}(y,\ b))^2$$

$$= \sum_{xy\notin G}\sum_{ab\in H}(n_2d_G(x)+n_1d_H(a)-2d_G(x)d_H(a))+(n_2d_G(y)+n_1d_H(b)$$

$$-2d_G(y)d_H(b))^2$$

$$= \sum_{xy\notin G}\sum_{ab\in H}(n_2(d_G(x)+d_G(y))+n_1(d_H(a)+d_H(b))-2d_G(x)d_H(a)$$

$$-2d_G(y)d_H(b))^2$$

$$S_{4,1} = 4m_2n_2^2\overline{HM}(G)+4\overline{m}_1n_1^2HM(H)+8F(H)[2(n_1-1)^2\overline{m}_1+F(\overline{G})$$

$$-2(n_1-1)M_1(\overline{G})]+8n_1n_2(2(n_1-1)\overline{m}_1-M_1(\overline{G}))M_1(H)$$

$$-8n_2M_1(H)(4(n_1-1)^2\overline{m}_1-4(n_1-1)M_1(\overline{G})+2M_2(\overline{G})+F(\overline{G}))$$

$$-8n_1(F(H)+2M_2(H))(2(n_1-1)\overline{m}_1-M_1(\overline{G}))$$

$$+32M_2(H)((n_1-1)^2\overline{m}_1-(n_1-1)M_1(\overline{G})+M_2(\overline{G}))$$

$$
\begin{aligned}
S_{4,2} &= \sum_{xy\notin G}\sum_{ab\in H}(d_{G\oplus H}(x,\ b)+d_{G\oplus H}(y,\ a))^2\\
&= \sum_{xy\notin G}\sum_{ab\in H}(n_2d_G(x)+n_1d_H(b)-2d_G(x)d_H(b))+(n_2d_G(y)+n_1d_H(a)\\
&\quad -2d_G(y)d_H(a))^2\\
&= \sum_{xy\notin G}\sum_{ab\in H}(n_2(d_G(x)+d_G(y))+n_1(d_H(a)+d_H(b))-2d_G(x)d_H(b)\\
&\quad -2d_G(y)d_H(a))^2\\
S_{4,2} &= 4m_2n_2^2\overline{HM}(G)+4\overline{m}_1n_1^2HM(H)+8F(H)[2(n_1-1)^2\overline{m}_1+F(\overline{G})\\
&\quad -2(n_1-1)M_1(\overline{G})]+8n_1n_2(2(n_1-1)\overline{m}_1-M_1(\overline{G}))M_1(H)\\
&\quad -8n_2M_1(H)(4(n_1-1)^2\overline{m}_1-4(n_1-1)M_1(\overline{G})+2M_2(\overline{G})+F(\overline{G}))\\
&\quad -8n_1(F(H)+2M_2(H))(2(n_1-1)\overline{m}_1-M_1(\overline{G}))\\
&\quad +32M_2(H)((n_1-1)^2\overline{m}_1-(n_1-1)M_1(\overline{G})+M_2(\overline{G}))\\
S_4 &= S_{4,1}+S_{4,2}\\
&= 8m_2n_2^2\overline{HM}(G)+8\overline{m}_1n_1^2HM(H)+16F(H)[2(n_1-1)^2\overline{m}_1+F(\overline{G})\\
&\quad -2(n_1-1)M_1(\overline{G})]+16n_1n_2(2(n_1-1)\overline{m}_1-M_1(\overline{G}))M_1(H)\\
&\quad -16n_2M_1(H)(4(n_1-1)^2\overline{m}_1-4(n_1-1)M_1(\overline{G})+2M_2(\overline{G})+F(\overline{G}))\\
&\quad -16n_1(F(H)+2M_2(H))(2(n_1-1)\overline{m}_1-M_1(\overline{G}))\\
&\quad +64M_2(H)((n_1-1)^2\overline{m}_1-(n_1-1)M_1(\overline{G})+M_2(\overline{G}))
\end{aligned}
$$

Adding S_1, S_2, S_3 and S_4 we get the desired result. □

4. THE HYPER ZAGREB INDEX OF DISJUNCTION OF GRAPHS

Here we compute the hyper Zagreb Index of disjunction of Graphs.

