Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 2, 2377-2402
https://doi.org/10.28919/jmcs/5426
ISSN: 1927-5307

SCREEN CAUCHY RIEMANN LIGHTLIKE SUBMERSIONS

S. S. SHUKLA, SHIVAM OMAR*

Department of Mathematics, University of Allahabad, Prayagraj-211002, India

Abstract. We introduce the notion of screen Cauchy Riemann (SCR) lightlike submersions from an indefinite K*ä*hler manifold onto a lightlike manifold. We prove that SCR-lightlike submersions include complex (invariant) and screen real (anti-invariant) lightlike submersions. We study some properties of proper SCR-lightlike submersions, their invariant and anti-invariant subcases. We also study the geometry of complex lightlike submersions and show that the radical distribution defines a totally geodesic foliation.

Keywords: submersion; lightlike manifold; lightlike submersion; K*ä*hler manifold.

2010 AMS Subject Classification: 53B15, 53C20, 53C55.

1. Introduction

A Riemannian submersion between Riemannian manifolds M and B is the mapping f from M onto B such that f has maximal rank and the derivative map f_* preserves the length of horizontal vectors. The idea of Riemannian submersion between Riemannian manifolds was introduced by O' Neill [9] and Gray [7]. Almost Hermitian submersions between almost Hermitian manifolds were introduced by B. Watson [14]. In this class, Riemannian submersion is also an almost complex mapping. Later, different classes of almost Hermitian submersions have been studied between different subclasses of almost Hermitian manifolds [5] and [6]. On the

*Corresponding author
E-mail address: shivamomar.2010@gmail.com
Received January 13, 2021

other hand, lightlike submanifolds of semi-Riemannian manifolds were introduced by Duggal and Bejancu in [1]. In [3], Duggal and Sahin introduced screen Cauchy-Riemannian (SCR)-lightlike submanifolds of an indefinite K*ä*hler manifold. In [10], O' Neill introduced the notion of semi-Riemannian submersions. If M and B are Riemannian manifolds then the fibres are always Riemannian manifolds. However, if M and B are semi-Riemannian manifolds, then the fibres may not be semi-Riemannian manifolds. Sahin introduced the notion of screen lightlike submersions from lightlike manifolds onto semi Riemannian manifolds in [12]. Later, Sahin and G*ü*nd*ü*zalp introduced lightlike submersions from semi-Riemannian manifolds onto lightlike manifolds in [13]. This motivated us to study screen Cauchy Riemann (SCR)-lightlike submersions. The paper is organized as follows:

In section 2, we give basic definitions related to this paper. In section 3, we introduce the notion of screen Cauchy Riemann (SCR)-lightlike submersions giving examples and prove two existence theorems. We prove that this class contains complex (invariant) and screen real (anti-invariant) subcases and obtain the integrability conditions of distributions involved in the definition of such submersions. In section 4, we study proper SCR-lightlike submersions with totally umbilical fibres and prove an existence theorem. In section 5, we study the geometry of complex lightlike submersions and prove that radical distribution defines totally geodesic foliation. In last section, we study screen real lightlike submersions with irrotational fibres and prove some results related to it.

2. Preliminaries

Let (M,J) be a 2m-dimensional almost complex manifold with an almost complex structure J and g be a semi-Riemannian metric of index $0 < r \leq 2m$. Then, the pair (J,g) is said to be an indefinite almost Hermitian structure on M, and M an indefinite almost Hermitian manifold, if

$$g(JX,JY) = g(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{2.1}$$

Further, if J defines a complex structure on M, then (J,g) and M are called indefinite Hermitian structure and indefinite Hermitian manifold respectively. Let (M,J,g) be an indefinite almost Hermitian manifold and ∇ be the Levi-Civita connection on M with respect to g. Then M is said to be an indefinite K*ä*hler manifold if

$$(\nabla_X J)Y = 0, \quad \forall X,Y \in \Gamma(TM). \tag{2.2}$$

A connected indefinite K*ä*hler manifold M with a constant holomorphic sectional curvature c is called an indefinite complex space form and is denoted by $M(c)$. The curvature tensor field of $M(c)$ is given by

$$R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\}, \quad \forall X,Y,Z \in \Gamma(TM). \tag{2.3}$$

The fundamental 2-form Ω of M is given by

$$\Omega(X,Y) = g(X,JY), \quad \forall X,Y \in \Gamma(TM). \tag{2.4}$$

For a K*ä*hler manifold M, $d\Omega = 0$, where d is the operator of exterior derivative.

Let (M,g) be a real m-dimensional smooth manifold, where g is a symmetric tensor field of type $(0,2)$. The radical or null space of T_pM, denoted by $Rad\ T_pM$, is defined as

$$Rad\ T_pM = \{\xi \in T_pM : g(\xi,X) = 0,\ \forall X \in T_pM\}.$$

The dimension of $Rad\ T_pM$ is said to be nullity degree of g. Suppose the mapping $RadTM : p \in M \to Rad\ T_pM$ defines a smooth distribution of rank $r > 0$ on M. If $0 < r \leq m$, then $Rad\ TM$ is said to be radical (null) distribution of M and the manifold M is said to be a lightlike manifold or an r-lightlike manifold.

Let (M,g) be a semi-Riemannian manifold of dimension m and (B,g') an r-lightlike manifold of dimension n. If $f : M \to B$ is a smooth submersion, then $f^{-1}(x)$ is a closed submanifold of M of dimension $(m-n)$, for any $x \in B$. The kernel of f_* at $p \in M$, is defined as

$$Ker\ f_{*p} = \{X \in T_pM : f_{*p}X = 0\},$$

and $(Ker\ f_{*p})^{\perp}$ is defined as

$$(Ker\ f_{*p})^{\perp} = \{Y \in T_pM : g(Y,X) = 0,\ \forall\ X \in Ker\ f_{*p}\}.$$

As T_pM is a semi-Riemannian vector space, $(Ker\ f_{*p})^{\perp}$ may not be a complementary space to $Kerf_{*p}$. Therefore we assume

$$\Delta_p = Ker\ f_{*p} \cap (Ker\ f_{*p})^{\perp} \neq \{0\}.$$

Then $\Delta : p \to \Delta_p$ is a distribution on M, called the radical distribution. Since Δ is a degenerate distribution, its orthogonal complementary distribution $S(Kerf_*)$ in $Ker\ f_*$ is non-degenerate. Thus we have the following orthogonal decomposition

$$Ker\ f_* = \Delta \perp S(Ker\ f_*). \tag{2.5}$$

Similarly

$$(Ker\ f_*)^{\perp} = \Delta \perp S(Ker\ f_*)^{\perp}, \tag{2.6}$$

where $S(Ker\ f_*)^{\perp}$ is the complementary distribution to Δ in $(Ker\ f_*)^{\perp}$. Moreover as $S(Ker\ f_*)$ is non-degenerate in $TM|_{Ker\ f_*}$, we have

$$TM|_{Ker\ f_*} = S(Ker\ f_*) \perp (S(Ker\ f_*))^{\perp},$$

where $(S(Kerf_*))^{\perp}$ is the complementary orthogonal distribution to $S(Kerf_*)$ in TM. Now, since $S(kerf_*)^{\perp}$ is non-degenerate in $(S(Kerf_*))^{\perp}$, we have

$$(S(ker\ f_*))^{\perp} = S(Ker\ f_*)^{\perp} \perp (S(Ker\ f_*)^{\perp})^{\perp}.$$

Suppose $dim\ \Delta = r > 0$. Since $\Delta \subset (S(Ker\ f_*)^{\perp})^{\perp}$ and $(S(Kerf_*)^{\perp})^{\perp}$ is non-degenerate, so there exists null vectors $N_1, N_2 ..., N_r$, such that

$$g(N_i, N_j) = 0, \quad g(\xi_i, N_j) = \delta_{ij}.$$

The distribution generated by vector fields $N_1, N_2 ..., N_r$ is denoted by $ltr(Ker\ f_*)$. Then, we take

$$tr(Kerf_*) = ltr(Kerf_*) \perp S(Kerf_*)^{\perp} \tag{2.7}$$

We observe that $ltr(Kerf_*)$ and $Kerf_*$ are not orthogonal to each other. Moreover

$$TM = Kerf_* \oplus tr(Kerf_*), \tag{2.8}$$

where $Kerf_*$ and $tr(Kerf_*)$ are not orthogonal to each other.

