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OPTIMAL ORDERING POLICY FOR SUBSTITUTABLE AND COMPLEMENTARY PRODUCTS UNDER QUADRATIC DEMAND

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Abstract: In this article, two mutually substitutable products are considered in the inventory. Out of which one of the product is assumed to be a bundle of two complementary components. The inventory of both products are depleted by its demand, which is assumed to be quadratic. In case of stock out, partial substitution is carried out by the other product to fulfil the demand of the customers with a rate of substitution. The unmet demand is assumed to be lost. The proposed model is formulated and a solution procedure is suggested to obtain the optimal ordering quantity. Sensitivity analysis has been carried out extensively for all the parameters in the model in order to analyse the behaviour of the model. We observe that, there is a substantial improvement in the total cost with substitution compared to that of total cost without substitution.

Keywords: inventory model; substitutable product; complementary product; quadratic demand.

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1. INTRODUCTION

Inventory management decides the performance of the company or firm. It involves majorly in planning and controlling the level of the stocks in the warehouse by determining two important aspects in the management, how much to order and when to order. The improper planning of the inventory affects the profit of the company. In addition, the customer's satisfaction and maintenance of the goodwill of the company rely on the management of the inventory. We often notice in super markets where two complementary products are bundle together as single product for utilization. For example, tooth paste and toothbrush, hardware and software, fountain pen and inkbottle, pencil and eraser. Usually a single component usage will not be sufficient to get the complete utility of the whole product, such products are called complementary products.

Often we come across in our daily life, where a consumer choose between two different products like different brands of hand sanitizer, different brands of face mask, different brand of mobile phone, different brand of AC, different brand of laptops, different brand of coffee , shampoo, bath soap etc. These products are called substitutable products, as one product is an alternate to another product. In general, when two products serves the need in all aspect like quality, price etc, even after interchanging are said to be substitutes. The advantage of product substitution is that it saves the purchasing time of the customers, when their preferred product is out of stock, customers are willing to accept the substitute as they does not wish to visit another store. Substitutable products benefits the company by minimizing its total cost. In addition, it further improves the good will of the company. The effectiveness of the system is established by the product substitution. However, in real life not all customers willing to accept the substitute product, such demands are assumed lost.

Many articles describes the substitutable product and complementary products which are as follows, Benkherouf et al [1] developed an inventory model for substitutable products with time varying demand and the planning horizon is assumed finite. Drenzner et al [2] proposed an inventory model with two substitutable products under EOQ model with joint replenishment policy. Durga, B. K., & Chandrasekaran, E [3] considered an inventory model consisting of two substitutable products having quadratic demand. This model also considers the cost of substitution of the item. Goyal [4] developed an inventory model with the option of substitution. Gurnani and Drezner [5] extended the work of Drenzner et al [2] for multiple products. Krommyda et al [6] considered stock dependant demand and partial substitution. Liu et al [7] considered one-way substitution with or without conversion cost for perishable products. Maddah et al [8] considered an EOQ model with multiple products with partial substitution. Mishra [9] also developed an inventory model for substitutable and deteriorating products with cost of substitution. Mishra and Shanker [10] proposed a model for substitutable products with cost of substitution. Mukhopadhyay, A., & Goswami [12] developed an EOQ inventory model with one-way substitution for imperfect quality of products. Salameh et al [13] extended the work of Drenzer et al [2] with partial substitution where only a fraction of customers is interested to buy the substitute item. Stavrulaki [14] developed a study of products, which are substitutable based on the level of inventory.. Zhang et al [17] considered correlated demand between products and partial backordering under EOQ framework. Zhang et al [18] developed EOQ model with substitution and partial backordering.

Yue et al [16] developed a Bertrand model of pricing of complementary goods under information asymmetry in which he considered two complementary goods as mixed bundle given by two different company. Mukhopadhyay et al [11] developed an optimum pricing strategy for complementary products by applying a game theoretical approach. Yan and Bandyopadhyay [15] developed a model on complementary products with bundle pricing in which he combine highly complementary products and change a relatively lower price.

It can be seen that not many structured work on substitutable and complementary products considered together in the literature. In this article, two mutually substitutable product is considered, where one of the product consists of two complementary components. The inventory depletes by its demand, which is assumed to be quadratic. The objective of this model is to find the optimal ordering policy. The products are assumed to be similar in quality and price. When one of the products becomes out of stock, the demand of it is partially met from the inventory of the other product.

The proposed model is described in detail in the subsequent sections. In Section 2 the assumptions and notations used in the model is presented. In section 3 the proposed model is mathematically formulated. In section 4 a solution procedure is suggested to obtain the optimal ordering quantity by showing that the total cost function is pseudo convex. Extensive sensitivity analysis has been done for all the parameter in the model to test the performance of the inventory system and presented in section 5. Finally in section 6 conclusion and future work is stated.

2. ASSUMPTIONS AND NOTATIONS

The assumptions made in the proposed model are listed below.

- The proposed inventory model consists of two products, which are substitutable.
- Product 2 is a bundle of two complementary products α and β.
- The demand of the product is quadratic.
- The demand of the product is partially substituted by the other product when stocks out with a rate of substitution. The unmet demand is assumed to be lost incurring a shortage cost.
- Both products are ordered jointly in each ordering cycle.
- The ordering cost and holding cost is constant.
- Lead-time is zero and instant replenishment.

