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PCCP OF WHEEL GRAPH FAMILY WITH NULL, CHAIN, FAN AND CYCLE GRAPH

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Abstract. A function f : P(G) ∪ L(G) ∪ R(G) → C is said to be perfect coloring of the graph G, if f(x) $\neq$ f(y) for any two adjoint or incident elements x,y ∈ P(G) ∪ L(G) ∪ R(G). And the PC number $\chi^P(G)$ is the least colors needed to color a graph by using perfect coloring. In this paper, we prove the results for perfect coloring of corona product(PCCP) of wheel graph family with null, chain, fan and cycle graph, which leads to perfect chromatic number equivalent to Δ+1, where Δ is the largest degree of the resultant graph in corona product.

Keywords: graph coloring; corona product; perfect coloring.

2010 AMS Subject Classification: 05C15.

1. INTRODUCTION

The graph coloring is preeminent element of graph theory. It is having huge implementations in abundant disciplines like aircraft scheduling, register allocation, sudoku, mobile networking etc. The four color theorem plays important role in graph coloring[5]. The result of four color theorem was proved using PRN of that graph by Bhapkar H R[3].The graph coloring basically deals with vertex, region and edge coloring. The coloring of element (vertex, region or edge) of a connected graph such that adjoining element should receive dissimilar colors is the graph

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coloring. And the least colors needed to color a graph is the chromatic number[10]. Behzad M proposed the concept of total coloring. In this type of coloring adjoint vertices and incident edges receives distinct colors[1][2]. Rosenfeld proved that every cubic graph is having total chromatic number five[4].Rong Luo proved that the face-edge chromatic number is equivalent to edge chromatic hence degree of the graph for any 2-connected planar graph with $\Delta \geq 24$[6]. S. Mohan et al. proposed the tight bounds of vizing's conjecture on total coloring for corona product of two graphs[7]. Bhange A A and Bhapkar H R proposed the concept of perfect coloring and proved the results for some standard graphs[12]. Bhange and Bhapkar proved that PCCP of cycle graph with null, circular and chain graph is Δ+1 of resultant graph[11].S Nada et al. stated the cordiality of the corona between cycle graphs and path graphs[8]. In this paper, the PCCP of wheel graph and it's family i.e. gear graph, helm graph etc. is determined with null, chain, fan and cycle graph.

2. PRELIMINARIES

All graphs assumed in this paper are directionless and plane graphs. In the paper,we have considered graph G with set of vertices and edges as $g_1, g_2, g_3 ... g_n$ and (g_1, g_2),(g_1, g_3), (g_2, g_3) ...(g_{n-1}, g_n) respectively. And $C\{g_i\}$ and $C\{g_i, g_j\}$ is the color of vertex g_i and edge (g_i, g_j) respectively.

Definition 2.1. Consider a graph G=(V(G),E(G),R(G)) having set of vertices V(G), set of edges E(G) and set of regions R(G), then perfect coloring is the mapping h : V(G) $\cup$ E(G) $\cup$ R(G)$\rightarrow$ S, where S is set of colors with following conditions:

(i) h(a) $\neq$ h(b),For any two adjoint vertices a and b of V(G),
(ii) $h(x) \neq h(y)$, For any two adjoint edges x and y of E(G),
(iii) $h(R_i) \neq h(R_j)$, where R_i and R_j are adjoint regions of R(G),
(iv) h(e) $\neq$ g(x) $\neq$ h(y), where e is edge connecting point x and y and
(v) $h(p_i) \neq h(l_j) \neq h(R)$, where p_i and l_j are frontier vertices and edges respectively creating region R.

PC number or Perfect chromatic number($\chi^P(G)$) is least number needed to color any graph which obeys conditions of perfect coloring[11][12].

Definition 2.2. Let M and N be two graphs. Consider a copy of graph M and $|P(M)|$ copies of N and keeping j^{th} vertex of M adjacent to each point of j^{th} copy of graph N, this gives corona product of graph M and N (MoN). Frucht and Harary defined this corona product [9].

3. Main Results

Theorem 3.1 The PCCP of wheel graph W_n and cycle graph C_m is $\Delta+1$ or m+n+1, $\forall$ n$\geq$5, m$\geq$5.

