Available online at http://scik.org

J. Math. Comput. Sci. 11 (2021), No. 2, 2174-2192

https://doi.org/10.28919/jmcs/5428

ISSN: 1927-5307

APPLICATIONS OF FRAMELETS FOR SOLVING THIRD KIND INTEGRAL-ALGEBRAIC EQUATIONS

YOUSEF AL-JARRAH*

Department of Mathematics, Tafila Technical University, Jordan

Abstract: This article is dedicated to obtain numerical solution for semi-explicit integral-algebraic equation of third kind (IAE). We used tight wavelet frames (framelets) to obtain the numerical solution. The framelets system generated by unitary extension principle and oblique extension principle based on the B-spline refinable function. We provide some numerical examples to show the efficiency of the proposed method. The obtained numerical results of our method compared with those obtained from Spline and discretized colocation methods. We confirm that, our proposed method achieves accurate and efficient results than others.

Keywords: framelets; integral-algebraic; B-spline; unitary extension principle; oblique extension principle.

2010 AMS Subject Classification: 65R10, 65R20; 65T60.

1. INTRODUCTION

Integral-algebraic equations of third kind are widely used in the development of many problems in engineering and applied sciences. For instance, it plays important rule in the theory of the homogenous transport problem [1], neutron transport [2] and scattering of particles [3]. The general form for the linear semi-explicit IAE is given by

$$A(t)X(t)=q(t)+\int_0^t K(t,s)X(s)ds, \quad t\in I=[0,T] \tag{1.1}$$

*Corresponding author

E-mail address: yjarrah@ttu.edu.jo

Received January 13, 2021

where $q(t)=\left(q_1(t),q_2(t)\right)^t$ and A is a singular matrix with

$$A=\begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t)\end{pmatrix}, K(t,s)=\begin{pmatrix} k_{11}(t,s) & k_{12}(t,s) \\ k_{21}(t,s) & k_{22}(t,s)\end{pmatrix} \text{and } X(t)=\left(y(t),z(t)\right)^t .$$

There are several different forms for system(1.1) according to the algebraic form of the matrix $A(t)$. Because of the singularity of the matrix A, $A(t)$ has two eigenvalues 0 and $\lambda(t)$. If $\lambda(t)\neq 0\ \forall t\in I$, then there exists a matrix $P(t)$ such that $P^{-1}(t)A(t)P(t)=\begin{pmatrix}\alpha(t) & \beta(t)\\ 0 & 0\end{pmatrix}$. Let $P^{-1}(t)X(t)=\left(y(t),\tilde{z}(t)\right)^t$ then system(1.1) becomes as

$$\begin{aligned}\alpha(t)y(t)+\beta(t)\tilde{z}(t)&=q_1(t)+\int_0^t\left(k_{11}(t,s)y(s)+k_{12}(t,s)\tilde{z}(s)\right)ds\\ 0&=q_2(t)+\int_0^t\left(k_{21}(t,s)y(s)+k_{22}(t,s)\tilde{z}(s)\right)ds\end{aligned} \tag{1.2}$$

This system is called semi-explicit linear integral-algebraic equation of index-1 when $\beta(t)=0$ and $\alpha(t)\neq 0$. Many methods considered the numerical solution for system(1.2), Kauthen [4] applied the spline collocation method, in [5] the authors used discretized collocation method and in [6] the authors used the Sinc collocation method.

In this article we consider the following form of system(1.1);

$$\begin{aligned}y(t)-\rho(t)z(t)&=f(t)+\int_0^t\left(k_{11}(t,s)y(s)+k_{12}(t,s)z(s)\right)ds\\ 0&=g(t)+\int_0^t\left(k_{21}(t,s)y(s)+k_{22}(t,s)z(s)\right)ds\end{aligned} \tag{1.3}$$

Where the data functions ρ, f, g and $k_{i,j},\ \ i,j=1,2$ are sufficiently smooth. System (1.3)is called third kind semi-explicit linear integral-algebraic equation, it has been investigated by [7] and [8], where the authors applied collocation and discretized collocations methods to approximate the exact solution in the discontinuous spline polynomial space $S_{m-1}^{-1}\left(\prod_N\right)$ of degree $m-1$.

In the literature, wavelet theory provides the major mathematical multiscale representation for analyzing functions has undergone extensive development for more than forty years. Wavelets have been used for solving different kinds of integral equation, see [9, 10, 11, 12]. These wavelet bases are typically nonredundant, and thus transform coefficients will be lose. Tight framelets are useful tool in applications that is because of its desirable features of redundancy and flexibility over orthogonal wavelets. For example, the undecimated wavelet transform using an orthogonal wavelet can be used to effectively remove white Gaussian noise in signals and images; interestingly, its underlying system is in fact a tight framelets with high redundancy. Using framelets for IAE solving gives better results with redundant systems, and it is easiest to use. Different types of integral equations have solved by using framelets, see [13, 14, 15].