The following notations are used throughout the article.

k_1, k_2	Ordering cost of Product 1 and 2
h_1, h_2	Holding cost of Product 1 and 2
D_1, D_2	Demand rate of Product 1 and 2
a_1, a_2	Usage rates of complementary components α and β
c_{s_1}, c_{s_2}	Shortage cost per unit of Product 1 and 2

T_1	Inventory cycle time of Product 1
T_2	Inventory cycle time of Product 2
t	Substitution time interval
Q_1	Order quantity of Product 1
q_1, q_2	Order quantity of two complementary component α and β
v_1	Rate of substitution of product 1 by product 2
v_2	Rate of substitution of product 2 by product 1
TCU_1	Total cost per unit time with substitution in case(i)
TCU_2	Total cost per unit time with substitution in case(ii)
TCU_{WOS}	Total cost per unit time without substitution in case (iii)

3. PROPOSED MODEL

Consider an inventory system with two products. Product 1 is composed of two complementary components α and β. These two components are manufactured separately and packed together and to be consumed together as product 1 with usage rates a_1 and a_2. Whereas product 2 is a single component. Initially q_1 units of component α and q_2 units of component β is ordered. The cycle time of product 1 is T_1. Whereas Q_2 units of product 2 is ordered initially and its cycle time is T_2. The demand of product 1(both component α and β) and product 2 are denoted by D_1 and D_2 respectively which is assumed to be quadratic. Two products are assumed to be substitutable products that is, they are similar in quality and price. Therefore, when one product becomes out of stock, the demand of the product is partially fulfilled from the inventory of the other product. Let v_1 and v_2 be the rate of substitution of product 1 and product 2 respectively.

That is, when product 1 runs out of stock, customer's demand is satisfied by product 1. Whereas, when product 2 is unavailable, the customer will wait for the packaging of complementary components α and β, together to be consumed as product and get satisfied with the substitution. Since the substitution is partially done, the unmet demand of the products is assumed to be lost. Two cases is considered for this model based on the inventory cycle of the products, ie. When $T_1 \le T_2$ and $T_1 \ge T_2$.

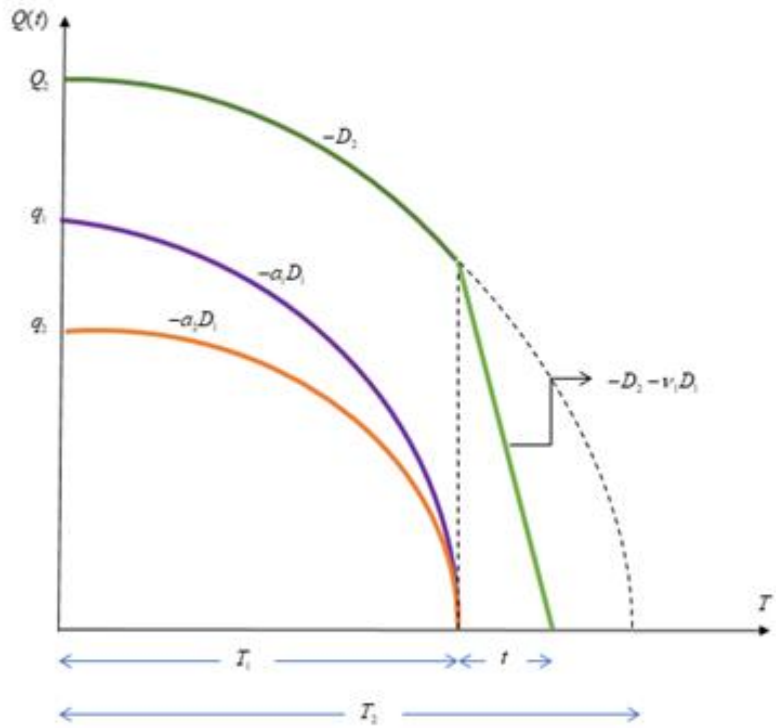


Fig 1 The inventory level for case (i) when $T_1 \le T_2$

Case (i): $T_1 \le T_2$

In this case, product 1 stocks out first. The future demand of this product is met from the leftover inventory of the product 2. The total cost in this case consists of the ordering cost of product 1 which is composed of two complementary components α and β, the ordering cost of product 2 , holding cost of product 1 and 2. Also, the unmet demand of product 1 which is assumed to be lost is included in the shortage cost.

1. Ordering cost:
    a) Product 1
        i. Component $\alpha = k_1$
        ii. Component $\beta = k_1$
    b) Product 2= k_2

2. Holding cost:
    a) Product 1
        i. Component $\alpha = h_1 \int_0^{T_1} \int_0^{q_1 - \frac{a_1^2 D_1^2 T^2}{q_1}} dQ\, dT = \frac{2}{3} \frac{h_1 q_1^2}{a_1 D_1}$
        ii. Component $\beta = h_1 \int_0^{T_1} \int_0^{q_2 - \frac{a_2^2 D_1^2 T^2}{q_2}} dQ\, dT = \frac{2}{3} \frac{h_1 q_2^2}{a_2 D_1}$
    b) Product 2= $h_2 \int_0^{T_1} \int_0^{q_2 - \frac{a_2^2 D_1^2 T^2}{q_2}} dQ\, dT = \frac{2}{3} \frac{h_2 q_2^2}{a_2 D_1}$
    c) Product 2 during $[T_1, t] = h_2 \int_{T_1}^{t} \int_0^{(q_1+q_2)+Q_2-[D_2+\nu_1(a_1+a_2)D_1]T} dQ\, dT$

$$= \frac{h_2 D_1^2 (a_1+a_2)^2 (\nu_1 - 1)^2 (Q_2 + q_1 + q_2)^2}{2\left(D_2 + D_1(a_1+a_2)\right)^2 \left(D_2 + \nu_1 D_1 (a_1+a_2)\right)}$$