Theorem 4.1. *Let G and H be connected graphs with n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then*

$$2HM(G\vee H)=8n_2^2m_2M_1(G)+2n_1^3HM(H)+2M_1(G)HM(H)$$

$$+16m_1n_1n_2M_1(H)-8n_2M_1(G)M_1(H)-8m_1n_1HM(H)+8m_2n_2^2\overline{HM}(G)$$
$$+8\overline{m}_1n_1^2HM(H)+4F(H)(2(n_1-1)^2\overline{m}_1+F(\overline{G})-2(n_1-1)M_1(\overline{G}))$$
$$+16n_1n_2(2(n_1-1)\overline{m}_1-M_1(\overline{G}))M_1(H)-8n_2M_1(H)(4(n_1-1)^2\overline{m}_1$$
$$-4(n_1-1)M_1(\overline{G})+F(\overline{G})+2M_2(\overline{G}))-8n_1(2M_2(H)+F(H))$$
$$(2(n_1-1)\overline{m}_1-M_1(\overline{G}))+16M_2(H)((n_1-1)^2\overline{m}_1-(n_1-1)M_1(\overline{G})+M_2(\overline{G}))$$
$$+2n_2^4HM(G)+2n_1^2m_1(8m_2^2+2n_2)M_1(H)+2n_2F(G)M_1(H)+16m_2n_1n_2^2M_1(G)$$
$$-8m_2n_2^2(F(G)+2M_2(G))-4n_1M_1(G)(n_2M_1(H)+4m_2^2)+16m_2^2M_2(G)$$

Proof:

$$\begin{aligned}2HM(G\vee H)&=\sum_{xy\in G}\sum_{a\in V(H)}\sum_{b\in V(H)}(d_{G\vee H}(x,\ a)+d_{G\vee H}(y,\ b))^2\\&\quad+\sum_{ab\in H}\sum_{x\in V(G)}(d_{G\vee H}(x,\ a)+d_{G\vee H}(x,\ b))^2\\&\quad+\sum_{xy\notin G}\sum_{ab\in H}((d_{G\vee H}(x,\ a)+d_{G\vee H}(y,\ b))^2+(d_{G\vee H}(x,\ b)+d_{G\vee H}(y,\ a))^2\\&=S_3+S_1+S_2\end{aligned}$$

where S_1,S_2 and S_3 denote the sums of the above terms in order. Next we calculate S_1

$$\begin{aligned}S_1&=\sum_{x\in V(G)}\sum_{ab\in H}(d_{G\vee H}(x,\ a)+d_{G\vee H}(x,\ b))^2\\&=\sum_{x\in V(G)}\sum_{ab\in H}(n_2d_G(x)+n_1d_H(a)-d_G(x)d_H(a)+n_2d_G(x)+n_1d_H(b)-d_G(x)d_H(b))^2\\&=\sum_{x\in V(G)}\sum_{ab\in H}(2n_2d_G(x)+n_1(d_H(a)+d_H(b))-d_G(x)(d_H(a)+d_H(b))^2\\&=8n_2^2m_2M_1(G)+2n_1^3HM(H)+2M_1(G)HM(H)+16m_1n_1n_2M_1(H)\\&\quad-8n_2M_1(G)M_1(H)-8m_1n_1HM(H)\end{aligned}$$

Now,

$$S_2=\sum_{ab\in H}\sum_{xy\notin G}((d_{G\vee H}(x,\ a)+d_{G\vee H}(y,\ b))^2+(d_{G\vee H}(x,\ b)+d_{G\vee H}(y,\ a))^2$$

$$= S_{2,1} + S_{2,2}$$

$$\begin{aligned}
S_{2,1} &= \sum_{xy\notin G}\sum_{ab\in H}(d_{G\vee H}(x,\ a)+d_{G\vee H}(y,\ b))^2\\
&= \sum_{xy\notin G}\sum_{ab\in H}(n_2d_G(x)+n_1d_H(a)-d_G(x)d_H(a)+n_2d_G(y)+n_1d_H(b)-d_G(y)d_H(b))^2\\
&= \sum_{xy\notin G}\sum_{ab\in H}(n_2(d_G(x)+d_G(y)+n_1(d_H(a)+d_H(b))-d_G(x)d_H(a)-d_G(y)d_H(b))^2\\
&= 4m_2n_2^2\overline{HM}(G)+4\overline{m}_1n_1^2HM(H)+2F(H)(2(n_1-1)^2\overline{m}_1+F(\overline{G})-2(n_1-1)M_1(\overline{G}))\\
&\quad +8n_1n_2(2(n_1-1)\overline{m}_1-M_1(\overline{G}))M_1(H)-4n_2M_1(H)[4(n_1-1)^2\overline{m}_1\\
&\quad -4(n_1-1)M_1(\overline{G})+F(\overline{G})+2M_2(\overline{G})]-4n_1(F(H)+2M_2(H))(2(n_1-1)\overline{m}_1-M_1(\overline{G}))\\
&\quad +8M_2(H)[(n_1-1)^2\overline{m}_1-(n_1-1)M_1(\overline{G})+M_2(\overline{G})]
\end{aligned}$$

