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ON TRUNCATED GENERALIZED CAUCHY DISTRIBUTION

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Abstract. In this paper, we propose a truncated version of *generalized Cauchy* distribution suggested by Rider[19] in a special setting. One possible use for the proposed model is in life-testing where the domain of definition is not only non-negative but also guarantees no failure before a given time (truncated parameter). The parameters, reliability and failure rate functions are estimated using the maximum likelihood and Bayes methods. The Bayes estimates ($BE's$) are obtained under the squared-error and liner exponential ($LINEX$) loss functions. The computations have been carried out using the Markov Chain Monte Carlo (MCMC) algorithm.

Keywords: Guarantee time, maximum likelihood and Bayes estimation; MCMC algorithm; squared error and $LINEX$ loss functions.

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1. Introduction

The Cauchy distribution is a symmetric distribution with bell-shaped density function as the normal distribution but with a greater probability mass in the tails. The distribution is often used in the cases which arise in outlier analysis.

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It is well-known that the Cauchy distribution can arise as the ratio of two independent normal variates. The probability density function (PDF) with location parameter β (representing the population median) and scale parameter γ (representing the semi-quartile range) is given by

$$f_X(x) = \frac{1}{\pi\gamma}\left[1+\left(\frac{x-\beta}{\gamma}\right)^2\right]^{-1}, -\infty < x < \infty, -\infty < \beta < \infty, \gamma > 0. \tag{1}$$

Balmer et al [2], Cane [4], Chan [5], and Howlader and Weiss [13-14] found the maximum likelihood and Bayes estimates of β, γ and the reliability function. Copas [6] and Gabrielsen [10] have established that the joint maximum is unique. Also, Hinkley [11] has carried out large-scale computer simulation of samples of sizes 20 and 25 and found that Newton-Raphson iteration method rarely failed to converge rapidly.

Ferguson [8] gave closed-form solutions for the maximum likelihood estimators of β and γ when $n < 5$.

Frank [9] studied the problem of testing the normal versus Cauchy distributions and Spiegelhalter [20] used Frank's results to obtain exact Bayes estimators for β and γ using a non-informative prior. Howlader and Weiss [12] used Lindley's approximation form to obtain the Bayes estimates of β and γ. The book by Johnson, Kotz and Balakrishnan [15] covers the Cauchy distribution in many of its aspects starting from the history, properties, developments and applications up to the most recent research done in the subject matter, to the date of the book's publication.

A random variable X is said to have a generalized Cauchy distribution (GCD) according to Rider[19], if its PDF, takes the form

$$f_X(x) = \frac{\delta\,\Gamma(\omega)}{2\,\Gamma(1/\delta)\,\Gamma(\omega-1/\delta)}\left[\,1+|x-\beta|^{\delta}\,\right]^{-\omega},$$

where

$$-\infty < x < \infty, -\infty < \beta < \infty,\ \delta,\ \omega > 0 \ \text{ and } \ \delta\,\omega > 1.$$

The Cauchy distribution has received applications in many areas, including biological analysis, clinical trials, stochastic modeling of decreasing failure rate life components, queueing theory, and reliability. For data from these areas, there is no reason to believe that empirical moments of any order should be infinite. Thus, the choice of the Cauchy distribution as a model is unrealistic since its moments of all orders are not finite.

The introduced truncated generalized Cauchy distribution can be a more appropriate model for the kind of data mentioned.

We suggest a left truncated version of Rider's GCD at β when $\delta = 2,\ \omega = \alpha + 1/2$ and introduce a scale parameter γ so that the PDF takes the form

$$f_X(x) = \frac{2}{\sqrt{\pi}} \; \frac{\Gamma(\alpha + 1/2)}{\gamma\, \Gamma(\alpha)} \left[1 + \left(\frac{x-\beta}{\gamma} \right)^2 \right]^{-\alpha-1/2}, x \geq \beta, (\beta, \gamma, \alpha > 0). \tag{2}$$

We shall write $X \sim TGCD(\beta, \gamma, \alpha)$ to denote that the random variable X has $PDF(1.2)$.

Ateya and Madhagi [1] introduced a multivariate version of $TGCD$, $MVTGCD$, and derived its moment generating function, conditional density functions, mixed moments and estimate its parameters using the maximum likelihood and Bayes methods.