3. Shortage cost:

SC= $\dfrac{C_s D_1^2 (a_1+a_2)^2 (q_1+q_2+Q_2)(\nu_1 - 1)^2}{\left(D_2 + \nu_1 D_1 (a_1+a_2)\right)\left(D_1(a_1+a_2)+D_2\right)}$

Therefore, sum of all the above components comprises of the total cost involved in case (i), which is given by

$$TC_1 = 2k_1 + k_2 + h_1\left(\frac{2}{3}\frac{h_1 q_1^2}{a_1 D_1} + \frac{2}{3}\frac{h_1 q_2^2}{a_2 D_1}\right) + \frac{2}{3}\frac{h_2 q_2^2}{a_2 D_1} + \frac{h_2 D_1^2 (a_1+a_2)^2 (\nu_1 - 1)^2 (Q_2 + q_1 + q_2)^2}{2\left(D_2 + D_1(a_1+a_2)\right)^2 \left(D_2 + \nu_1 D_1 (a_1+a_2)\right)} + \frac{C_s D_1^2 (a_1+a_2)^2 (q_1 + q_2 + Q_2)(\nu_1 - 1)^2}{\left(D_2 + \nu_1 D_1 (a_1+a_2)\right)\left(D_1(a_1+a_2) + D_2\right)}$$

Since the inventory cycles of the two complementary components α and β are equal, we have the relation $q_1 = \left(\frac{a_1}{a_2}\right) q_2$. Therefore, the total cost equation is rewritten as

$$TC_1 = 2k_1 + k_2 + \frac{2}{3}\frac{h_1 q_2^2}{a_2 D_1}\left(1 + \frac{a_1}{a_2}\right) + \frac{2}{3}\frac{h_2 Q_2^2}{D_2} + \frac{h_2 D_1^2 (a_1+a_2)^2 (\nu_1 - 1)^2 \left(Q_2 + q_2\left(1 + \frac{a_1}{a_2}\right)\right)^2}{2\left(D_2 + D_1(a_1+a_2)\right)^2 \left(D_2 + \nu_1 D_1 (a_1+a_2)\right)} + \frac{C_s D_1^2 (a_1+a_2)^2 (\nu_1 - 1)^2 \left(Q_2 + q_2\left(1 + \frac{a_1}{a_2}\right)\right)}{\left(D_2 + \nu_1 D_1 (a_1+a_2)\right)\left(D_1(a_1+a_2) + D_2\right)}$$

The total cost per unit time is obtained by dividing the total cost by

$$t = \frac{Q_2 + q_2\left(1 + \frac{a_1}{a_2}\right)}{D_2 + \nu_1 D_1 (a_1+a_2)}$$

$$TCU_1 = \frac{D_2 + \nu_1 D_1 (a_1+a_2)}{Q_2 + q_2\left(1 + \frac{a_1}{a_2}\right)}\left[2k_1 + k_2 + \frac{2}{3}\frac{h_1 q_2^2}{a_2 D_1}\left(1 + \frac{a_1}{a_2}\right) + \frac{2}{3}\frac{h_2 Q_2^2}{D_2} + \frac{h_2 D_1^2 (a_1+a_2)^2 (\nu_1 - 1)^2 \left(Q_2 + q_2\left(1 + \frac{a_1}{a_2}\right)\right)^2}{2\left(D_2 + D_1(a_1+a_2)\right)^2 \left(D_2 + \nu_1 D_1 (a_1+a_2)\right)} + \frac{C_s D_1^2 (a_1+a_2)^2 (\nu_1 - 1)^2 \left(Q_2 + q_2\left(1 + \frac{a_1}{a_2}\right)\right)}{\left(D_2 + \nu_1 D_1 (a_1+a_2)\right)\left(D_1(a_1+a_2) + D_2\right)}\right]$$