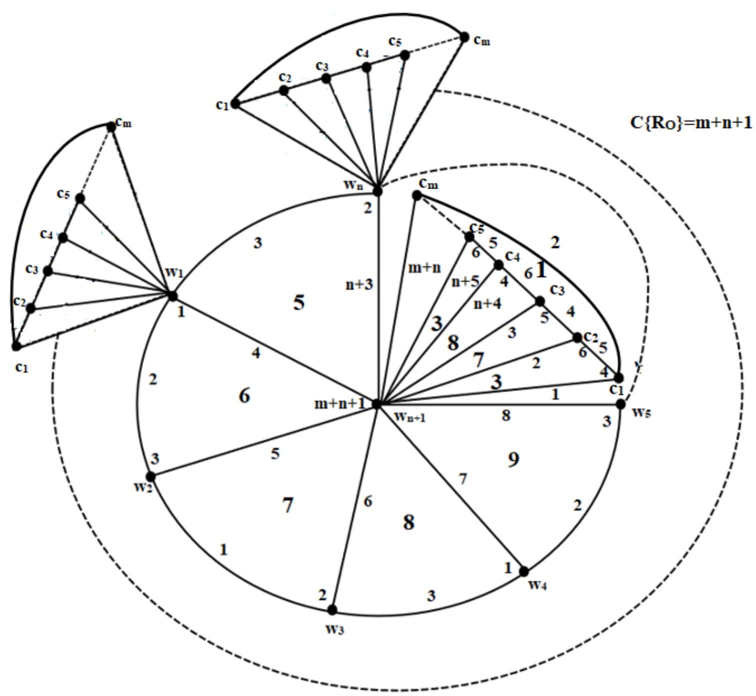

Figure 1. PCCP Of W_n with C_m

Proof. Firstly consider the corona product of wheel graph W_n with n vertices and cycle graph C_m with m vertices(figure 1). Assign colors to all vertices and edges of the resultant graph as

$$\forall j = 1 : n/3 \begin{cases} C\{w_{3j-2}\} = C\{w_{3j-1}, w_{3j}\} = 1, \\ C\{w_{3j-1}\} = C\{w_n, w_1\} = 3, \\ C\{w_{3j}\} = C\{w_{3j-2}, w_{3j-1}\} = 2. \end{cases} \tag{1}$$

(2) $$C\{w_{n+1}, w_j\} = j+3; \forall j = 1:n.$$

(3) $$C\{w_{3j}, w_{3j+1}\} = 3; \forall j = 1:(n-1)/3$$

Consider the corona product at central vertex of W_n i.e. w_{n+1} with cycle graph C_m, as it is highest degree vertex among all vertices.

(4) $$\forall k = 1:m/3 \begin{cases} C\{c_{3k-2}\} = C\{c_{3k-1}, c_{3k}\} = 4, \\ C\{c_{3k}\} = C\{c_{3k-2}, c_{3k-1}\} = 5, \\ C\{c_{3k-1}\} = 6, C\{c_m, c_1\} = 2. \end{cases}$$

Also,

(5) $$C\{w_{n+1}, c_k\} = \begin{cases} k; \forall k = 1:3 \\ n+k; \forall k = 4:m \end{cases}$$

and

(6) $$C\{w_{n+1}\}\text{=m+n+1}$$

Also,

$$C\{c_{3k}, c_{3k+1}\} = 6; \forall k = 1:(m-1)/3.$$

Hence total coloring is

$$\chi^{''}(W_n o C_m) = m+n+1 = \Delta+1.$$

Also, color the regions of the graph to get perfect coloring as

(7) $$\forall j = 1:n-2 \begin{cases} C\{R_{wj}\} = C\{w_{n+1}, w_{j+2}\}, \\ C\{R_{w(n-1)}\} = C\{w_{n+1}, w_1\}, \\ C\{R_{w(n)}\} = C\{w_{n+1}, w_2\}. \end{cases}$$

And

$$\forall k = 1 : (m-1)/3 \begin{cases} C\{R_{wc(3k-2)}\} = 3, \\ C\{R_{wc(3k-1)}\} = 7, \\ C\{R_{wc(3k}\} = 8. \end{cases} \tag{8}$$

Where, R_{wj} is internal bounded region of wheel graph W_n and R_{wck} is internal region between c_k and w_j.

$$C\{R_{bc}\} = 1, \tag{9}$$

$$C\{R_O\} = m+n+1. \tag{10}$$

where R_{bc} is bounded region between c_1 and c_m and R_O is open unbounded region. Hence

$$\chi^P(W_n o C_m) = m+n+1 = \Delta+1.$$

□

Theorem 3.2 The PCCP of wheel graph W_n and chain graph C_m is $\Delta+1$ or m+n+1, for all n≥5, m ≥4.