One reason for truncation at β is that, in industry, we sometimes require a minimum time β before which no failure occurs. This minimum time β is known as the *guarantee time.*Another use for truncation at $\beta > 0$ is in epidemiological or biomedical applications where β may represent the latent period of some disease. For example, in cancer research problems, β is regarded as the time elapsed between first exposure to carcinogen and the appearance of tumer.

A special case of (2) may be obtained when $\alpha = k - 1/2,\ k = 1, 2, ...$ in this case, the PDF becomes

$$f_X(x) = \frac{2\,\Gamma(k)}{\sqrt{\pi}\,\gamma\,\Gamma(k-1/2)} \left[1 + \left(\frac{x-\beta}{\gamma} \right)^2 \right]^{-k}, x \geq \beta. \tag{3}$$

If $k = 1,\ f_X(x)$ is then the left truncated version of the Cauchy $PDF(1)$ that takes the form

$$f_X(x) = \frac{2}{\pi\gamma} \left[1 + \left(\frac{x-\beta}{\gamma} \right)^2 \right]^{-1}, x \geq \beta. \tag{4}$$

Dahiya et al [7] studied the maximum likelihood estimates ($MLE's$) of the parameters of a doubly truncated Cauchy distribution.

If $k \geq 2$ we then have

$$f_X(x) = \frac{2^k\,(k-1)!}{\gamma\,\pi[1.3.5...(2k-3)]}\left[1+\left(\frac{x-\beta}{\gamma}\right)^2\right]^{-k}, x \geq \beta. \tag{5}$$

Another special case of (2) is when $\gamma^2 = 2\alpha = k$ so that

$$f_X(x) = \frac{2\,\Gamma(\frac{k+1}{2})}{\Gamma(k/2)\,\sqrt{k\pi}}\left[1+\frac{(x-\beta)^2}{k}\right]^{-\frac{k+1}{2}}, x \geq \beta. \tag{6}$$

This is the PDF of a left truncated t-distribution with k degrees of freedom.

1.1. Properties of the $TGCD$

The PDF (2) of the $TGCD$ is monotone decreasing on the interval $[\beta, \infty)$. The maximum value of f is attained at $x = \beta$ and $f(\beta) = 2\,\Gamma(\alpha + 1/2)/[\sqrt{\pi}\,\gamma\,\Gamma(\alpha)]$.
Wile the moment generating function (MGF) of the Cauchy PDF (1)(and the moments of any order) do not exist, the MGF of the $TGCD$ and moments of all orders do exist. In fact, it can be shown that if $X \sim TGCD(\beta,\,\gamma,\,\alpha)$, then

$$M_X(t) = \frac{2}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha)}\int_0^{\pi/2}\exp[(\beta+\gamma\tan\phi)t]\,(\cos\phi)^{2\alpha-1}\,d\phi.$$

For $r = 1,\,2, ...$ such that $r < 2\alpha$,

$$E(X^r) = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha)}\sum_{i=0}^{r}\binom{r}{i}\,\gamma^{\,i}\,\beta^{\,r-i}\,\mathrm{Beta}(\alpha-\frac{i}{2},\frac{i+1}{2}),$$

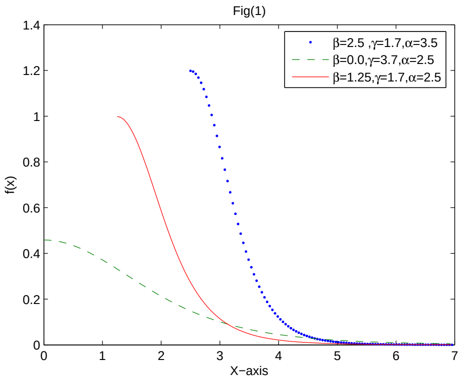
where $B(a,b) = \int_0^1 z^{a-1}\,(1-z)^{b-1}\,dz$, is the standard beta integral.
Furthermore, the cumulative distribution function (CDF) takes the form