A Riemannian submersion $f : M \to B$ is called

(a) r-lightlike submersion if

$$dim\,\Delta = dim\{(Ker\ f_*)\cap(Ker\ f_*)^{\perp}\} = r, \quad 0<r<min\{dim(Ker\ f_*), dim(Ker\ f_*)^{\perp}\},$$

(b) co-isotropic submersion if $dim\,\Delta = dim(Ker\ f_*)^{\perp} < dim(Ker\ f_*)$,

(c) isotropic submersion if $dim\,\Delta = dim(Ker\ f_*) < dim(Ker\ f_*)^{\perp}$,

(d) totally lightlike submersion if $dim\,\Delta = dim(Ker\ f_*)^{\perp} = dim(Ker\ f_*)$.

From (2.5), (2.7) and (2.18) we have

$$TM = (Ker\ f_*)\oplus tr(Ker\ f_*) = \Delta\oplus(ltr(Ker\ f_*))\oplus S(Ker\ f_*)\oplus S(Ker\ f_*)^{\perp}. \tag{2.9}$$

Now we assume the following local quasi-orthogonal field of frames of M along $Ker\ f_*$

$$\{\xi_1,\ \dots,\ \xi_r,\ N_1,\dots,\ N_r,\ X_{r+1},\dots,X_m,\ W_{r+1},\dots,W_n\}, \tag{2.10}$$

where $\{\xi_1,\ \dots,\xi_r\}$ and $\{N_1,\dots,\ N_r\}$ are lightlike basis of $\Gamma(\Delta)$ and $\Gamma(ltr(Ker\ f_*))$ respectively. Also, $\{X_{r+1},\dots,X_m\}$ and $\{W_{r+1},\dots,W_n\}$ are orthonormal basis of $\Gamma(S(Ker\ f_*))$ and $\Gamma(S(Ker\ f_*)^{\perp})$ respectively.

The geometry of Riemannian and semi-Riemannian submersions is characterized by O' Neill's tensors T and A, defined as

$$T_XY = h\nabla_{\nu X}\nu Y + \nu\nabla_{\nu X}hY, \tag{2.11}$$

$$A_XY = \nu\nabla_{hX}hY + h\nabla_{hX}\nu Y. \tag{2.12}$$

Tensor fields T and A are skew symmetric in Riemannian submersions but it is not generally valid for lightlike submersions. Moreover T and A both reverse the vertical and horizontal subspaces and are vertical and horizontal tensors respectively, that is,

$$T_XY = T_{\nu X}Y, \quad A_XY = A_{hX}Y \tag{2.13}$$

Also, T has symmetric property for vertical vector fields, that is,

$$T_XY = T_YX, \quad \forall\, X,Y\in\Gamma(Ker\ f_*) \tag{2.14}$$

Now we state the following lemma proved in [13].

Lemma 2.1. Let $f:(M,g)\to(B,g')$ be an r-lightlike submersion. Then

(a) $g(T_VX,Y)=-g(T_VY,X)$,

(b) $g(A_XV,W)=-g(A_XW,V))$,

for any $V\in\Gamma(Ker\ f_*),\ X,Y\in\Gamma(Ker\ f_*)^{\perp}$ and $W\in\Gamma(\Delta)$.

3. SCR-LIGHTLIKE SUBMERSIONS

In this section, we introduce screen Cauchy Riemann (SCR) lightlike submersions from an indefinite K*ä*hler manifold onto a lightlike manifold and give several examples.

Definition. An r-lightlike submersion $f:(M,g,J)\to(B,g')$ from a 2m-dimensional indefinite K*ä*hler manifold M onto a lightlike manifold B is called a SCR-lightlike submersion if

(i) Δ is invariant with respect to J, that is, $J\Delta=\Delta$,

(ii) $S(Ker\ f_*)$ contains a non-null distribution D, such that

$$S(Ker\ f_*)=D\oplus D^{\perp},\quad JD^{\perp}\subset S(Ker\ f_*)^{\perp},\quad D\cap D^{\perp}=\{0\},$$

where $D^{\perp}$ is the orthogonal complementary distribution to D in $S(Ker\ f_*)$,

(iii) D is invariant with respect to J, that is, $JD=D$.

From the definition, it is clear that

$$J\Delta=\Delta,\ JD=D,\ Jltr(Ker\ f_*)=ltr(Ker\ f_*) \tag{3.1}$$

$$Ker\ f_*=D'\oplus D^{\perp},\quad D'=D\perp\Delta. \tag{3.2}$$

Let D_0 be the orthogonal complement to $JD^{\perp}$ in $S(Ker\ f_*)^{\perp}$. Then, we have

$$tr(Ker\ f_*)=ltr(Ker\ f_*)\perp JD^{\perp}\perp D_0. \tag{3.3}$$

If $D\neq\{0\}$ and $D^{\perp}\neq\{0\}$, then we say that f is a proper SCR-lightlike submersion. Note the following important features of SCR-lightlike submersions:

(a) Condition (i) implies that $dim\ \Delta=2k\geq 2$

(b) For a proper SCR-lightlike submersion f, (2.10) and (3.1) imply that $dim(D)=2l\geq 2,\ dim(D^{\perp})\geq 1$ and $dim(ltr(Ker\ f_*))=dim(\Delta)$. It follows that, $dim(Ker\ f_*)\geq 5,\ dim(M)\geq 8$.

Proposition 3.1. A SCR-lightlike submersion $f:(M,g)\rightarrow(B,g')$ from an indefinite K*ä*hler manifold M onto a lightlike manifold B is a complex (resp. screen real) lightlike submersion if and only if $D^{\perp}=\{0\}$ (resp. $D=\{0\}$).

Proof. Let f be a complex (invariant) lightlike submersion, that is, $J(Ker\ f_*)=Ker\ f_*$. As Δ and D are invariant with respect to J, it follows that, $D^{\perp}=\{0\}$. Conversely if $D^{\perp}=\{0\}$ then $J(S(Ker\ f_*))=S(Ker\ f_*)$. Moreover, f is a SCR-lightlike submersion implies that $J(Ker\ f_*)=Ker\ f_*$. In the same way, the other assertion can be proved. □

Proposition 3.2. A SCR co-isotropic or isotropic or totally lightlike submersion f from an indefinite K*ä*hler manifold M onto a lightlike manifold B is a complex (invariant) lightlike submersion.

Proof. Let f be a SCR-lightlike submersion. If f is co-isotropic, then, $S(Ker\ f_*)^{\perp}=\{0\}$ which implies $J(Ker\ f_*)=Ker\ f_*$, that is, f is invariant lightlike submersion. If f is isotropic, then $S(Ker\ f_*)=\{0\}$ which implies f is invariant lightlike submersion. Similarly, if f is totally lightlike submersion, then $\Delta=Ker\ f_*=(Ker\ f_*)^{\perp}$ which also implies f is invariant lightlike submersion. □

From proposition (3.2), we can see that proper SCR and screen real lightlike submersions must be r-lightlike. Thus we have the following proposition:

Proposition 3.3. There exist no proper SCR co-isotropic or isotropic or totally lightlike submersions from an indefinite K*ä*hler manifold onto a lightlike manifold.

Now, we give examples of proper-SCR, complex and screen real lightlike submersions.

Denote by $\mathbb{R}^n_{r,q,p}$ the space $\mathbb{R}^n$ equipped with the semi-Riemannian metric g defined by $g(e_i,e_j)_{r,q,p}=(G_{r,q,p})_{ij},\ i\in\{1,...,n\}$, where e_i is the standard basis of $\mathbb{R}^n$, and $G_{r,q,p}$ is the diagonal matrix determined by g, that is, G_{ij} = diagonal$(\underbrace{0,...,0}_{\text{r-times}},\underbrace{-1,...,-1}_{\text{q-times}},\underbrace{1,...,1}_{\text{p-times}})$.