The remainder of this article is organized as follows. The next section is allocated to providing some preliminary background of frames, some notations and the B-spline function. In Section 3, we introduce a general discrete framelets transform that is based on the unitary extension principles and oblique extension principle, which allows us to increase vanishing moments of high-pass filters in a filter bank. In section 4, we stablish a matrix formulation of the proposed method. Numerical examples and graphical illustration are presented in section 5.

`

2. PRELIMINARIES

Frame was established in 1952 in work of Duffin and Schaeffer [16]. Daubecheis [17] constructed examples of univariate wavelet frames and constructed the necessary and sufficient conditions for wavelets to generate frame. The general characterization of framelets was given by Ron and Shen [18]. We consider

The normalized Fourier transform for a given function $f \in L^2(\mathbb{R})$ is defined by

$f(\omega) = \int_{\mathbb{R}} f(x) e^{-i\omega x} dx,\ \ \omega \in \mathbb{R}$, and the discrete Fourier transform for a sequence

$\{h_k, k \in \mathbb{Z}, k = 0,...,m\}$ is given by $h(\omega) = \sum_{i=0}^{m-1} h(k) e^{-i\omega k/m}$.

Definition: A countable family of elements $\{ f_k \}_{k=1}^{\infty}$ in $L^2(\mathbb{R})$ is a frame for $L^2(\mathbb{R})$ if there exist positive constants A, B such that

$$A\|f\|^2 \le \sum_{k=1}^{\infty} \left|\langle f, f_k \rangle\right| \le B\|f\|^2$$

The numbers A, B are called frame bounds, the frame is tight if $A = B$. There are different methods for framelets constructions; which can be derived by frame multiresolution analysis (FMRA), unitary extension principles (UEP) and its generalization, oblique extension principle

(OEP). The derivation of framelets system based on a compactly supported refinable function, in this article, we used the B-spline function of order 2 and 4 to construct framelets. The compactly supported function $\varphi(x)$ is refinable if it satisfies the refinable equation

$$\varphi(x) = 2\sum_{k\in\mathbb{Z}} h_0[k]\varphi(2x-k) \tag{1.4}$$

for some finite supported sequence $h_0[k] \in l^2(\mathbb{Z})$. The sequence h_0 is called the refinable mask or the low pass filter for φ. The refinable function (1.4) can be written via its Fourier transform as:

$$\varphi(\omega) = h_0\left(\frac{\omega}{2}\right)\varphi\left(\frac{\omega}{2}\right) \tag{1.5}$$

The B-spline function of degree $n \geq 1$, is defined to a continuous piece-wise polynomial function of degree m of a real variable with continuous derivatives up to order $m-1$. The normalized B-spline function of order m is defined by

$$B_m(x) = B_{m-1}(x) * B_1(x) = \int_{\left[\frac{-1}{2},\frac{1}{2}\right]} B_{m-1}(x-t)dt,\ x \in \mathbb{R}$$

where $*$ is the convolution operator and $B_1(x) = \begin{cases} 1 & \frac{-1}{2} \leq x \leq \frac{1}{2} \\ 0 & x \geq \frac{1}{2} \,\&\, x \leq \frac{-1}{2} \end{cases} = \chi_{\left[\frac{-1}{2},\frac{1}{2}\right]}(x)$. The B-spline function $B_m(x)$ is a compactly supported, such that, $\sup(B_m) = \left[\frac{-m}{2}, \frac{m}{2}\right]$.The Fourier transform for the B-spline of order m is given by the following equation:

$$B_m(\omega) = \exp\left(\frac{-ij\omega}{2}\right)\left(\operatorname{sinc}\left(\frac{\omega}{2}\right)\right)^m \tag{1.6}$$

such that, $\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ and $j=1$ when m is odd and zero otherwise, and its refinable mask is defined to be;

$$h_0(\omega) = \exp\left(\frac{-ij\omega}{2}\right)\left(\cos\left(\frac{\omega}{2}\right)\right)^m \tag{1.7}$$

The B-spline function of order 1 to 4 is shown in the figure 1.