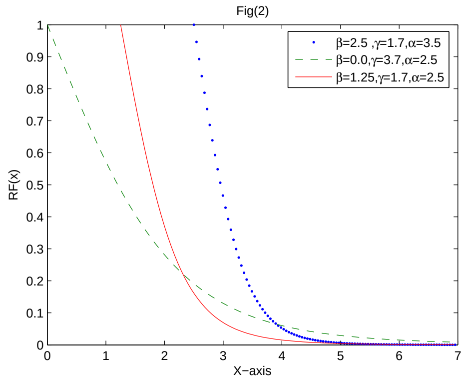
$$F_X(x) = \frac{2}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha)}\int_0^{\tan^{-1}(\frac{x-\beta}{\gamma})}(\cos\phi)^{2\alpha-1}\,d\phi.$$

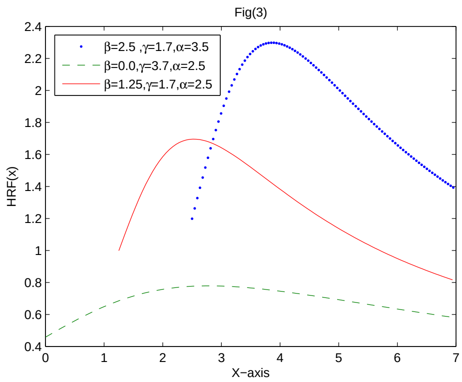
The reliability function (RF) and hazard rate function (HRF), are defined, respectively, at time x, by

$$R_X(x) = 1 - F_X(x) \quad \text{and} \quad h_X(x) = f_X(x)/R_X(x). \tag{7}$$

Graphs of $f_X(x)$, $R_X(x)$ and $h_X(x)$ are shown in Figures 1, 2 and 3 for different choices of $(\beta,\, \gamma,\, \alpha)$







2. Maximum likelihood estimation

Let $X_1, ..., X_n$ be a random sample drawn from a population having a PDF given by (2). The likelihood function (LF) is then given, for $x_i \geq \beta$, $i = 1, 2, ..., n$, by

$$L(\beta, \gamma, \alpha|\boldsymbol{x}) = \prod_{i=1}^{n} \left[\frac{2}{\sqrt{\pi}} \; \frac{\Gamma(\alpha + 1/2)}{\gamma\, \Gamma(\alpha)} \left[1 + \left(\frac{x_i - \beta}{\gamma} \right)^2 \right]^{-\alpha - 1/2} \right]. \tag{8}$$

where $\boldsymbol{x} = (x_1, ..., x_n)$ is the vector of observations (realization of $X_1, ..., X_n$). The LF (8) is a monotone increasing function of the parameter β on the interval $(0, \min\{x_i\})$, so that, the maximum likelihood estimator of the parameter β, denoted by $\hat{\beta}$, is given by

$$\hat{\beta} = \min\{X_i\}. \tag{9}$$

The logarithm of (8) is given by

$$\ell(\beta,\gamma,\alpha|\,\boldsymbol{x}\,) = n\ \ln(2) - \frac{n}{2}\ \ln(\pi) - n\ \ln(\gamma) - n\ \ln(\Gamma(\alpha)) + n\ \ln(\Gamma(\alpha+1/2)) - (\alpha+1/2)\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i-\beta}{\gamma}\right)^2\right]. \tag{10}$$

Replacing the parameter β by $\hat{\beta}$ in (10), differentiating with respect to γ and α and then setting to zero, we obtain the two likelihood equations ($LE's$)

$$\begin{cases} \frac{\partial\ell}{\partial\gamma} = 0 = -n/\hat{\gamma} + (2\hat{\alpha}+1)\sum_{i=1}^{n}(x_i-\hat{\beta})^2/\{\hat{\gamma}[\hat{\gamma}^2+(x_i-\hat{\beta})^2]\}, \\ \frac{\partial\ell}{\partial\alpha} = 0 = -n\ \psi(\hat{\alpha}) + n\ \psi(\hat{\alpha}+1/2) - \sum_{i=1}^{n}\ln\left[1+\left([x_i-\hat{\beta}]/\hat{\gamma}\right)^2\right]. \end{cases} \tag{11}$$

where

$\psi(z) = \frac{d}{dz}\ln\ \Gamma(z)$, is the digamma function.