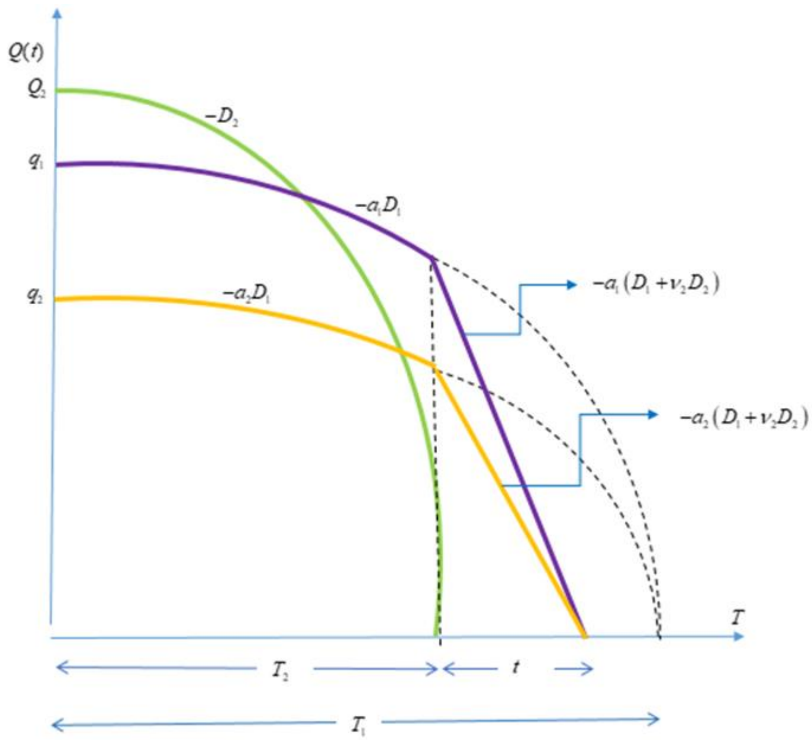


Fig 2 The inventory level for case (ii) when $T_1 \geq T_2$

Case(ii): $T_1 \geq T_2$

In this case, product 2 stocks out first, the future demand of it is met from the leftover inventory of product 1. The total cost equation is derived similar to case(i), which is given as

$$TC_2 = 2k_1 + k_2 + \frac{2}{3}\frac{h_1 q_2^2}{a_2 D_1}\left(1+\frac{a_1}{a_2}\right) + \frac{2}{3}\frac{h_2 Q_2^2}{D_2} + \frac{h_1 D_2^2 (\nu_2 - 1)^2 \left(Q_2 + q_2\left(1+\frac{a_1}{a_2}\right)\right)^2}{2\left(D_2 + D_1(a_1+a_2)\right)^2 \left(D_1(a_1+a_2)+\nu_2 D_2\right)} +$$
$$+\frac{C_{s_2} D_2^2 (\nu_2 - 1)^2 \left(Q_2 + q_2\left(1+\frac{a_1}{a_2}\right)\right)}{\left(D_1(a_1+a_2)+\nu_2 D_2\right)\left(D_1(a_1+a_2)+D_2\right)}$$

Similarly, the total cost per unit time is obtained for case(ii) by dividing the total cost equation by its cycle time $t = \dfrac{Q_2 + q_2\left(1+\frac{a_1}{a_2}\right)}{D_1(a_1+a_2)+\nu_2 D_2}$ which is given as,

$$TUC_2=\frac{D_1\left(a_1+a_2\right)+\nu_2 D_2}{Q_2+q_2\left(1+\frac{a_1}{a_2}\right)}\left[\begin{array}{l}2k_1+k_2+\frac{2}{3}\frac{h_1 q_2^2}{a_2 D_1}\left(1+\frac{a_1}{a_2}\right)+\frac{2}{3}\frac{h_2 Q_2^2}{D_2}+\frac{h_1 D_2^2\left(\nu_2-1\right)^2\left(Q_2+q_2\left(1+\frac{a_1}{a_2}\right)\right)^2}{2\left(D_2+D_1\left(a_1+a_2\right)\right)^2\left(D_1\left(a_1+a_2\right)+\nu_2 D_2\right)}+\\ \qquad +\frac{C_{s_2} D_2^2\left(\nu_2-1\right)^2\left(Q_2+q_2\left(1+\frac{a_1}{a_2}\right)\right)}{\left(D_1\left(a_1+a_2\right)+\nu_2 D_2\right)\left(D_1\left(a_1+a_2\right)+D_2\right)}\end{array}\right]$$

Case(iii) : No substitution

Consider the case, when substitution of products is not allowed. The total cost equation involves the ordering cost and holding cost. This equation can be derived very easily,

$$TC_{WOS}=2k_1+k_2+\frac{2}{3}\frac{h_1 q_2^2}{a_2 D_1}\left(1+\frac{a_1}{a_2}\right)+\frac{2}{3}\frac{h_2 Q_2^2}{D_2}$$

The total cost per unit time is given as

$$TCU_{WOS}=\frac{D_1\left(a_1+a_2\right)}{Q_2+q_2\left(1+\frac{a_1}{a_2}\right)}\left[2k_1+k_2+\frac{2}{3}\frac{h_1 q_2^2}{a_2 D_1}\left(1+\frac{a_1}{a_2}\right)+\frac{2}{3}\frac{h_2 Q_2^2}{D_2}\right]$$

4. SOLUTION PROCEDURE

In this section, we propose a solution procedure to obtain the optimal ordering quantities of the inventory model. We state the following theorem.

Theorem1 TCU_1 is pseudo convex.

Proof: Appendix A

Theorem2 TCU_2 is pseudo convex.

Proof: Appendix B

We now present an algorithm to find the optimal order quantity for the proposed model.

Algorithm:

Step1 Initialize the parametric values of the inventory model.

Step 2 Solve the following constrained nonlinear optimization problem and obtain the optimal values.

i. $\min_{q_2, Q_2} TCU_1 \quad subject\,to \quad T_1 \le T_2$

ii. $\min_{q_2, Q_2} TCU_1 \quad subject\,to \quad T_1 \ge T_2$

Step 3 Compare the optimal values obtained in step 2 and hence the required optimal ordering quantity.

5. Sensitivity Analysis

In this section, numerical illustration of the proposed model is presented. Initialize the parameter of the model with the following.