Let $\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},...,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_n}\right\}$ be a canonical basis of $\mathbb{R}^{2n}$. Then we define J as

$$J\Big\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},...,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_n}\Big\}=\Big\{-\frac{\partial}{\partial y_1},\frac{\partial}{\partial x_1},...,-\frac{\partial}{\partial y_n},\frac{\partial}{\partial x_n}\Big\}.$$

Example 3.1. Let $\mathbb{R}^{12}_{0,2,10}$ and $\mathbb{R}^{6}_{2,0,4}$ endowed with the semi-Riemannian metric

$$g = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2 + (dx_8)^2 + (dx_9)^2 + (dx_{10})^2 + (dx_{11})^2 + (dx_{12})^2,$$

and degenerate metric $g' = (dy_3)^2 + (dy_4)^2 + (dy_5)^2 + (dy_6)^2$, where $x_1, \dots, x_{12}$ and $y_1, \dots, y_6$ are the canonical coordinates on $\mathbb{R}^{12}$ and $\mathbb{R}^6$, respectively. Define the map $f : (\mathbb{R}^{12}, g) \to (\mathbb{R}^6, g')$ as

$$(x_1, \dots, x_{12}) \longmapsto \Big(x_1 - x_3,\ x_2 - x_4,\ \frac{x_5 - x_7}{\sqrt{2}},\ \frac{x_6 - x_8}{\sqrt{2}},\ x_{10},\ x_{12}\Big).$$

Then, we have

$$Kerf_* = Span\Big\{V_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, V_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}, V_3 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}\Big), V_4 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8}\Big), V_5 = \frac{\partial}{\partial x_9}, V_6 = \frac{\partial}{\partial x_{11}}\Big\},$$

and

$$(Kerf_*)^\perp = Span\Big\{V_1, V_2, X_1 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7}\Big), X_2 = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8}\Big), X_3 = \frac{\partial}{\partial x_{10}}, X_4 = \frac{\partial}{\partial x_{12}}\Big\}.$$

Thus f is a 2-lightlike submersion with $\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\{V_1, V_2\}$. Since $\mathbb{R}^{12}$ has complex structure, we can see easily that $JV_1 = V_2$ and $JV_3 = V_4$, i.e., $D = Span\{V_3, V_4\}$ and Δ are invariant with respect to J. Moreover, $D^\perp = Span\{V_5, V_6\}$ and $JD^\perp \subset S(Ker\ f_*)^\perp$. Hence, f is a proper SCR-lightlike submersion. Furthermore, we can see that lightlike transversal distribution $ltr(Ker\ f_*)$ is spanned by $N_1 = \frac{1}{2}\Big(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\Big)$ and $N_2 = \frac{1}{2}\Big(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\Big)$. It is easy to see that $JN_1 = N_2$. Therefore, $ltr(Ker\ f_*)$ is invariant with respect to J.

Example 3.2. Let $\mathbb{R}^{8}_{0,2,6}$ and $\mathbb{R}^{4}_{2,0,2}$ be endowed with the semi-Riemannian metric

$$g = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2 + (dx_8)^2,$$

and degenerate metric $g' = (dy_3)^2 + (dy_4)^2$, where $x_1, \dots, x_8$ and $y_1, \dots, y_4$ are the canonical coordinates on $\mathbb{R}^8$ and $\mathbb{R}^4$, respectively. Define the map $f : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as

$$(x_1, \dots, x_8) \longmapsto \Big(x_1 - x_3,\ x_2 - x_4, x_5, x_6\Big).$$

Then

$$Ker\ f_* = Span\Big\{U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, U_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}, U_3 = \frac{\partial}{\partial x_7}, U_4 = \frac{\partial}{\partial x_8}\Big\},$$

and

$$(Ker\ f_*)^\perp = Span\Big\{U_1, U_2, X_1 = \frac{\partial}{\partial x_5}, X_2 = \frac{\partial}{\partial x_6}\Big\}$$

Thus f is a 2-lightlike submersion with $\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\{U_1, U_2\}$. It is easy to see that $JU_1 = U_2$, so Δ is invariant with respect to J. Now, since $JU_3 = U_4$, we see that $D = S(Ker\ f_*) = Span\{U_3, U_4\}$ is invariant under J. Hence, f is complex lightlike submersion. Further, $ltr(Ker\ f_*)$ is spanned by $N_1 = \frac{1}{2}\Big(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\Big)$ and $N_2 = \frac{1}{2}\Big(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\Big)$. By easy calculation we can see that $JN_1 = N_2$, which implies $ltr(Ker\ f_*)$ is also invariant with respect to J.

Example 3.3. Let $\mathbb{R}^8_{0,2,6}$ and $\mathbb{R}^4_{2,0,2}$ be endowed with the semi-Riemannian metric

$$g = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2 + (dx_8)^2,$$

and degenerate metric $g' = (dy_3)^2 + (dy_4)^2$, where $x_1, \dots, x_8$ and $y_1, \dots, y_4$ are the canonical coordinates on $\mathbb{R}^8$ and $\mathbb{R}^4$, respectively. Define the map $f : (\mathbb{R}^8, g) \to (\mathbb{R}^4, g')$ as

$$(x_1, \dots, x_8) \longmapsto \Big(x_1 - x_5,\ x_2 - x_6,\ x_3,\ x_7\Big).$$

Then, we have

$$Kerf_* = Span\Big\{U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, U_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, U_3 = \frac{\partial}{\partial x_4}, U_4 = \frac{\partial}{\partial x_8}\Big\},$$

and

$$(Kerf_*)^\perp = Span\Big\{U_1, U_2, X_1 = \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_7}\Big\}.$$

Therefore, f is a 2-lightlike submersion with $\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\{U_1, U_2\}$. Since $JU_1 = U_2$, Δ is invariant with respect to J. Clearly $S(Ker\ f_*) = Span\{U_3, U_4\}$ and $S(Ker\ f_*)^\perp =$

$Span\{X_1, X_2\}$. We can see easily that $JU_3 = X_1$ and $JU_4 = X_2$. Thus, $S(Ker\ f_*) = D^\perp$ and $JD^\perp = S(Ker\ f_*)^\perp$. Hence f is screen real lightlike submersion. Moreover, we obtain

$$ltr(Ker\ f_*) = Span\Big\{N_1 = \frac{1}{2}\Big(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}\Big),\ N_2 = \frac{1}{2}\Big(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}\Big)\Big\},$$

which is invariant with respect to J.

Theorem 3.1. Let $f : (M(c), g) \to (B, g')$ be a lightlike submersion, where $M(c)$ is an indefinite complex space form with $c \neq 0$ and B an r-lightlike manifold. Then, f is an SCR-lightlike submersion with $D \neq \{0\}$ if and only if

(a) The maximal complex subspaces of $Ker\ f_{*p},\ p \in M$ define a distribution $D' = D \perp \Delta$, where D is a holomorphic distribution.

(b) $g(R(X,Y)U,V) = 0$, for any $X,Y \in \Gamma(D')$ and $U,V \in \Gamma(D^\perp)$, where $D^\perp$ is orthogonal complementary distribution to D in $S(Ker\ f_*)$.

Proof. Let f be a SCR-lightlike submersion, then $D' = D \perp \Delta$. If $X \in \Gamma(Ker\ f_*)$ and $Y \in \Gamma(\Delta)$, then (b) follows from (2.3). If $X \in \Gamma(D')$ and $Y \in \Gamma(D)$ then, as $D^\perp$ is orthogonal to D in $S(Ker\ f_*)$, (b) follows again from (2.3). Conversely, (a) implies that $J\Delta$ is a distribution on M and since D is a holomorphic distribution we have $J\Delta \cap D = \{0\}$. Therefore, we have $JD' = D'$, which implies Δ is invariant with respect to J. From (2.1), (2.3) and (b) we obtain

$$0 = g(R(JX,X)U,V) = \Big(\frac{c}{2}\Big) g(X,X) g(JU,V),$$

for any $X \in \Gamma(D)$ and $U,V \in \Gamma(D^\perp)$. Thus $JD^\perp$ is orthogonal to $D^\perp$. Also, as D is holomorphic, $JD^\perp$ is orthogonal to D. At the end, since $ltr(Ker\ f_*)$ is invariant with respect to J, we get $JD^\perp \subset \Gamma(S(Ker\ f_*)^\perp)$. This completes the proof. □

On the existence of a proper SCR-lightlike submersions, we prove the following result.