Equations (11) represent two nonlinear equations which can be solved using some iteration scheme, such as Newton-Raphson, to obtain the $MLE's$ of γ and α, denoted by $\hat{\gamma}$ and $\hat{\alpha}$. The invariance property of $MLE's$ can be applied to obtain $MLE's$ for the RF and HRF, $R_X(x^*)$ and $h_X(x^*)$, at some x^*.

3. Bayes estimation

Let $u(\boldsymbol{\theta})$ be a general function of the vector of parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_m)$. Under the **squared error loss** function (SEL), $L^* = [\hat{u}(\theta) - u(\theta)]^2$, the Bayes estimate of $u(\boldsymbol{\theta})$ is given by

$$\hat{u}_S(\boldsymbol{\theta}) = E(u(\boldsymbol{\theta})|\,\boldsymbol{x}) = \int ... \int u(\boldsymbol{\theta})\,\pi^*(\boldsymbol{\theta}|\boldsymbol{x})\,d\theta_1...d\theta_m. \tag{12}$$

The integrals are taken over the m-dimensional space.

The SEL function has probably been the most popular loss function used in literature. The symmetric nature of SEL function gives equal weight to over- and underestimation of the parameter(s) under consideration. However, in life testing, overestimation may be more serious than underestimation or vice-versa. Research has then been directed towards asymmetric loss functions and Varian [22] suggested the use of the linear-exponential($LINEX$) loss function to be of the form

$$L^*(\Delta) = b\Big[e^{a\,\Delta} - a\,\Delta - 1\Big],$$

where $|a| \neq 0, b \geq 0, \Delta = \hat{u}(\theta) - u(\theta)$.

Thompson and Basu [21] generalized the $LINEX$ loss function to the squared-exponential ($SQUAREX$) loss function to be in the form

$$L^*(\Delta) = b\Big[e^{a\,\Delta} - d\,\Delta^2 - a\,\Delta - 1\Big],$$

where $d \neq 0$, a, b and Δ are as before.

Indeed, the $SQUAREX$ loss function reduces to the $LINEX$ loss function if $d = 0$.

If $a = 0$, the $SQUAREX$ loss function reduces to SEL function.

We shall use the $LINEX$ loss function since it is simpler to use than the $SQUAREX$ loss function. Notice that in $LINEX$ loss function, for $\hat{u}(\theta) - u(\theta) = 0, L^*(\Delta) = 0$. For $a > 0$, the loss declines almost exponentially for $\hat{u}(\theta) - u(\theta) > 0$ and rises approximately linearly when $\hat{u}(\theta) - u(\theta) < 0$. For $a < 0$, the reverse is true. By expanding $e^{a\Delta}$, $L^*(\Delta)$ can be approximated to the SEL function when $\hat{u}(\theta) - u(\theta)$ is small. Without loss of generality we shall take $b = 1$.

Using the $LINEX$ loss function, the Bayes estimate of $u(\boldsymbol{\theta})$ is given by

$$\hat{u}_L(\boldsymbol{\theta}) = \frac{-1}{a}\,\ln[E(e^{-a\,u(\boldsymbol{\theta})}|\boldsymbol{x})] = \frac{-1}{a}\ln\,\Big[\int ... \int e^{-a\,u(\boldsymbol{\theta})}\,\pi^*(\boldsymbol{\theta}|\boldsymbol{x})\,d\theta_1...d\theta_m\Big], \tag{13}$$

where $\pi^*(\boldsymbol{\theta}|\,\boldsymbol{x}) \propto \pi(\boldsymbol{\theta})L(\boldsymbol{\theta}|\,\boldsymbol{x})$ is the posterior PDF of the vector of parameters $\boldsymbol{\theta}$ given the vector of observations $\boldsymbol{x}$, $\pi(\boldsymbol{\theta})$ is a prior density function of $\boldsymbol{\theta}$ and $L(\boldsymbol{\theta}|\,\boldsymbol{x})$ is the LF of $\boldsymbol{\theta}$ given $\boldsymbol{x}$. The integrals are taken over the m-dimensional space R^m.