$D_1 = 100\,units, D_2 = 200\,units, h_1 = Rs.10, h_2 = Rs.10, k_1 = Rs.75, k_2 = Rs.150, a_1 = 1, a_2 = 2,$

$v_1 = 0.2, v_2 = 0.8, c_{s_1} = 1, c_{s_2} = 1$.. Apply step 2 of the algorithm, we have the optimal value of case(i) as $q_2^* = 66.26\,units, Q_2^* = 22.09\,units, TCU_1^* = Rs.1415$. and for case (ii) the optimal values is btained as $q_2^* = 60.86\,units, Q_2^* = 25.22\,units, TCU_2^* = Rs.2116$. On comparing, we observe that $TCU_2^* > TCU_1^*$ therefore the optimal solution of the proposed model is $q_2^* = 66.26\,units, Q_2^* = 22.09\,units, TCU_1^* = Rs.1415$. Now, to show that the substitution between products is reasonably effective than the basic model, optimal values are obtained for case(iii) and compared with that of the inventory model consisting of substitution and complementary products so s to understand the advantage of the proposed model. The optimal values for case (iii) is obtained as $q_2^* = 111.63\,units, Q_2^* = 37.21\,units, TCU_{WOS}^* = Rs.2052.21$ .Thus, there is 31% improvement in the total cost of the inventory model consisting of substitution and complementary products with that of the basic model with no substitution. Sensitivity analysis has been extensively carried out and presented in Table 1-6. The graphical representation of the analysis is presented in Fig 3-8.

Table 1 Sensitivity analysis of ordering cost

ordering cost k_1	With substitution			Without substitution			% improvement
	q_2	Q_2	$TC(WS)$	q_2	Q_2	$TC(WOS)$	
75	46.85	15.62	1097.48	78.94	26.31	1451.13	24.37
85	49.88	16.63	1147.10	84.03	28.01	1544.85	25.75
95	52.73	17.58	1193.87	88.84	29.61	1633.20	26.90
115	58.02	19.34	1280.55	97.74	32.58	1796.91	28.74
125	60.49	20.16	1321.06	101.90	33.97	1873.40	29.48
145	65.15	21.72	1397.47	109.76	36.59	2017.72	30.74
165	69.50	23.17	1468.76	117.08	39.03	2152.38	31.76
185	73.59	24.53	1535.86	123.97	41.32	2279.09	32.61
215	79.33	26.44	1630.02	133.65	44.55	2456.95	33.66
255	86.40	28.80	1745.88	145.55	48.52	2675.76	34.75
285	91.34	30.45	1826.90	153.87	51.29	2828.78	35.42
335	99.03	33.01	1952.97	166.83	55.61	3066.89	36.32
495	120.37	40.12	2303.02	202.79	67.60	3728.03	38.22

Table 2 Sensitivity analysis of holding cost

holding cost h_1	With substitution			Without substitution			%improvement
	q_2	Q_2	$TC(WS)$	q_2	Q_2	$TC(WOS)$	
2	148.17	49.39	815.08	249.62	83.21	917.78	11.19
3	120.98	40.33	924.29	203.81	67.94	1124.04	17.77
4	104.77	34.92	1016.36	176.50	58.83	1297.93	21.69
5	93.71	31.24	1097.48	157.87	52.62	1451.13	24.37
6	85.54	28.51	1170.81	144.12	48.04	1589.64	26.35
7	79.20	26.40	1238.25	133.42	44.48	1717.00	27.88
8	74.08	24.69	1301.02	124.81	41.60	1835.55	29.12
9	69.85	23.28	1359.97	117.67	39.22	1946.90	30.15
10	66.26	22.09	1415.73	111.63	37.21	2052.21	31.01

Table 3 Sensitivity analysis of Shortage Cost

Shortage Cost c_{s_1}	With substitution			Without substitution			%improvement
	q_2	Q_2	$TC(WS)$	q_2	Q_2	$TC(WOS)$	
0.15	66.26	22.09	1135.96	111.63	37.21	2052.21	44.65
0.25	66.26	22.09	1168.87	111.63	37.21	2052.21	43.04
0.35	66.26	22.09	1201.79	111.63	37.21	2052.21	41.44
0.45	66.26	22.09	1234.70	111.63	37.21	2052.21	39.84
0.55	66.26	22.09	1267.61	111.63	37.21	2052.21	38.23
0.65	66.26	22.09	1300.53	111.63	37.21	2052.21	36.63
0.75	66.26	22.09	1333.44	111.63	37.21	2052.21	35.02
0.85	66.26	22.09	1366.36	111.63	37.21	2052.21	33.42
0.95	66.26	22.09	1399.27	111.63	37.21	2052.21	31.82
1.05	66.26	22.09	1432.19	111.63	37.21	2052.21	30.21
1.15	66.26	22.09	1465.10	111.63	37.21	2052.21	28.61
1.25	66.26	22.09	1498.01	111.63	37.21	2052.21	27.00
1.35	66.26	22.09	1530.93	111.63	37.21	2052.21	25.40
1.45	66.26	22.09	1563.84	111.63	37.21	2052.21	23.80
1.55	66.26	22.09	1596.76	111.63	37.21	2052.21	22.19
1.65	66.26	22.09	1629.67	111.63	37.21	2052.21	20.59
1.75	66.26	22.09	1662.59	111.63	37.21	2052.21	18.99
1.85	66.26	22.09	1695.50	111.63	37.21	2052.21	17.38
1.95	66.26	22.09	1728.41	111.63	37.21	2052.21	15.78