Proof. The theorem can be proved using analogy of theorem 3.1 and by eliminating edge between c_1 and c_m. □

Theorem 3.3 The PCCP of wheel graph W_n and null graph N_m is $\Delta+1$ or m+n+1, for all n≥5, m ≥1.

Proof. The theorem can be proved using analogy of theorem 3.1 and by eliminating edges between all $c_i's$ for all i=1:m. □

Theorem 3.4 The PCCP of gear graph G_n and cycle graph C_m is $\Delta+1$ or m+n+1, for all n≥5, m≥5.

Proof. Let G_n be gear graph with n vertices and C_m be cycle graph with m vertices. consider the corona product of these graphs (figure 2). Assign Colors to all edges and vertices of the

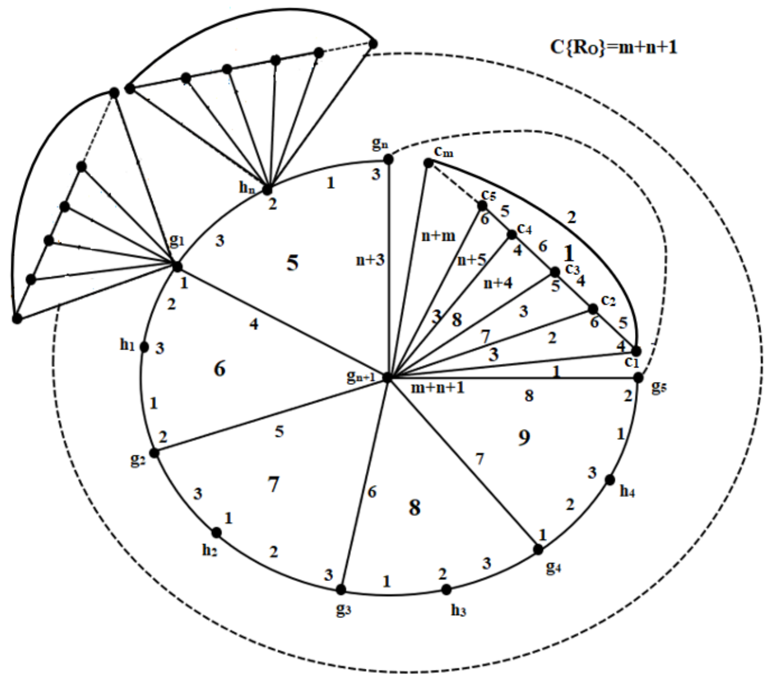

FIGURE 2. PCCP Of G_n with C_m

resultant graph as ∀ j=1:n/3

$$C\{g_{3j-2}\} = C\{h_{3j-1}\} = C\{h_{3j-2}, g_{3j-1}\} = C\{g_{3j}, h_{3j}\} = 1,$$

$$C\{g_{3j-1}\} = C\{h_{3j}\} = C\{h_{3j-1}, g_{3j}\} = C\{g_{3j-2}, h_{3j-2}\} = 2,$$

$$C\{g_{3j}\} = C\{h_{3j-2}\} = C\{g_{3j-1}, h_{3j-1}\} = C\{h_n, g_1\} = 3.$$

and

$$C\{g_{n+1}, g_j\} = j+3; \forall j = 1 : n.$$

Let us consider the corona product of central vertex of graph G_{n+1} i.e. g_{n+1} with cycle graph C_m, as it is highest degree vertex.

Hence for all k=1:m/3,

Color the attached cycle graph using analogy of equation (4), (5) and (6). Also,

$$C\{g_{n+1}\} = m+n+1.$$

Hence total coloring is

$$\chi''(G_n o C_m) = \Delta + 1 = m+n+1.$$

Also, color the internal regions of the graph using analogy of equation (7).

Finally, color the internal region of cycle graph and open unbounded region using analogy of equation (8), (9) and (10). Which gives

$$\chi^P(G_n o C_m) = \Delta + 1 = m + n + 1.$$

□

Theorem 3.5 The PCCP of gear graph G_n and chain graph C_m is Δ+1 or m+n+1, for all m≥5, n≥5.