To compute the integrals, we use $Markov\ Chain\ Monte\ Carlo$ ($MCMC$), method to generate a random sample $[\boldsymbol{\theta}^i = (\theta_1^i, ..., \theta_m^i),\ i = 1, 2, ..., k]$ from the posterior density function $\pi^*(\boldsymbol{\theta}|\,\boldsymbol{x})$ and then write (3.1) and (3.2), respectively in the forms,

$$\hat{u}_S(\boldsymbol{\theta}) = \frac{\sum_{i=1}^{k} u(\boldsymbol{\theta}^i)}{k} \tag{14}$$

and

$$\hat{u}_L(\boldsymbol{\theta}) = (-1/a)\,\ln\left[\frac{1}{k}\sum_{i=1}^{k} e^{-a\,u(\boldsymbol{\theta}^i)}\right]. \tag{15}$$

The $MCMC$ method is described in Kandu and Howlader [16] and Press [18].

3.1. Bayes estimation of $\beta, \gamma, \alpha, R_X(x^*)$ and $h_X(x^*)$ under squared error loss function

In this subsection the $BE's$ of $\beta, \gamma, \alpha, R_X(x^*)$ and $h_X(x^*)$ are obtained under squared error loss function in case of informative and non-informative priors. To estimate these parameters and functions we define a function $u(\beta, \gamma, \alpha)$ as

$$u(\beta, \gamma, \alpha) = \beta^{\delta_1}\,\gamma^{\delta_2}\,\alpha^{\delta_3}\,(f(x^*))^{\delta_4}\,(R_X(x^*))^{\delta_5}. \tag{16}$$

The Bayes estimate of $u(\beta, \gamma, \alpha)$ is obtained in five cases:

(1) when $\delta_1 = 1, \delta_2 = \delta_3 = \delta_4 = \delta_5 = 0$, which is equivalent to estimating β,

(2) when $\delta_2 = 1, \delta_1 = \delta_3 = \delta_4 = \delta_5 = 0$, which is equivalent to estimating γ,

(3) when $\delta_3 = 1, \delta_1 = \delta_2 = \delta_4 = \delta_5 = 0$, which is equivalent to estimating α.

(4) when $\delta_5 = 1, \delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, which is equivalent to estimating $R_X(x^*)$.

(5) when $\delta_4 = 1, \delta_5 = -1, \delta_1 = \delta_2 = \delta_3 = 0$, which is equivalent to estimating $h_X(x^*)$.

3.1.1. Bayes estimation in case of informative prior

Suppose that the prior belief of the experimenter is measured by a function $\pi(\beta, \gamma, \alpha)$, where α is assumed to be independent of β and γ, so that the prior density function is given by

$$\begin{aligned} \pi(\beta, \gamma, \alpha) &= \pi_1(\beta, \gamma)\,\pi_2(\alpha) \\ &= \pi_{11}(\beta \mid \gamma)\,\pi_{12}(\gamma)\,\pi_2(\alpha). \end{aligned} \tag{17}$$

Suppose that $\pi_{11}(\beta \mid \gamma)$ is Gamma (c_1, γ), $\pi_{12}(\gamma)$ is Gamma (c_2, c_3) and $\pi_2(\alpha)$ is Gamma(c_4, c_5), with respective densities

$$\pi_{11}(\beta|\gamma) \propto \gamma^{\,c_1}\, \beta^{c_1-1} \exp(-\gamma\, \beta),\ \beta,\ \gamma\ > 0,\ (c_1 > 0),$$

$$\pi_{12}(\gamma) \propto \gamma^{\,c_2-1}\ \exp(-c_3\, \gamma),\ \gamma\ > 0,\ (c_2,\ c_3 > 0),$$

$$\pi_2(\alpha) \propto \alpha^{\,c_4-1}\ \exp(-c_5\, \alpha),\ \alpha\ > 0,\ (c_4,\ c_5 > 0),$$