Table 4 Sensitivity analysis of Demand rate

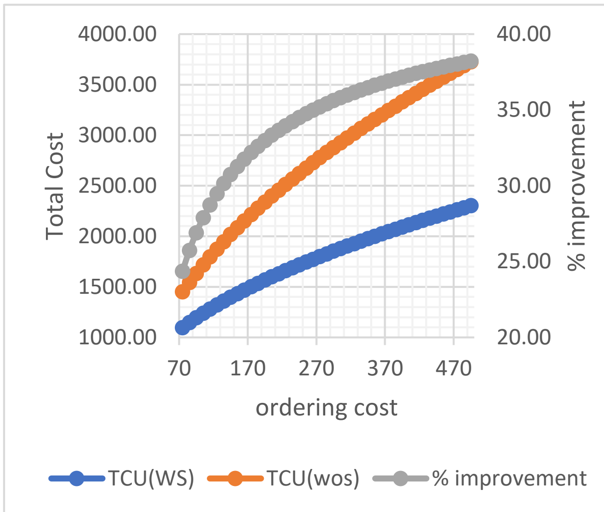
Demand rate D_2	With substitution			Without substitution			%improvement
	q_2	Q_2	$TC(WS)$	q_2	Q_2	$TC(WOS)$	
20	47.32	15.77	1339.91	74.74	24.91	2714.83	50.64
40	54.07	18.02	1328.38	92.34	30.78	2268.34	41.44
60	58.81	19.60	1350.71	101.69	33.90	2124.15	36.41
80	62.78	20.93	1381.50	107.57	35.86	2068.83	33.22
100	66.26	22.09	1415.73	111.63	37.21	2052.21	31.01
120	69.40	23.13	1451.84	114.61	38.20	2056.01	29.39
140	71.60	25.06	1489.13	116.89	38.96	2071.93	28.13
160	72.31	28.92	1526.02	118.69	39.56	2095.63	27.18
180	72.87	32.79	1562.17	120.15	40.05	2124.64	26.47
200	73.30	36.65	1597.66	121.36	40.45	2157.44	25.95

Table 5 Sensitivity analysis of Usage rate

Usage rate $a_1 = a_2$	With substitution			Without substitution			%improvement
	q_2	Q_2	$TC(WS)$	q_2	Q_2	$TC(WOS)$	
1.00	44.07	22.04	1184.99	85.81	28.60	1498.30	20.91
1.25	48.41	19.36	1299.46	93.83	31.28	1644.29	20.97
2.00	56.59	18.86	1636.58	111.63	37.21	2073.15	21.06
3.00	63.98	21.33	2075.45	127.28	42.43	2626.40	20.98
3.25	65.63	21.88	2183.07	130.32	43.44	2762.38	20.97
3.50	67.22	22.41	2289.92	133.11	44.37	2897.67	20.97
3.75	68.75	22.92	2396.05	135.68	45.23	3032.34	20.98
5.25	76.97	25.66	3020.05	147.67	49.22	3830.82	21.16
6.00	80.58	26.86	3325.41	152.13	50.71	4225.77	21.31
8.00	88.99	29.66	4124.12	161.00	53.67	5270.73	21.75
9.25	93.55	31.18	4614.56	165.06	55.02	5919.85	22.05
9.50	94.41	31.47	4712.01	165.78	55.26	6049.42	22.11
10.00	96.08	32.03	4906.32	167.13	55.71	6308.34	22.22

Table 6 Sensitivity analysis of Substitution rate

Substitution rate v_1	With substitution			Without substitution			%improvement
	q_2	Q_2	$TC(WS)$	q_2	Q_2	$TC(WOS)$	
0.10	56.22	18.74	1347.97	111.63	37.21	2052.21	34.32
0.20	66.26	22.09	1415.73	111.63	37.21	2052.21	31.01
0.30	74.53	24.84	1481.60	111.63	37.21	2052.21	27.80
0.40	81.22	27.07	1555.21	111.63	37.21	2052.21	24.22
0.50	86.47	28.82	1642.53	111.63	37.21	2052.21	19.96
0.60	90.41	30.14	1747.43	111.63	37.21	2052.21	14.85
0.70	93.19	31.06	1872.46	111.63	37.21	2052.21	8.76
0.80	94.97	31.66	2019.20	111.63	37.21	2052.21	1.61



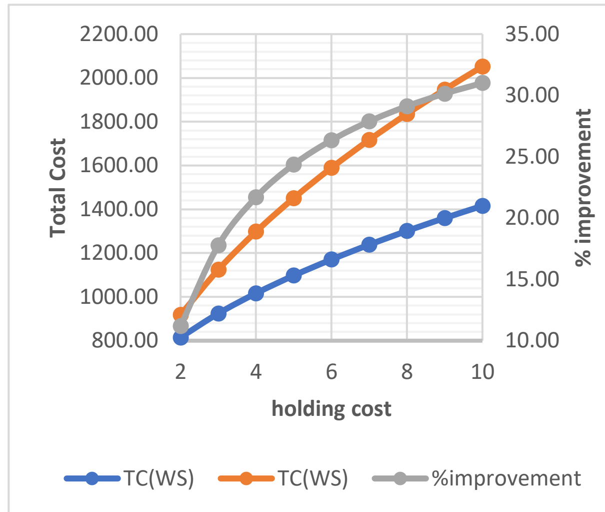
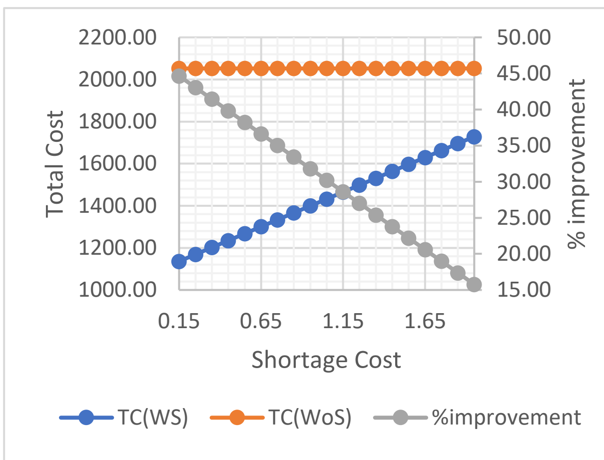