Proof. The result can be proved using analogy of theorem 3.4. □

Theorem 3.6 The PCCP of gear graph G_n and null graph N_m is Δ+1 or m+n+1, for all m≥1, n≥5.

Proof. The result can be proved using analogy of theorem 3.4. □

Theorem 3.7 The PCCP of helm graph H_n and cycle graph C_m is m+n+1, for all m≥4, n≥5.

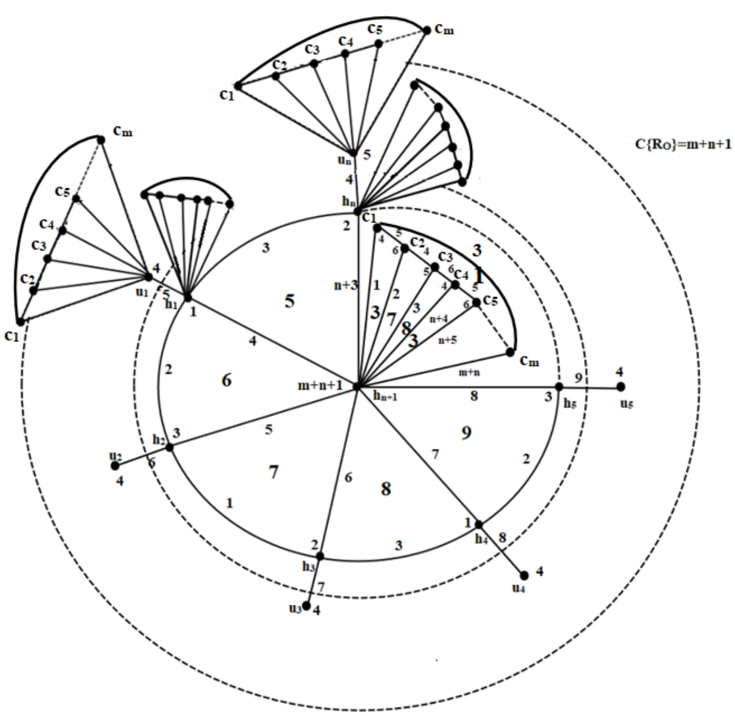

FIGURE 3. PCCP Of H_n with C_m

Proof. Consider a helm graph H_n with n vertices and cycle graph C_m with m vertices. Consider the corona product of these graphs as shown in figure 3. Assign Colors to the vertices and edges of the graphs using analogy of theorem 3.1(eq.(1)-(10)). Also, color the rest of the edges and vertices of helm graph as follows

$$\forall j = 1 : n-1 \begin{cases} C\{h_j, u_j\} = C\{h_{n+1}, h_{j+1}\}, \\ C\{u_j\}=4, \\ C\{u_n\} = 5, \\ C\{h_n, u_n\} = C\{h_{n+1}, h_1\}. \end{cases}$$

Which gives, total coloring as well as perfect coloring as

$$\chi^{''}(H_n o C_m) = \chi^{P}(H_n o C_m) = \Delta + 1 = m + n + 1.$$

□

Theorem 3.8 The PCCP of helm graph H_n and chain graph C_m is Δ +1 or m+n+1, for all m≥4, n≥5.

Proof. This result can be proved using analogy of theorem 3.7. □

Theorem 3.9 The PCCP of helm graph H_n and null graph N_m is Δ +1 or m+n+1, for all m≥1, n≥5.

Proof. This result can be proved using analogy of theorem 3.7. □

Theorem 3.10 The PCCP of helm graph H_n and fan graph F_m is Δ +1 or m+n+2, for all n≥5, m≥4.

Proof. Consider the helm graph H_n with n vertices and fan graph F_m with m vertices. Consider the corona product of them as shown in figure 4. Assign colors to the helm graph using analogy of theorem 3.1 and 3.7. As central vertex H_n is highest degree vertex, let us consider corona product with fan graph at that point.