It then follows that the prior density of β, γ and α is given by

$$\begin{aligned} &\pi(\beta,\gamma,\alpha) \propto \alpha^{\,c_4-1}\ \beta^{c_1-1}\ \gamma^{c_1+c_2-1}\ \exp(-\gamma\,\beta - c_3\,\gamma - c_5\,\alpha), \\ &\alpha,\ \beta,\ \gamma > 0,\ (c_1, c_2, c_3, c_4, c_5 > 0), \end{aligned} \tag{18}$$

where, c_1, c_2, c_3, c_4 and c_5 are the prior parameters (also known as hyperparameters). From (8) and (18), the posterior density function can be written in the form

$$\begin{aligned} &\pi^*(\beta,\gamma,\alpha|\,\boldsymbol{x}) = A\,\alpha^{\,c_4-1}\ \beta^{c_1-1}\ \gamma^{c_1+c_2-1}\ \exp(-\gamma\,\beta - c_3\,\gamma - c_5\,\alpha). \\ &\prod_{i=1}^{n}\left[\frac{2}{\sqrt{\pi}}\ \frac{\Gamma(\alpha+1/2)}{\gamma\,\Gamma(\alpha)}\left[1+\left(\frac{x_i-\beta}{\gamma}\right)^2\right]^{-\alpha-1/2}\right], \\ &\alpha,\ \beta,\ \gamma > 0,\ (c_1, c_2, c_3, c_4, c_5 > 0), \end{aligned} \tag{19}$$

where A is a normalizing constant.
Using $MCMC$ method, we get the $BE's$ of the considered parameters and functions.

3.1.2. Bayes estimation in case of non-informative prior

In this case, we consider independent non-informative priors of the parameters β, γ and α in the forms

$$\pi_1(\beta) \propto 1/\beta,\ \beta > 0,$$

$$\pi_2(\gamma) \propto 1/\gamma,\ \gamma > 0,$$

$$\pi_3(\alpha) \propto 1/\alpha,\ \alpha > 0.$$

so that

$$\pi(\beta,\ \gamma,\ \alpha) \propto\ (\alpha\,\beta\,\gamma)^{-1}. \tag{20}$$

Using this prior and the LF (8), the posterior density function of β, γ and α can be written in the form

$$\pi^*(\beta,\gamma,\alpha|\,\boldsymbol{x}) = A_1\,\alpha^{-1}\,\beta^{-1}\,\gamma^{-1}\prod_{i=1}^{n}\left[\frac{2}{\sqrt{\pi}}\;\frac{\Gamma(\alpha+1/2)}{\gamma\,\Gamma(\alpha)}\left[1+\left(\frac{x_i-\beta}{\gamma}\right)^2\right]^{-\alpha-1/2}\right], \tag{21}$$

$$\alpha,\ \beta,\ \gamma > 0,$$

where A_1 is a normalizing constant.

Using $MCMC$ method, we can obtain the $BE's$ of β, $\gamma, \alpha, R_X(x^*)$ and $h_X(x^*)$.

3.2. Bayes estimation of $\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{R_X}(\boldsymbol{x}^*)$ and $\boldsymbol{h_X}(\boldsymbol{x}^*)$ under $LINEX$ loss function

The $MCMC$ method is used to generate a random sample of size k, $[\boldsymbol{\theta}^i = (\theta_1^i, ..., \theta_m^i),\ i = 1, 2, ..., k]$ by using the posteriors (19) and (21). Equation (15),(a=1, 7), is then used to compute the $BE's$ of the parameters and functions of such parameters under $LINEX$ loss function.

4. Simulation study

In this section the maximum likelihood and Bayes estimates of β, $\gamma, \alpha, R_X(x^*)$ and $h_X(x^*)$ are obtained as follows :

(1) For a given set of prior parameters, generate the population parameters β, γ, α.

(2) Making use of the generated population parameters, generate random samples of different sizes $(15, 25, 40)$ from the population distribution under study.

(3) The maximum likelihood estimate (MLE) of the parameter β is the minimum value of the random sample.

(4) The MLE $\hat{\beta}$ of β, given by (9), on the basis of the samples of sizes 15, 25, 40, obtained in step 2. The estimate $\hat{\beta}$ is then substituted in the nonlinear equations (11). Solving these equations we get the maximum likelihood estimates $\hat{\gamma}$ of γ and $\hat{\alpha}$ of α.

The use of the invariance property of the $MLE's$ yields $MLE's$ of RF and HRF, given by (7).

(5) The $BE's$ of $\beta, \gamma, \alpha, R_X(x^*)$ and $h_X(x^*)$ are computed under the SEL and $LINEX$ loss functions using the function u, defined in (16), for different values of $\delta_i,\ i = 1, 2, ..., 5.$ the $MCMC$ technique is used in the computations.