Fig 3 Sensitivity with respect to ordering cost

Fig 4 Sensitivity with respect to holding cost



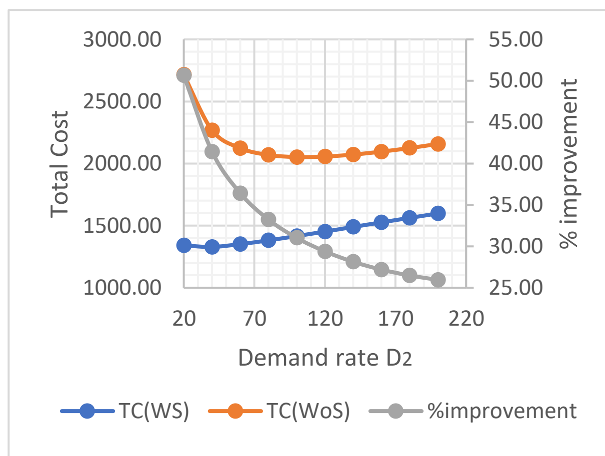
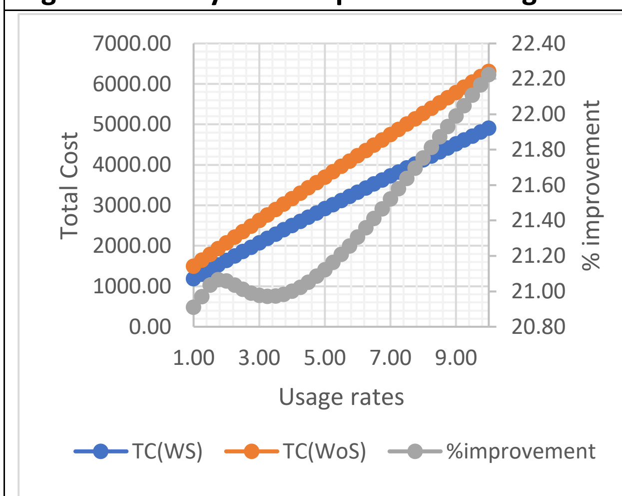


Fig 5 Sensitivity with respect to Shortage cost

Fig 6 Sensitivity with respect to Demand rate



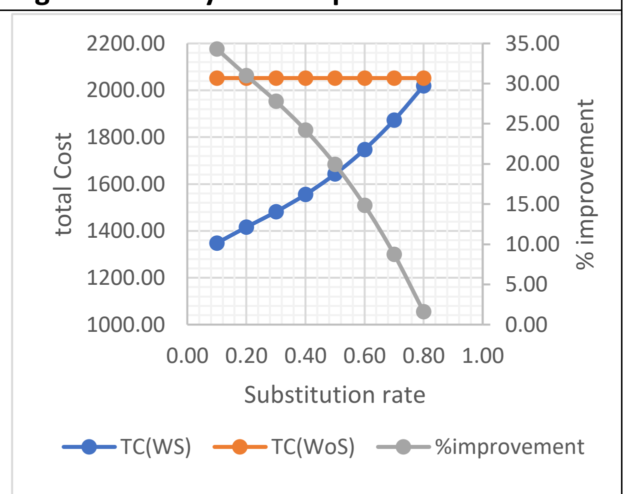


Fig 7 Sensitivity with respect to Usage rate

Fig 8 Sensitivity with Substitution rate

Table 7 Summary of Sensitivity analysis

Parameter	**Variation**	**TCU(WS)**	**TCU(WOS)**	**% improvement**
Ordering cost (k_1, k_2)	Increases	Increases	Increases	Increases
Holding cost (h_1, h_2)	Increases	Increases	Increases	Increases
Shortage cost (c_{s_1}, c_{s_2})	Increases	Increases	Constant	Decreases
Demand rate (D_1, D_2)	Increases	Increases	Increases	Decreases
Usage rate (a_1, a_2)	Increases	Increases	Increases	Increases
Substitution rate (ν_1, ν_2)	Increases	Increases	Constant	Decreases

6. CONCLUSION

An inventory system with two mutually substitutable products has been considered, where one of the product is composed of two complementary components α and β. When one product becomes out of stock, the future demand of the product is met partially from the leftover inventory of the other product. This model is applicable to similar items consisting of complementary products such as, different sim cards and mobile phone, different laptop with graphic software, coffee and milk, different brands of toothpaste and brush etc,. The pseudo convexity of the total cost function for two different cases has been derived. Analysis of this proposed model shows that substitution between products saves in the total cost. Sensitivity analysis is carried out extensively to validate the proposed model. The summary of the results is presented in Table 7 .When the substitution rates increases, the percentage improvement in the total cost of substitution over without substitution decreases. Whereas when there is an increment in ordering cost and holding cost, the percentage improvement in the total cost of substitution over without substitution also increases. This article can be further extended by considering, cost of substitution, both products as complementary products, deterioration of items, also one can extend the work by considering multiple product.