Similarly,

$$\begin{aligned}
S_{2,2} &= \sum_{xy\notin G}\sum_{ab\in H}(d_{G\vee H}(x,\ b)+d_{G\vee H}(y,\ a))^2\\
&= \sum_{xy\notin G}\sum_{ab\in H}(n_2d_G(x)+n_1d_H(b)-d_G(x)d_H(b)+n_2d_G(y)+n_1d_H(a)-d_G(y)d_H(a))^2\\
&= \sum_{xy\notin G}\sum_{ab\in H}(n_2(d_G(x)+d_G(y)+n_1(d_H(a)+d_H(b))-d_G(x)d_H(b)-d_G(y)d_H(a))^2\\
&= 4m_2n_2^2\overline{HM}(G)+4\overline{m}_1n_1^2HM(H)+2F(H)(2(n_1-1)^2\overline{m}_1+F(\overline{G})-2(n_1-1)M_1(\overline{G}))\\
&\quad +8n_1n_2(2(n_1-1)\overline{m}_1-M_1(\overline{G}))M_1(H)-4n_2M_1(H)[4(n_1-1)^2\overline{m}_1\\
&\quad -4(n_1-1)M_1(\overline{G})+F(\overline{G})+2M_2(\overline{G})]-4n_1(F(H)+2M_2(H))(2(n_1-1)\overline{m}_1-\\
&\quad M_1(\overline{G}))+8M_2(H)[(n_1-1)^2\overline{m}_1-(n_1-1)M_1(\overline{G})+M_2(\overline{G})]
\end{aligned}$$

$$\begin{aligned}
S_2 &= S_{2,1}+S_{2,2}\\
&= 8m_2n_2^2\overline{HM}(G)+8\overline{m}_1n_1^2HM(H)+4F(H)(2(n_1-1)^2\overline{m}_1+F(\overline{G})-2(n_1-1)M_1(\overline{G}))\\
&\quad +16n_1n_2(2(n_1-1)\overline{m}_1-M_1(\overline{G}))M_1(H)-8n_2M_1(H)[4(n_1-1)^2\overline{m}_1\\
&\quad -4(n_1-1)M_1(\overline{G})+F(\overline{G})+2M_2(\overline{G})]-8n_1(F(H)+2M_2(H))(2(n_1-1)\overline{m}_1-M_1(\overline{G}))
\end{aligned}$$

$$+16M_2(H)[(n_1-1)^2\overline{m}_1-(n_1-1)M_1(\overline{G})+M_2(\overline{G})]$$

Now,

$$\begin{aligned}
S_3 &= \sum_{xy\in G}\sum_{a\in V(H)}\sum_{b\in V(H)}(d_{G\vee H}(x,a)+d_{G\vee H}(y,b))^2\\
&= \sum_{xy\in G}\sum_{a\in V(H)}\sum_{b\in V(H)}(n_2d_G(x)+n_1d_H(a)-d_G(x)d_H(a)+n_2d_G(y)+n_1d_H(b)-d_G(y)d_H(b))^2\\
&= \sum_{xy\in G}\sum_{a\in V(H)}\sum_{b\in V(H)}(n_2(d_G(x)+d_G(y))+n_1(d_H(a)+d_H(b))-d_G(x)d_H(a)-d_G(y)d_H(b))^2\\
&= 2n_2^4HM(G)+2n_1^2m_1(8m_2^2+2n_2)M_1(H)+2n_2F(G)M_1(H)\\
&\quad +16m_2n_1n_2^2M_1(G)-8m_2n_2^2(F(G)+2M_2(G))-4n_1M_1(G)(n_2M_1(H)+4m_2^2)\\
&\quad +16m_2^2M_2(G)
\end{aligned}$$

By adding S_1, S_2 and S_3 the desired result follows. □

5. Conclusion

In this paper, some basic mathematical expressions for the Hyper Zagreb index of Product Graphs containing the symmetric difference, disjunction and tensor product are derived.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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