(6) Steps $2 - 5$ are repeated $m = 1000$ times.

(7) If $\hat{\theta}_j$ is an estimate of θ, based on sample j, $j = 1, 2, ..., m$, then the average estimate over the m samples is given by
$\bar{\hat{\theta}} = \frac{1}{m}\sum_{j=1}^{m}\hat{\theta}_j.$

(8) The variance of $\hat{\theta}$, $V(\hat{\theta})$, over the m samples is given by
$V(\hat{\theta}) = \frac{1}{m}\sum_{j=1}^{m}(\hat{\theta}_j - \bar{\hat{\theta}})^2.$

Using steps (7) and (8), compute $\bar{\hat{\beta}}, \bar{\hat{\gamma}}, \bar{\hat{\alpha}}, \bar{\hat{R}}(x^*), \bar{\hat{h}}(x^*), V(\hat{\beta}), V(\hat{\gamma}), V(\hat{\alpha}), V(\hat{R}(x^*))$ and $V(\hat{h}(x^*))$.

In our study, Table(1) displays the average estimates and variances of the $MLE's$ and $BE's$, using informative and non-informative priors, under squared error loss function, based on samples of different sizes n and for m =1000 repetitions. Tables(2) and (3) display the same data as Table(1) under $LINEX$ loss function in case of $a = 1$ and $a = 7$. The given vector of haperparamters is $(c_1 = 1.7, c_2 = 1.0, c_3 = 1.8, c_4 = 2.0, c_5 = 2.7)$ and the generated population parameters are $(\beta = 2.5, \gamma = 1.7, \alpha = 3.5)$. The population reliability and hazard rate functions are computed at $x^* = 3.5$, using (7) and the population parameters. Their values are $R_X(x^*) = 0.1636$ and $h_X(x^*) = 2.2317$.

Table(1): Maximum likelihood and Bayes estimation under **SEL** function.

n	Method		$\bar{\hat{\beta}}$ $V(\hat{\beta})$	$\bar{\hat{\gamma}}$ $V(\hat{\gamma})$	$\bar{\hat{\alpha}}$ $V(\hat{\alpha})$	$\bar{\hat{R}}(x^*)$ $V(\hat{R}(x^*))$	$\bar{\hat{h}}(x^*)$ $V(\hat{h}(x^*))$
15	B	Informative Prior	2.6805	1.7941	3.4027	0.2351	1.9703
			0.1998	0.0917	0.2186	0.0516	0.8215
		Non-Informative Prior	2.6947	1.8013	3.3901	0.2607	1.9133
			0.2018	0.0994	0.2397	0.0651	0.8913
	ML		2.7132	1.8215	3.3156	0.2939	1.8925
			0.2215	0.1084	0.2447	0.0733	0.9761
25	B	Informative Prior	2.5311	1.7508	3.4713	0.2183	2.1625
			0.0878	0.0685	0.1366	0.0492	0.5917
		Non-Informative Prior	2.5303	1.7759	3.4435	0.2206	2.1316
			0.1251	0.0692	0.1514	0.0588	0.6351
	ML		2.6105	1.7936	3.4236	0.2364	2.005
			0.1423	0.0790	0.1678	0.0633	0.7009
40	B	Informative Prior	2.5093	1.72110	3.5116	0.1701	2.2013
			0.0735	0.04719	0.0925	0.0297	0.4902
		Non-Informative Prior	2.5108	1.7194	3.4911	0.1739	2.1933
			0.0882	0.05032	0.1051	0.0381	0.5922
	ML		2.5210	1.7131	3.4871	0.1751	2.1655
			0.0990	0.0534	0.1303	0.0433	0.6313

Table(2): Maximum likelihood and Bayes estimation under $LINEX$ loss function ($a = 1$).