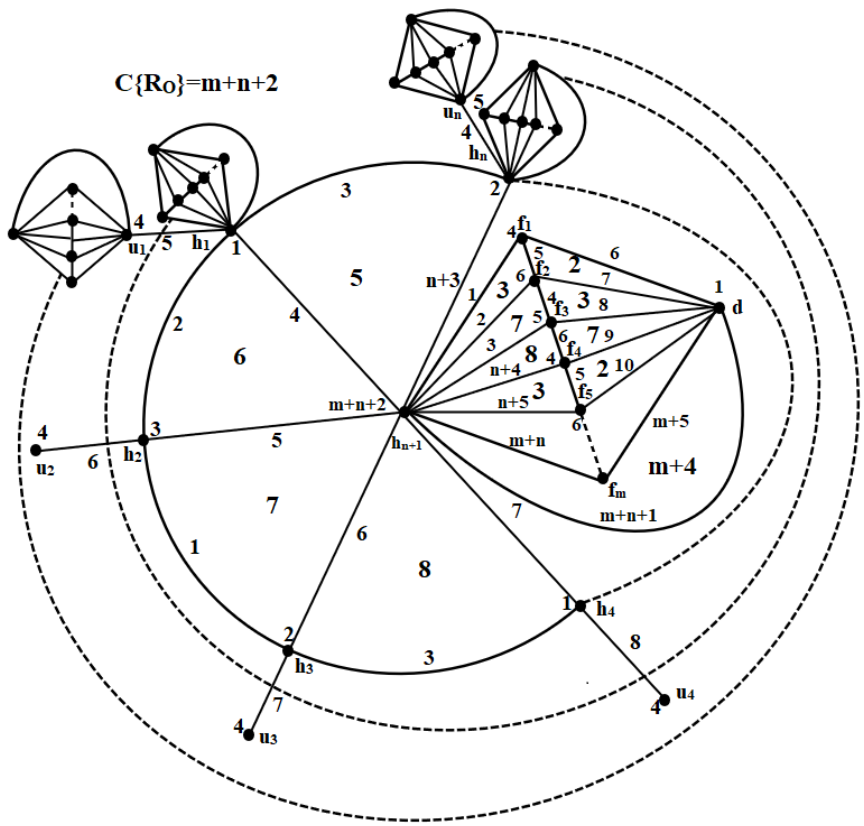

FIGURE 4. PCCP Of H_n and F_m

$$\forall\ \text{k=1:m/3} \begin{cases} C\{f_{3k-2}\} = C\{f_{3k-1}, f_{3k}\} = 4, \\ C\{f_{3k}\} = C\{f_{3k-2}, f_{3k-1}\} = 5, \\ C\{f_{3k-1}\} = 6. \end{cases}$$

And

$$C\{f_{3k}, f_{3k+1}\} = 6; k = 1 : (m-1)/3.$$

$$C\{h_{n+1}, f_k\} = \begin{cases} k & \text{if } 1 \le k \le 3 \\ n+k & \text{if } 4 \le k \le m \end{cases}$$

Also,

$$C\{h_{n+1}, d\} = m+n+1,$$

$$C\{h_{n+1}\} = m+n+2,$$

$$C\{d, f_j\} = k+5; \forall \text{ k=1:m}$$

and

$$C\{d\} = 1$$

Hence

$$\chi^{''}(H_n o F_m) = m+n+2 = \Delta+1.$$

To deduce PC number, color the regions as below

Assign color to R_{hi} i.e. internal bounded region of helm graph H_n using analogy of equation (7).

If R_{hfk} is internal region formed by corona product between F_m and h_{n+1} and R_{dfk} is internal region between F_m and d, then color them as below

$$\text{For k=1:(m-1)/3} \begin{cases} C\{R_{hf(3k-2)}\} = C\{R_{df(3k-1)}\} = 3, \\ C\{R_{hf(3k-1)}\} = C\{R_{df(3k)}\} = 7, \\ C\{R_{hf(3k)}\} = 8, \\ C\{R_{df(3k-2)}\} = 2. \end{cases}$$

Finally,

$$C\{R_{dh}\} = m+4,$$

and

$$C\{R_O\} = m+n+2$$

where R_{dh} is region between d and h_{n+1} and R_O is open unbounded region.

Which proves,

$$\chi^{P}(H_n o F_m) = m+n+2 = \Delta+1.$$

□

Theorem 3.11 The PCCP of wheel graph W_n and fan graph F_m is Δ +1 or m+n+2, for all n≥5, m≥4.

Proof. The theorem can be proved using analogy of theorem 3.10 and by elimination of outer edges of the helm graph. □

Theorem 3.12 The PCCP of gear graph G_n and fan graph F_m is Δ +1 or m+n+2, for all n$\geq$5, m$\geq$4.

Proof. The theorem can be proved using analogy of theorem 3.10. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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