Theorem 3.2. Let $f : (M, g, J) \to (B, g')$ be a 2r-lightlike submersion from an indefinite Kähler manifold M onto a lightlike manifold B and $dim(Ker\ f_*) = m$. Let $dim(S(Ker\ f_*)^\perp) = 1$. If Δ is invariant with respect to J and $2r < m$ then f is a proper SCR-lightlike submersion.

Proof. Since $dim(S(Ker\ f_*)^\perp) = 1$ and $g(JW,W) = 0$ for any $W \in \Gamma(S(Ker\ f_*)^\perp)$, we have $J(S(Ker\ f_*)^\perp) \cap S(Ker\ f_*)^\perp = \{0\}$. The radical distribution Δ is invariant under J, implies invariance of $ltr(Ker\ f_*)$ under J and since JW is a non-radical vector field, we have $JW \in S(Ker\ f_*)$. Assume that $D^{\perp} = Span\{JW\}$ and D be a orthogonal complementary distribution to $D^{\perp}$ in $S(Kerf_*)$. Moreover, as Δ is invariant, we have $g(JU,\xi) = -g(U,J\xi) = 0$, for any $U \in \Gamma(D)$, $\xi \in \Gamma(\Delta)$. Thus, $JD \cap \Delta = \{0\}$. As $D^{\perp}$ is anti-invariant, we get $g(JU,V) = -g(U,JV) = 0$, for any $U \in \Gamma(D), V \in \Gamma(D^{\perp})$. Therefore, $JD \cap D^{\perp} = \{0\}$. In the same way, $g(JU,W) = -g(U,JW) = 0$, for any $U \in \Gamma(D)$, $W \in \Gamma(S(Ker\ f_*)^{\perp})$. Now, since D and $D^{\perp}$ are orthogonal to each other, we have $JD \cap S(Ker\ f_*)^{\perp} = \{0\}$. At the end, non-degenerate D and invariant $ltr(Ker\ f_*)$ imply that $g(JU,N) = -g(U,JN) = 0$, for any $U \in \Gamma(D)$, $N \in \Gamma(ltr(Ker\ f_*))$. Thus, we have $JD \cap ltr(Ker\ f_*) = \{0\}$. All the above results imply that D is invariant with respect to J. This completes the proof. □

Let f be a lightlike submersion from a real (m + n)-dimensional semi-Riemannian manifold (M,g), where $m,n > 1$, onto a lightlike manifold (B,g'). Also, assume that $Ker\ f_*$ is an m-dimensional lightlike distribution of M and $tr(Ker\ f_*)$ is the complementary distribution to $Ker\ f_*$ in M with respect to the pair $\{S(Ker\ f_*), S(Ker\ f_*)^{\perp}\}$. Also, denote by ∇ the Levi-Civita connection on M and by $\hat{g}$ the induced metric on $Ker\ f_*$ of g. Then from (2.11), for any $U,V \in \Gamma(Ker\ f_*)$ and $X \in \Gamma(Ker\ f_*)^{\perp}$, we write

$$\nabla_U V = \hat{\nabla}_U V + T_U V, \tag{3.4}$$

$$\nabla_U X = T_U X + \nabla^{\perp}_U X, \tag{3.5}$$

where $\hat{\nabla}_U V = v\nabla_U V$ and $\nabla^{\perp}_U X = h\nabla_U X$, $\{\hat{\nabla}_U V, T_U X\}$ and $\{T_U V, \nabla^{\perp}_U X\}$ belong to $\Gamma(Ker\ f_*)$ and $\Gamma(tr(Ker\ f_*))$, respectively. Here $\hat{\nabla}$ is a linear connection on $Ker\ f_*$. Let $S(Ker\ f_*)^{\perp} \neq \{0\}$. Next, we assume that L and S the projection of $tr(Ker\ f_*)$ on $ltr(Ker\ f_*)$ and $S(Ker\ f_*)^{\perp}$, respectively. Then we write

$$\nabla_U V = \hat{\nabla}_U V + T^l_U V + T^s_U V, \tag{3.6}$$

$$\nabla_U X = T_U X + D^{\perp l}_U X + D^{\perp s}_U X, \tag{3.7}$$

where $T^l_U V = L(T_U V)$, $T^s_U V = S(T_U V)$ and $D^{\perp l}_U X = L(D^{\perp}_U X), D^{\perp s}_U X = S(D^{\perp}_U X)$. We call T^l and T^s lightlike second fundamental form and the screen second fundamental form of f respectively.

It is important to note that $D^{\perp l}$ and $D^{\perp s}$ do not define linear connections on $tr(Ker\ f_*)$, although we can see easily that these are otsuki connections.

By virtue of above otsuki connections, we define the following differential operators

$$\nabla_U^{\perp l} \quad : \quad \Gamma(ltr(Ker\ f_*)) \longrightarrow \Gamma(ltr(Ker\ f_*)); \quad \nabla_U^{\perp l}(LX) = D_U^{\perp l}(LX), \tag{3.8}$$

and

$$\nabla_U^{\perp s} \quad : \quad \Gamma(S(Ker\ f_*)^{\perp}) \longrightarrow \Gamma(S(Ker\ f_*)^{\perp}); \quad \nabla_U^{\perp s}(SX) = D_U^{\perp s}(SX), \tag{3.9}$$

for any $U \in \Gamma(Ker\ f_*)$ and $X \in \Gamma(tr(Ker\ f_*))$. We can see by easy calculation that both $\nabla_U^{\perp l}$ and $\nabla_U^{\perp s}$ are linear connections on $ltr(Ker\ f_*)$ and $S(Ker\ f_*)^{\perp}$, respectively. Moreover, we define the mappings

$$D^{\perp l}\ :\ \Gamma(Ker\ f_*) \times \Gamma(S(Ker\ f_*)^{\perp}) \longrightarrow \Gamma(ltr(Ker\ f_*));\ D^{\perp l}(U,SX) = D_U^{\perp l}(SX), \tag{3.10}$$

and

$$D^{\perp s}\ :\ \Gamma(Ker\ f_*) \times \Gamma(ltr(Ker\ f_*)) \longrightarrow \Gamma(S(Ker\ f_*)^{\perp});\ D^{\perp s}(U,LX) = D_U^{\perp s}(LX), \tag{3.11}$$

$U \in \Gamma(Ker\ f_*)$ and $X \in \Gamma(tr(Ker\ f_*))$. Using (3.8)-(3.11), (3.7) becomes

$$\nabla_U X = T_U X + \nabla_U^{\perp l}(LX) + D^{\perp l}(U,SX) + \nabla_U^{\perp s}(SX) + D^{\perp s}(U,LX). \tag{3.12}$$

Thus for any $U \in \Gamma(Ker\ f_*)$, $N \in \Gamma(ltr(Ker\ f_*))$ and $W \in \Gamma(S(Ker\ f_*)^{\perp})$, from (3.12) we obtain

$$\nabla_U N = T_U N + \nabla_U^{\perp l} N + D^{\perp s}(U,N), \tag{3.13}$$

$$\nabla_U W = T_U W + D^{\perp l}(U,W) + \nabla_U^{\perp s} W. \tag{3.14}$$

By using (3.6), (3.13) and taking into account that ∇ is a metric connection, we get

$$g(T_U^s V, W) + g(V, D^{\perp l}(U,W)) = -\hat{g}(T_U W, V), \tag{3.15}$$

and using (3.13) and (3.14), we get

$$g(D^{\perp s}(U,N), W) = -g(N, T_U W) \tag{3.16}$$

If f is a co-isotropic or totally lightlike submersion, then there is no screen transversal part, i.e., $S(Ker\ f_*)^{\perp} = \{0\}$. So equations (3.6) and (3.13) reduced to

$$\nabla_U V = \hat{\nabla}_U V + T^l_U V, \tag{3.17}$$

$$\nabla_U N = T_U N + \nabla^{\perp l}_U N, \tag{3.18}$$

for any $U, V \in \Gamma(Ker\ f_*)$ and $N \in \Gamma(ltr(Ker\ f_*))$.