n	Method		$\bar{\hat{\beta}}$ $V(\hat{\beta})$	$\bar{\hat{\gamma}}$ $V(\hat{\gamma})$	$\bar{\hat{\alpha}}$ $V(\hat{\alpha})$	$\bar{\hat{R}}(x^*)$ $V(\hat{R}(x^*))$	$\bar{\hat{h}}(x^*)$ $V(\hat{h}(x^*))$
15	B	Informative Prior	2.6908 0.2001	1.8061 0.0977	3.4011 0.2285	0.1905 0.0531	2.0132 0.8735
		Non-Informative Prior	2.7023 0.2199	1.8090 0.1015	3.3170 0.2502	0.2103 0.0691	1.9074 0.9105
	ML		2.7132 0.2230	1.8215 0.1084	3.3156 0.2447	0.2939 0.0733	1.8925 0.9761
25	B	Informative Prior	2.5516 0.0920	1.7861 0.0714	3.4605 0.1509	0.1822 0.0516	2.1930 0.6220
		Non-Informative Prior	2.5702 0.1413	1.7905 0.0721	3.4310 0.1622	0.1952 0.0611	2.1441 0.6616
	ML		2.6105 0.1423	1.7936 0.0790	3.4236 0.1678	0.2364 0.0633	2.0056 0.7009
40	B	Informative Prior	2.5116 0.0792	1.7583 0.0504	3.5311 0.1134	0.1628 0.0325	2.2218 0.5133
		Non-Informative Prior	2.5174 0.0891	1.7201 0.0511	3.4893 0.1262	0.1601 0.0404	2.1915 0.6227
	ML		2.5210 0.0990	1.7131 0.0534	3.4871 0.1303	0.1751 0.0433	2.1655 0.6313

Table(3): Maximum likelihood and Bayes estimation under $LINEX$ loss function ($a = 7$).

n	Method		$\bar{\hat{\beta}}$ $V(\hat{\beta})$	$\bar{\hat{\gamma}}$ $V(\hat{\gamma})$	$\bar{\hat{\alpha}}$ $V(\hat{\alpha})$	$\bar{\hat{R}}(x^*)$ $V(\hat{R}(x^*))$	$\bar{\hat{h}}(x^*)$ $V(\hat{h}(x^*))$
15	B	Informative Prior	2.6295	1.7714	3.4420	0.1873	2.0513
			0.1806	0.0804	0.2060	0.0451	0.5911
		Non-Informative Prior	2.6701	1.7910	3.4421	0.1903	1.9833
			0.1942	0.0815	0.2331	0.0603	0.7005
	ML		2.7132	1.8215	3.3156	0.2939	1.8925
			0.2230	0.1084	0.2447	0.0733	0.9761
25	B	Informative Prior	2.5234	1.7340	3.4726	0.1791	2.1766
			0.0815	0.0407	0.1035	0.0382	0.4613
		Non-Informative Prior	2.5291	1.7415	3.4603	0.1811	2.1012
			0.1153	0.0570	0.1203	0.0476	0.5312
	ML		2.6105	1.7936	3.4236	0.2364	2.0056
			0.1423	0.0790	0.1678	0.0633	0.7009
40	B	Informative Prior	2.5043	1.7021	3.5016	0.1651	2.2165
			0.0692	0.0381	0.0813	0.0170	0.3855
		Non-Informative Prior	2.5094	1.7092	3.5063	0.1706	2.1702
			0.0735	0.0470	0.0948	0.0232	0.4622
	ML		2.5210	1.7131	3.4871	0.1751	2.1655
			0.0990	0.0534	0.1303	0.0433	0.6313

5. Concluding remarks

In our study, observe the following:

(1) the variances of the $BE's$ (against the proposed subjective (informative) or objective (non-informative) prior) are smaller than the corresponding variances of the

$MLE's$. This means that the $BE's$ (against the proposed priors) are better than the $MLE's$,

(2) the variances of the $BE's$ in case of informative prior are smaller than the corresponding variances in case of non-informative prior,

(3) under $LINEX$ loss function, $a = 1$, the variances of the $BE's$ are greater than the variances under SEL function. That is when $a = 1$, the use of SEL leads to better estimates than the $LINEX$ loss function.

(4) under $LINEX$ loss function, $a = 7$, the variances of the $BE's$ are smaller than the variances under SEL function. That is when $a = 7$, the use of $LINEX$ loss function leads to better estimates than the SEL function.

(5) in Bayesian estimation, if the hyperparameters are unknown, they can be estimated by using the empirical Bayes method [see Maritz and Lwin [17]] or the hierarchical method [see Bernardo and Smith[3]].

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