APPENDIX A

Proof of Theorem 1:

Consider $TC\left(q_2,Q_2\right)$, calculating the following partial derivatives

$$\frac{\partial^2 TC_1}{\partial q_2^2}=\frac{4h_1\left(\frac{a_1}{a_2}+1\right)}{3D_1a_2}+\frac{D_1^2h_2\left(a_1+a_2\right)^2\left(\nu_1-1\right)^2\left(\frac{a_1}{a_2}+1\right)^2}{\left(D_2+D_1\left(a_1+a_2\right)\right)^2\left(D_2+D_1\nu_1\left(a_1+a_2\right)\right)}\geq 0$$

$$\frac{\partial^2 TC_1}{\partial Q_2^2}=\frac{4h_2}{3D_2}+\frac{D_1^2h_2\left(a_1+a_2\right)^2\left(\nu_1-1\right)^2}{\left(D_2+D_1\left(a_1+a_2\right)\right)^2\left(D_2+D_1\nu_1\left(a_1+a_2\right)\right)}\geq 0$$

$$\frac{\partial^2 TC_1}{\partial Q_2 q_2}=\frac{D_1^2h_2\left(a_1+a_2\right)^2\left(\nu_1-1\right)^2\left(\frac{a_1}{a_2}+1\right)}{\left(D_2+D_1\left(a_1+a_2\right)\right)^2\left(D_2+D_1\nu_1\left(a_1+a_2\right)\right)}\geq 0$$

Also, determinant of the Hessian matrix $\begin{vmatrix}\frac{\partial^2 TC_1}{\partial q_2^2} & \frac{\partial^2 TC_1}{\partial Q_2 q_2}\\ \frac{\partial^2 TC_1}{\partial Q_2 q_2} & \frac{\partial^2 TC_1}{\partial Q_2^2}\end{vmatrix}\geq 0$

Since the Hessian matrix is positive semi-definite, $TC\left(q_2,Q_2\right)$ is a convex function. We make use of the fact that, ratio of the positive convex function over a linear function is pseudo convex. Hence the Proof.

APPENDIX B

Proof of Theorem 2: Similar to Proof of Theorem 1

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] L. Benkherouf, K. Skouri, I. Konstantaras, Inventory decisions for a finite horizon problem with product substitution options and time varying demand, Appl. Math. Model. 51 (2017), 669–685.

[2] Z. Drezner, H. Gurnani, B.A. Pasternack, An EOQ Model with Substitutions Between Products, J. Oper. Res. Soc. 46 (1995), 887–891.

[3] B.K. Durga1, E. Chandrasekaran, Optimal ordering policy for substitutable products under quadratic demand with cost of substitution, J. Critical Rev. 7 (4) (2020), 200-205.

[4] S.K. Goyal, An EOQ model with substitutions between products, J. Oper. Res. Soc. 47 (3) (1996), 476-477.

[5] H. Gurnani, Z. Drezner, Deterministic hierarchical substitution inventory models, J. Oper. Res. Soc. 51 (1) (2000), 129-133.

[6] I.P. Krommyda, K. Skouri, I. Konstantaras, Optimal ordering quantities for substitutable products with stock-dependent demand, Appl. Math. Model. 39 (1) (2015), 147-164.

[7] L. Liu, I. Adan, D. Perry, Two perishable inventory systems with one-way substitution, Ann. Oper. Res. (2016). https://doi.org/10.1007/s10479-016-2128-1.

[8] B. Maddah, M. Kharbeche, S. Pokharel, A. Ghoniem, Joint replenishment model for multiple products with substitution, Appl. Math. Model. 40 (2016), 7678–7688.

[9] V.K. Mishra, Optimal ordering quantities for substitutable deteriorating items under joint replenishment with cost of substitution, J. Ind. Eng. Int. 13 (2017), 381–391.

[10] V.K. Mishra, K. Shanker, Optimal ordering quantities for substitutable items under joint replenishment with cost of substitution, Oper. Res. Decisions, 27 (2017), 77-104.

[11] A. Mukhopadhyay, A. Goswami, An inventory model with shortages for imperfect items using substitution of two products, Int. J. Oper. Res. 30 (2) (2017), 193-219.

[12] S.K. Mukhopadhyay, X. Yue, X. Zhu, A Stackelberg model of pricing of complementary goods under information asymmetry, Int. J. Product. Econ. 134 (2011), 424–433.

[13] M.K. Salameh, A.A. Yassine, B. Maddah, L. Ghaddar, Joint replenishment model with substitution, Appl. Math. Model. 38 (2014), 3662–3671.

[14] E. Stavrulaki, Inventory decisions for substitutable products with stock-dependent demand, Int. J. Product. Econ.

129 (2011), 65–78.

[15] R. Yan, S. Bandyopadhyay, The profit benefits of bundle pricing of complementary products, J. Retail. Consumer Services. 18 (2011), 355–361.

[16] X. Yue, S.K. Mukhopadhyay, X. Zhu, A Bertrand model of pricing of complementary goods under information asymmetry, J. Bus. Res. 59 (2006), 1182–1192.

[17] R. Zhang, I. Kaku, Y. Xiao, Deterministic EOQ with partial backordering and correlated demand caused by cross-selling, Eur. J. Oper. Res. 210 (2011), 537–551.

[18] R. Zhang, Y. Huang, Y. Xiao, A two-item deterministic EOQ model with partial backordering and substitution, International Conference on Economics, Business and Management, Manila, Philippines. 2010.