Let M be an indefinite K*ä*hler manifold, B an r-lightlike manifold and $f : M \to B$ be an r-lightlike submersion. Then for any $U \in \Gamma(Ker\ f_*)$ and $X \in \Gamma(Ker\ f_*)^{\perp}$, we write

$$JU = \phi U + \omega U, \tag{3.19}$$

$$JX = BX + CX, \tag{3.20}$$

where $\{\phi U, BX\}$ and $\{\omega U, CX\}$ belong to $\Gamma(Ker\ f_*)$ and $\Gamma(Ker\ f_*)^{\perp}$, respectively. If f is a SCR-lightlike submersion from M onto B, then by using (3.2) and (3.3) we have $\phi U \in \Gamma(D')$ and $\omega U \in \Gamma(JD^{\perp})$. Applying J to (3.19) and (3.20) we get

$$\phi^2 = -I - B\omega,\ \omega\phi + C\omega = 0,$$

from which we obtain $\phi^3 + \phi = 0$, that is, ϕ is an f-structure of constant rank [15].

Next, we consider a coordinate neighbourhood $\mathscr{U}$ of M and define locally the differential 1-forms

$$\eta_i(U) = g(U, N_i), \quad \forall\ U \in \Gamma(Ker\ f_*)_{|\mathscr{U}},\ N_i \in \Gamma(ltr(Ker\ f_*)) \quad i \in \{1, ..., r\}. \tag{3.21}$$

Then any vector field $U \in \Gamma(Ker\ f_*)$ can be expressed on $\mathscr{U}$ as:

$$U = \phi U + \sum_{i=1}^{r} \eta_i(U)\xi_i, \tag{3.22}$$

that is, $\{\eta_1, ..., \eta_r\}$ define locally the screen distribution $S(Ker\ f_*)$.

Now, we define new geometric objects induced by the screen distribution of f. Let f be either r-lightlike ($r < min(m,n)$) or co-isotropic submersion. Then, using (3.4), (3.5) and (3.22), we write

$$\hat{\nabla}_U \phi V = \hat{\nabla}^*_U \phi V + T^*_U \phi V, \tag{3.23}$$

$$\hat{\nabla}_U \xi = T^*_U \xi + \nabla^{*\perp}_U \xi, \tag{3.24}$$

$\forall\, U,V \in \Gamma(Ker\ f_*)$, $\xi \in \Gamma\Delta$, where $\{\hat{\nabla}^*_U \phi V,\ T^*_U \xi\}$ and $\{T^*_U \phi V,\ \nabla^{*\perp}_U \xi\}$ belong to $\Gamma(S(Ker\ f_*))$ and $\Gamma\Delta$ respectively. Thus, $\hat{\nabla}^*$ and $\nabla^{*\perp}$ are linear connections on $S(Ker\ f_*)$ and Δ respectively. Moreover, $T^*_U \phi V$ and $T^*_U \xi$ are $\Gamma\Delta$ and $\Gamma(S(Ker\ f_*))$-valued bilinear forms on $\Gamma(Kerf_*) \times \Gamma(S(Ker\ f_*))$ and $\Gamma(Kerf_*) \times \Gamma(\Delta)$ respectively. Using (3.6), (3.24) and taking into account that ∇ is a metric connection, we have

$$g(T^l_U \xi, \xi) = 0,\ T^*_\xi \xi = 0,\ \forall\, U \in \Gamma(Ker\ f_*),\ \xi \in \Gamma(\Delta) \tag{3.25}$$

Generally, the induced linear connection $\hat{\nabla}$ on $Ker\ f_*$ is not a metric connection. Moreover, using (3.6) and that ∇ is a metric, we get

$$(\hat{\nabla}_U \hat{g})(V,W) = g(T^l_U V, W) + g(T^l_U W, V), \quad \forall\, U,V,W \in \Gamma(Ker\ f_*). \tag{3.26}$$

Now, we study integrability of distributions involved in the definition of SCR-lightlike submersions.

Theorem 3.3. Let $f : (M,g) \to (B,g')$ be a SCR-lightlike submersion from an indefinie K*ä*hler manifold M onto a lightlike manifold B. Then, $S(Ker\ f_*)$ is integrable if and only if following conditions are satisfied:

$$g(T_V N, JU) = g(T_U N, JV), \quad U,V \in \Gamma(D), \tag{3.27}$$

$$g(T_V N, JU) = g(D^{\perp s}(U,N), JV), \quad U \in \Gamma(D),\ V \in \Gamma(D^\perp), \tag{3.28}$$

$$g(D^{\perp s}(U,N), JV) = g(D^{\perp s}(V,N), JU), \quad U,V \in \Gamma(D^\perp), \tag{3.29}$$

for any $N \in \Gamma(ltr(Ker\ f_*))$.

Proof. Let M be an indefinite K*ä*hler manifold and Ω is the fundamental 2-form of M. Then, we have $d\Omega(U,V,N) = 0$, $\forall\, U,V \in \Gamma(S(Ker\ f_*))$ and $N \in \Gamma(ltr(Ker\ f_*))$. It follows that

$$U(\Omega(V,N)) + V(\Omega(N,U)) + N(\Omega(U,V)) - \Omega([U,V],N) - \Omega([V,N],U) - \Omega([N,U],V) = 0.$$

Using (2.4), above equation gives

$$Ng(U,JV) - g([U,V],JN]) - g([V,N],JU) - g([N,U],JV) = 0.$$

By using (2.2), (2.9), (3.13) and torsion free ∇, we obtain

$$\begin{aligned} g([U,V],JN) &= -g(\nabla_V N, JU) + g(\nabla_U N, JV) \\ &= -g(T_V N, JU) - g(D^{\perp s}(V,N), JU) + g(T_U N, JV) + g(D^{\perp s}(U,N), JV). \end{aligned}$$

Since $S(Ker\ f_*) = D \oplus D^{\perp}$, for any $U,V \in \Gamma(D)$ we get (3.27), for $U \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$ we get (3.28) and for $U,V \in \Gamma(D^{\perp})$ we get (3.29). □

Theorem 3.4. Let $f : (M,g) \to (B,g')$ be a SCR-lightlike submersion from an indefinite K*ä*hler manifold M onto a lightlike manifold B. Then, the distribution $D' = D \perp \Delta$ is integrable if and only if

$$T_U JV = T_V JU, \quad \forall\, U,V \in \Gamma(D')$$

Proof. By using (2.2), (3.6), (3.19) and (3.20) we get

$$T_U JV = \omega \hat{\nabla}_U V + C T_U V,\ \forall\, U,V \in \Gamma(D')$$

Then, since ∇ is torsion free, we get

$$T_U JV - T_V JU = \omega [U,V].$$

This completes the proof. □

Theorem 3.5. Let $f : (M,g) \to (B,g')$ be a SCR-lightlike submersion from an indefinite K*ä*hler manifold M onto a lightlike manifold B. Then the following conditions are equivalent:

(a) $T_Z JW = T_W JZ$, for any $Z,W \in \Gamma(D^{\perp})$,

(b) $\hat{g}(W, T_Z N) = \hat{g}(T_W N, Z)$, for any $Z,W \in \Gamma(D^{\perp})$, $N \in \Gamma(ltr(Ker\ f_*))$,

(c) $D^{\perp}$ is integrable.

Proof. (a) $\Rightarrow$ (b): Assume that $T_Z JW = T_W JZ$, $\forall\, Z,W \in \Gamma(D^{\perp})$. Then, by using (3.14), we have

$$g(\nabla_Z JW, JN) = g(\nabla_W JZ, JN),\ \forall\, N \in \Gamma(ltr(Ker\ f_*)).$$

From (2.1) and (2.2), it follows that $g(\nabla_Z W, N) = g(\nabla_W Z, N)$. Now, since ∇ is a metric connection, we get $g(W, \nabla_Z N) = g(Z, \nabla_W N)$. Hence, by using (3.13), we obtain (b).

(b) $\Rightarrow$ (c): The distribution $D^\perp$ is integrable if and only if $g([Z,W], JN) = g([Z,W], JV) = 0$, $\forall\, Z, W \in \Gamma(D^\perp), V \in \Gamma(D)$ and $N \in \Gamma(ltr(Ker\ f_*))$. Since ∇ is a metric connection, we obtain $g([Z,W], JN) = g(W, \nabla_Z JN) - g(Z, \nabla_W JN)$. Since $ltr(Ker\ f_*)$ is invariant, using (b) and (3.13), we obtain

$$g([Z,W], JN) = \hat{g}(W, T_Z JN) - \hat{g}(Z, T_W JN) = 0.$$

Moreover, from (3.15) we get

$$\hat{g}(T_Z JW, V) = -g(T_Z^s V, JW),\ \forall\, Z, W \in \Gamma(D^\perp),\ V \in \Gamma(D). \tag{3.30}$$

Using (3.6) and (3.14) we obtain

$$\hat{g}(T_Z^s V, JW) = -g(T_V JZ, W). \tag{3.31}$$

Then, from (3.30) and (3.31) we obtain

$$\hat{g}(T_Z JW, V) = \hat{g}(T_V JZ, W).$$

Now, using (2.14) and lemma (2.1), above equation gives

$$\hat{g}(T_Z JW, V) = \hat{g}(T_W JZ, V),\ \forall\, Z, W \in \Gamma(D^\perp), V \in \Gamma(D).$$

Since ∇ is a metric connection, using (2.35), above equation implies

$$\hat{g}([Z,W], JV) = \hat{g}(T_W JZ, V) - \hat{g}(T_Z JW, V) = 0.$$

Hence, $D^\perp$ is integrable.

(c) $\Rightarrow$ (a): Using (2.2), (3.6), (3.7), (3.14), (3.19) and (3.20) we obtain

$$T_Z JW = \phi \hat{\nabla}_Z W + B T_Z W,\ \forall\, Z, W \in \Gamma(D^\perp).$$

Then, using (2.14), we obtain

$$T_Z JW - T_W JZ = \phi [Z, W].$$

Then using the integrability of $D^\perp$, we get (a). □

Theorem 3.6. Let $f:(M,g)\to(B,g')$ be a SCR-lightlike submersion from an indefinite K$\ddot{a}$hler manifold M onto a lightlike manifold B. Then the radical distribution Δ is integrable if and only if the following conditions hold:

$$g(T^l_\xi W,\xi')=g(T^l_{\xi'}W,\xi),$$

$$g(T^s_\xi JZ,J\xi')=g(T^s_{\xi'}JZ,J\xi),$$

where $\xi,\xi'\in\Gamma(\Delta), Z\in\Gamma(D^\perp)$ and $W\in\Gamma(D)$.

Proof. From the definition of SCR-lightlike submersion, Δ is integrable if and only if $\hat{g}([\xi,\xi'],Z)=\hat{g}([\xi,\xi'],W)=0$, $\forall\ \xi,\xi'\in\Gamma(\Delta),W\in\Gamma(D)$, and $Z\in\Gamma(D^\perp)$. Using (2.2), (3.6) and Lemma 2.1 (a), we have

$$\hat{g}([\xi,\xi'],W)=g(T^l_\xi W,\xi')-g(T^l_{\xi'}W,\xi), \tag{3.32}$$

$$\hat{g}([\xi,\xi'],Z)=g(T^s_\xi JZ,J\xi')-g(T^s_{\xi'}JZ,J\xi). \tag{3.33}$$

From (3.32) and (3.33), the proof follows. ☐

Theorem 3.7. Let $f:(M,g)\to(B,g')$ be a SCR-lightlike submersion from an indefinite K$\ddot{a}$hler manifold M onto a lightlike manifold B. Then, the following conditions are equivalent

(a) T_UJV has no components in D', for any $U,V\in\Gamma(D^\perp)$,

(b) T^s_UJZ and $D^{\perp s}(U,JN)$ have no components in $JD^\perp$, for any $U\in\Gamma(D^\perp)$, $Z\in\Gamma(D)$ and $N\in\Gamma(ltr(Ker\ f_*))$,

(c) $D^\perp$ defines totally geodesic foliation on $Ker\ f_*$.

Proof. (a) $\Rightarrow$ (b): Using (a) and (3.15), we get

$$0=\hat{g}(T_UJV,JZ)=-g(T^s_UJZ,JV)-g(JZ,D^{\perp l}(U,JV))=-g(T^s_UJZ,JV),$$

for any $U,V\in\Gamma(D^\perp)$, $Z\in\Gamma(D)$. Thus, T^s_UJZ has no components in $JD^\perp$. Also, from (3.16) we have

$$0=g(T_UJV,JN)=-g(D^{\perp s}(U,JN),JV),\quad U,V\in\Gamma(D^\perp),\ N\in\Gamma(ltr(Ker\ f_*)),$$

which implies $D^{\perp s}(U,JN)$ has no components in $JD^\perp$.

(b) $\Rightarrow$ (c): The distribution $D^{\perp}$ is parallel if and only if $\hat{g}(\hat{\nabla}_U V, Z) = g(\hat{\nabla}_U V, N) = 0$, for any $U, V \in \Gamma(D^{\perp}), Z \in \Gamma(D)$ and $N \in \Gamma(ltr(Ker\ f_*))$. From (2.1), (2.2), (3.6) and the fact that ∇ is a metric connection, we obtain $\hat{g}(\hat{\nabla}_U V, Z) = -g(T^s_U JZ, JV) = 0$. Moreover, from (3.13) we have $g(\hat{\nabla}_U V, N) = -g(D^{\perp s}(U, JN), JV) = 0$, which implies (c).

(c) $\Rightarrow$ (a): Using (c) it follows that $D^{\perp}$ is parallel and using (2.1), (2.2), (3.6) and (3.14) we obtain

$$0 = \hat{g}(\hat{\nabla}_U V, Z) = g(T_U JV, JZ),\ U, V \in \Gamma(D^{\perp}),\ Z \in \Gamma(D).$$

In the same way

$$0 = g(\hat{\nabla}_U V, N) = g(T_U JV, JN),\ U, V \in \Gamma(D^{\perp}),\ N \in \Gamma(ltr(Ker\ f_*)).$$

Hence, $T_U JV$ has no components in D'. □

4. Proper SCR-Lightlike Submersions

In this section, we study proper SCR-lightlike submersions with totally umbilical fibres from an indefinite Kähler manifold M onto a lightlike manifold B.

Definition. Let $f : (M, g) \to (B, g')$ be a SCR lightlike submersion from an indefinite K*ähler* manifold M onto a lightlike manifold B. Then the fibres are totally umbilical if

$$T_U V = g(U, V)\mathscr{H} \tag{4.1}$$

for any $U, V \in \Gamma(Ker\ f_*)$. Here $\mathscr{H}$ is the mean curvature vector field belonging to $\Gamma(tr(Ker\ f_*))$.

From (4.6), we can see that fibres are totally umbilical, if and only if on every coordinate neighborhood $\mathscr{U}$ there exist smooth curvature vector fields $\mathscr{H}^l \in \Gamma(ltr(Ker\ f_*))$ and $\mathscr{H}^s \in \Gamma(S(Ker\ f_*)^{\perp})$ and smooth functions $\mathscr{H}^l_i \in \mathscr{F}(ltr(Ker\ f_*))$ and $\mathscr{H}^s_j \in \Gamma(\mathscr{F}(S(Ker\ f_*)^{\perp}))$, where $\mathscr{F}(Ker\ f_*)^{\perp}$ denotes the algebra of smooth functions on $(Ker\ f_*)^{\perp}$, such that

$$T^l_U V = \mathscr{H}^l g(U,V), \tag{4.2}$$

$$T^s_U V = \mathscr{H}^s g(U,V), \tag{4.3}$$

$$(T^l_U)_i V = \mathscr{H}^l_i g(U,V), \tag{4.4}$$

$$(T^s_U)_j V = \mathscr{H}^s_j g(U,V), \tag{4.5}$$

for any $U,V \in \Gamma(Ker\ f_*)$. It is important to note that above definition does not depend on $S(Ker\ f_*)$ and $S(Ker\ f_*)^{\perp}$.

Theorem 4.1. There exists a Levi-Civita connection on the fibres of a proper SCR-lightlike submersion $f:(M,g)\to(B,g')$ with totally umbilical fibres from an indefinite K*ä*hler manifold M onto a lightlike manifold B.

Proof. Using (2.2) (3.6), (3.14), (3.19) and (3.20), we have

$$T^l_U JV = JT^l_U V, \quad U,V \in \Gamma(Ker\ f_*).$$

Moreover, as fibres are totally umbilical we obtain $\mathscr{H}^l g(U,JV) = J\mathscr{H}^l g(U,V)$. Interchanging U and V we have $\mathscr{H}^l g(V,JU) = J\mathscr{H}^l g(V,U)$. Since g is symmetric, using (2.1) we have $\mathscr{H}^l g(U,JV) = 0$. If $U,V \in \Gamma(D)$ we get $\mathscr{H}^l = 0$. Therefore from (4.2), it follows that T^l vanishes identically. Thus, the proof follows from (3.26). □

Theorem 4.2. Let $f:(M,g)\to(B,g')$ be a SCR-lightlike submersion with totally umbilical fibres and integrable screen distribution $S(Ker\ f_*)$ whose leaf is $\tilde{M}$, where M is an indefinite K*ä*hler manifold and B is a lightlike manifold. If $\tilde{M}$ is a totally geodesic submanifold of M then $Ker\ f_*$ is also a totally geodesic submanifold of M.

Proof. Using (2.42), we have

$$\hat{g}(U,V) = \hat{g}\Big(\phi U + \sum_{i=1}^{r}\eta_i(U)\xi_i,\ \phi V + \sum_{j=1}^{r}\eta_j(V)\xi_j\Big) = \hat{g}(\phi U,\phi V) = \tilde{g}(U,V),$$

for any $U,V \in \Gamma(Ker\ f_*)$, where $\tilde{g}$ is the semi-Riemannian metric of $\tilde{M}$. Since fibres are totally umbilical, using (4.2) and (4.3), we have $T^l_U\xi = 0$ and $T^s_U\xi = 0$ for any $U \in \Gamma(Ker\ f_*), \xi \in \Gamma(\Delta)$.

Moreover, from (3.6), we obtain

$$\nabla_U V = \tilde{\nabla}_U V + \tilde{T}_U V, \ \forall\, U,V \in \Gamma(S(Ker\ f_*)), \tag{4.6}$$

where $\tilde{\nabla}$ and $\tilde{T}$ are the metric connection and second fundamental form of $\tilde{M}$, respectively. Hence, from (3.6), (3.23) and (4.6), we obtain

$$\tilde{T}_U V = T^*_U V + T^s_U V, \ \forall\, U,V \in \Gamma(S(Ker\ f_*)),$$

which completes the proof. □

Theorem 4.3. Let $f : (M,g) \to (B,g')$ be a proper SCR-lightlike submersion from an indefinite K*ä*hler manifold M onto a lightlike manifold B. Then, the fibres are totally umbilical only if $D^{\perp}$ is one-dimensional.

Proof. From (2.2), (3.6), (3.19) and (3.20), we obtain

$$T^s_U JU = \omega \hat{\nabla}_U U + CT^s_U U, \quad U \in \Gamma(D).$$

Now, since fibres are totally umbilical using (4.3), we have

$$\omega \hat{\nabla}_U U = 0, \quad 0 = CT^s_U U = C\mathscr{H}^s g(U,U) = g(U,U)C\mathscr{H}^s.$$

Thus, we have

$$\nabla_U U \in \Gamma(D'), \ \mathscr{H}^s \in \Gamma(JD^{\perp}). \tag{4.7}$$

On the other hand, using (2.2), (3.6), (3.14), (3.19) and (3.20), we have

$$T_Z JW = \phi \hat{\nabla}_Z W + BT^s_Z W,$$

for $Z,W \in \Gamma(D^{\perp})$. Using above equation we get

$$\hat{g}(T_Z JW, Z) = -\hat{g}(T^s_Z W, JZ). \tag{4.8}$$

From (3.15), (4.3) and (4.8), we get

$$\hat{g}(Z,Z) g(\mathscr{H}^s, JW) = \hat{g}(Z,W)\hat{g}(\mathscr{H}^s, JZ). \tag{4.9}$$

Interchanging Z and W in (4.9), we have

$$\hat{g}(W,W) g(\mathscr{H}^s, JZ) = \hat{g}(W,Z)\hat{g}(\mathscr{H}^s, JW). \tag{4.10}$$

Using (4.9) and (4.10), we obtain

$$g(\mathscr{H}^s, JZ) = \frac{\hat{g}(Z,W)^2}{\hat{g}(Z,Z)\hat{g}(W,W)} g(\mathscr{H}^s, JZ). \tag{4.11}$$

If Z and W are non-null vector fields on $D^{\perp}$, then from (4.7) and (4.11) we conclude that either $\mathscr{H}^s = 0$ or Z and W are linearly dependent. Hence $dim(D^{\perp}) = 1$. □

5. Complex Lightlike Submersions

From proposition (3.1), we see that a SCR-lightlike submersion from an indefinite Kähler manifold (M,g) onto a lightlike manifold (B,g'), is a complex (invariant) lightlike submersion if and only if $D^{\perp} = \{0\}$, that is, $S(Ker\ f_*)$ is invariant under J. In fact, isotropic and totally lightlike submersions are trivial cases of complex lightlike submersions. So we assume that a complex lightlike submersion is a co-isotropic submersion.

Lemma 5.1. Let $f : (M,g) \to (B,g')$ be a complex lightlike submersion from an indefinite Kähler manifold M onto a lightlike manifold B .Then

$$T^l_U JV = JT^l_U V = T^l_{JU} V,\ \forall\ U,V \in \Gamma(Ker\ f_*). \tag{5.1}$$

Proof. Using (3.17), we have

$$\nabla_U JV = \hat{\nabla}_U JV + T^l_U JV$$

$$J\nabla_U V = J\hat{\nabla}_U V + JT^l_U V.$$

Using (2.2), (2.14) and above equations, we get (5.1). □

Let $f : (M,g,J) \to (B,g')$ be a SCR-lightlike submersion from a (2m + 2)-dimensional indefinite Kähler manifold M onto a lightlike manifold and M. Further, suppose that $Ker\ f_*$ is a 2m-dimensional indefinite complex submanifold of M, that is, $Ker\ f_*$ is a complex m-dimensional hypersurface of M, with 1 dimensional null holomorphic subbundle $\mathscr{H}$ of the complexified tangent bundle $(Ker\ f_*)^c$, such that

$$\Delta = Re(\mathscr{H} + \bar{\mathscr{H}}) = (Ker\ f_*)^{\perp} = Span\{\xi, J\xi\}.$$

Theorem 5.1. Let $f:(M,g)\to(B,g')$ be a complex lightlike submersion from an indefinite Kähler manifold M onto a lightlike manifold B and $Ker\ f_*$ be a complex hypersurface of M. Then the radical distribution Δ defines a totally geodesic foliation on $Ker\ f_*$.

Proof. Since Δ is spanned by ξ and $J\xi$, we write

$$U=p_1\xi+q_1J\xi,\ V=p_2\xi+q_2J\xi,$$

for any $U,V\in\Gamma(\Delta)$. Using the linearity of connection $\hat{\nabla}$, we get

$$\begin{aligned}(5.2)\quad \hat{g}(\hat{\nabla}_UV,\phi W)&=p_1p_2\hat{g}(\hat{\nabla}_\xi\xi,\phi W)+p_1q_2\hat{g}(\hat{\nabla}_\xi J\xi,\phi W)\\&\quad+q_1p_2\hat{g}(\hat{\nabla}_{J\xi}\xi,\phi W)+q_1q_2\hat{g}(\hat{\nabla}_{J\xi}J\xi,\phi W).\end{aligned}$$

Using (3.25), we obtain

$$\hat{g}(\hat{\nabla}_UV,\phi W)=p_1q_2\hat{g}(\hat{\nabla}_\xi J\xi,\phi W)+q_1p_2\hat{g}(\hat{\nabla}_{J\xi}\xi,\phi W).$$

Therefore, by using (3.6) and taking into account that ∇ is a metric connection, we get

$$\begin{aligned}\hat{g}(\hat{\nabla}_UV,\phi W)&=-p_1q_2g(J\xi,\nabla_\xi\phi W)-q_1p_2g(\xi,\nabla_{J\xi}\phi W)\\&=-p_1q_2g(J\xi,T^l_\xi\phi W)-q_1p_2g(\xi,T^l_{J\xi}\phi W).\end{aligned}$$

Then, using (2.1) and (5.1) we obtain

$$\hat{g}(\hat{\nabla}_UV,\phi W)=p_1q_2g(\xi,T^l_\xi J\phi W)-q_1p_2g(\xi,T^l_\xi J\phi W)=(p_1q_2-q_1p_2)g(\xi,T^l_\xi\phi W).$$

Finally, using (2.14) and (3.25) we get

$$\hat{g}(\hat{\nabla}_UV,\phi W)=0,$$

which implies $\hat{\nabla}_UV\in\Delta$, for any $U,V\in\Delta$. Hence Δ defines a totally geodesic foliation on $Ker\ f_*$. □

Let (M,g,J) be an indefinite Kähler manifold, (B,g') a lightlike manifold and $f:M\to B$ be a lightlike submersion with integrable screen distribution $S(Ker\ f_*)$. Let $(\check{M},\check{g})$ be an integral manifold of $S(Ker\ f_*)$, where $\check{g}$ is the induced non-degenerate semi-Riemannian metric on $\check{M}$. We say that f is screen Kähler submersion if $(\check{M},\check{g},\check{J})$ has *Kähler* structure induced by the almost complex operator $\check{J}$ on $\check{M}$.

Theorem 5.2. Let $f:(M,g)\rightarrow(B,g')$ be a complex lightlike submersion from an indefinite K*ä*hler manifold M onto a lightlike manifold B, whose screen distribution $S(Ker\ f_*)$ is integrable. Then f is a screen K*ä*hler submersion.

Proof. Since f is a complex lightlike submersion, therefore $S(Ker\ f_*)$ and Δ are invariant with respect to J. Let $(\check{M},\check{g})$ be an integral submanifold of $S(Kerf_*)$, where $\check{g}$ is the semi-Riemannian metric on $\check{M}$. Then $\check{M}$ is non degenerate manifold. We suppose that $\check{J}$ is the induced almost complex structure on $\check{M}$ and immersion $\check{f}:\check{M}\rightarrow M$ is an almost complex mapping. Then, we have

$$J\check{f}_*=\check{f}_*\check{J}. \tag{5.3}$$

Since $\check{g}$ is a semi Riemannian metric, therefore we have

$$\check{g}(U,V)=\check{f}_*g(U,V)=g(\check{f}_*U,\check{f}_*V),\ \forall\ U,V\in\Gamma(Ker\check{f}_*). \tag{5.4}$$

Using (2.1) and (5.4), we obtain

$$\check{g}(U,V)=g(J\check{f}_*U,J\check{f}_*V). \tag{5.5}$$

Using (5.3) and (5.5), we get

$$\check{g}(U,V)=\check{g}(\check{f}_*\check{J}U,\check{f}_*\check{J}V),$$

Then, from (5.4) we obtain

$$\check{g}(U,V)=\check{g}(\check{J}U,\check{J}V),$$

which implies $\check{g}$ is a Hermitian metric. Next, we assume that Ω and $\check{\Omega}$ be the fundamental 2-form of M and $\check{M}$, respectively. Then, from (5.3) and (5.4) we get

$$\Omega(\check{f}_*U,\check{f}_*V)=\check{\Omega}(U,V).$$

Since Ω is closed and $\hat{\nabla}^*$ is a metric connection on $\check{M}$, it follows that $\check{\Omega}$ is also closed. Hence, $\check{M}$ is a K*ä*hler manifold, which completes the proof. □

6. Screen Real Submersions

In this section, we study screen real lightlike submersions from an indefinite K*ä*hler manifold (M,g) onto a lightlike manifold (B,g'). Our main focus in this section is screen real lightlike submersions with irrotational fibres, that is, $\nabla_U \xi \in \Gamma(Ker\ f_*)$, for any $U \in \Gamma(Ker\ f_*)$ and $\xi \in \Delta$ [8, 11]. Using (3.6), we can see that this condition is equivalent to $T^l_U \xi = T^s_U \xi = 0$.

Theorem 6.1. There exists a Levi-Civita connection on the fibres of a screen real lightlike submersion $f : (M,g) \to (B,g')$ with irrotational fibres from an indefinite K*ä*hler manifold M onto a lightlike manifold B.

Proof. Let f be the screen real lightlike submersion with irrotational fibres. Then, we have

$$T^l_U \xi = T^s_U \xi = 0,$$

for any $U \in \Gamma(Ker\ f_*)$ and $\xi \in \Delta$. Using (3.6), we have

$$g(T^l_U V, \xi) = g(\nabla_U V, \xi),$$

for any $U,V \in \Gamma(S(Ker\ f_*))$ Then, using (2.1) and (2.2) above relation gives

$$g(T^l_U V, \xi) = g(\nabla_U JV, J\xi).$$

Now, since ∇ is a metric connection we get

$$g(T^l_U V, \xi) = -g(\nabla_U J\xi, JV).$$

Using (3.6), above equation implies

$$g(T^l_U V, \xi) = -g(T^s_U J\xi, JV),$$

for any $U,V \in S(Ker\ f_*)$. Since Δ is invariant under J and fibres are irrotational, we get

$$g(T^l_U V, \xi) = 0.$$

Thus $T^l_U V = 0$, for any $U,V \in \Gamma S((Ker\ f_*))$. Hence, the proof is completed by using (3.26). □

Theorem 6.2. Let $f : (M,g) \to (B,g')$ be a screen real lightlike submersion with irrotational fibres from an indefinite K*ä*hler manifold M onto a lightlike manifold B. Then the radical distribution Δ defines a totally geodesic foliation on $Ker\ f_*$.

Proof. The radical distribution Δ defines a totally geodesic foliation on $Ker\ f_*$ if and only if $\hat{g}(\hat{\nabla}_{\xi_1}\xi_2, Z) = 0$ for any $\xi_1, \xi_2 \in \Gamma(\Delta)$ and $Z \in \Gamma(S(Ker\ f_*))$. Using (3.6) and taking account that ∇ is a metric connection, we obtain

$$\hat{g}(\hat{\nabla}_{\xi_1}\xi_2, Z) = g(\nabla_{\xi_1}\xi_2, Z) = -g(\xi_2, \nabla_{\xi_1} Z) = -g(\xi_2, T^l_{\xi_1} Z).$$

Now, since fibres are irrotational using (2.14), we have $T^l_{\xi_1} Z = 0$, which implies

$$\hat{g}(\hat{\nabla}_{\xi_1}\xi_2, Z) = 0.$$

Thus the proof follows. □

Theorem 6.3. Let $f : (M, g) \to (B, g')$ be a screen real lightlike submersion from an indefinite K*ä*hler manifold M onto a lightlike manifold B. Then the following conditions are equivalent

(a) $S(Ker\ f_*)$ is parallel.

(b) $T_U JV$ is $S(Ker\ f_*)$ valued, $\forall\ U, V \in \Gamma(S(Ker\ f_*))$.

(c) $D^{\perp s}(U, N)$ is D_0 valued, $\forall\ U \in \Gamma(Ker\ f_*)$ and $N \in \Gamma(ltr(Ker\ f_*))$.

Proof. Screen distribution $S(Ker\ f_*)$ is parallel if and only if $g(\hat{\nabla}_U V, N) = 0$, for any $U, V \in \Gamma(S(Ker\ f_*))$ and $N \in \Gamma(ltr(Ker\ f_*))$. Using (3.6), we obtain

$$g(\hat{\nabla}_U V, N) = g(\nabla_U V, N).$$

Then, from (2.1) , (2.2)and (3.14), above equation gives

$$g(\hat{\nabla}_U V, N) = g(\nabla_U JV, JN) = g(T_U JV, JN).$$

It follows that (a) $\Leftrightarrow$ (b). Also, using (3.16) we have

$$(D^{\perp s}(U, N), JV) = -g(T_U JV, N).$$

Using this we get (b) $\Leftrightarrow$ (c). □

ACKNOWLEDGEMENT

This research was supported by council of scientific and industrial research (CSIR), New Delhi.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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