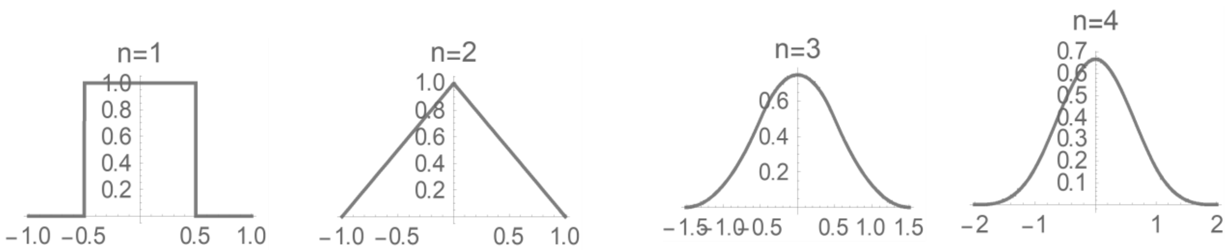


FIGURE 1: The graph of the B-spline function of different orders

3. Framelets via Extension Principles

In this section, we briefly introduce the UEP and OEP, which lead to explicit constructions of tight wavelet frames based on the multiresolution analysis generated by a refinable function φ. For a given function $f(x)$, we consider the operators $D^j f(.) = 2^{j/2} f(2^j.)$ and $T_k f(.) = f(.-k)$, which are called the dilation and translation operators respectively. Consider the set of functions $\Psi := \{\psi_1, \psi_2, ..., \psi_n\} \subset L^2(\mathbb{R})$, the framelets system for $L^2(\mathbb{R})$ is defined as

$$X(\Psi) = \{D^j T_k \psi_l,\ \ 1 \le i \le n,\ j,k \in \mathbb{Z}\} \tag{1.8}$$

where

$$\psi_l(x) = 2\sum_{k\in\mathbb{Z}} h_l[k]\varphi(2x-k),\ l = 1,...,n \tag{1.9}$$

$h_l, l = 1,...,n$ are called the high pass filter. According to equation(1.9), to construct the framelets system we need only to find the high pass filters h_l. So, the next theorem gives the sufficient conditions for the high pass filters in which the system $\{D^j T_k \psi_l\}_{l=1,...,n}$ is framelets.

Theorem 2: [19]*Let* φ *be a compactly supported refinable function with the refinable mask* h_0, *and let* $\{h_l, l = 1,...,r\}$ *be a set of finitely supported sequence, and* $\psi_l(.) = 2\sum_{k\in\mathbb{Z}} h_l[k]\varphi(2.-k)$. *Then the system*

$$X(\psi) = \{D^j T^k \psi_l,\ 1 \le l \le r,\ j,k \in \mathbb{Z}\} \tag{1.10}$$

forms a framelets for $L_2(\mathbb{R})$ *provided that*

$$\sum_{l=0}^{r}\left|h_l(\omega)\right|^2=1 \text{ and } \sum_{l=0}^{r}\left|h_l(\omega)\overline{h_l(\omega+\pi)}\right|^2=0 \tag{1.11}$$

Firstly, to use the UEP, the low pass filter must satisfies the following condition

$$\left|h_0(\omega)\right|^2+\left|h_0(\omega+\pi)\right|^2\leq 1 \tag{1.12}$$

Condition (1.11) means the two vectors $\left(h_0(\omega),h_1(\omega),...,h_r(\omega)\right)$,

$\left(h_0(\omega+\pi),h_1(\omega+\pi),...,h_r(\omega+\pi)\right)$ are orthogonal, and the condition can be written in terms of

$$\sum_{l=0}^{r}\sum_{k\in\mathbb{Z}}\overline{h_l[k]}h_l[k-p]=\delta_{p,0},\quad p\in\mathbb{Z}, \tag{1.13}$$

Equation(1.13) is equivalent to the first condition, and the second condition is equivalent to

$$\sum_{l=0}^{r}\sum_{k\in\mathbb{Z}}(-1)^{k-p}\,\overline{h_l[k]}h_l[k-p]=0,\quad p\in\mathbb{Z}, \tag{1.14}$$

In fact, a function approximation could be obtained by a framelets, and to have a good approximation for a function $f\in V_j\subseteq L_2(\mathbb{R})$ we need an approximation schemes, and we consider quasi-interpolation scheme which is defined as follows:

$$P_n: f\to\sum_{k\in\mathbb{Z}}\left\langle f,\varphi_{n,k}\right\rangle\varphi_{n,k} \tag{15}$$

Now, for applying the quasi-interpolatory operator on the framelets, we consider the following two lemmas, the proof exist in [19].

Lemma 1: *Let* $\varphi(x)\in L_2(\mathbb{R})$ *be a refinable function with a refinable mask* h_0 *, and the sequence* $\{h_0,h_1,...,h_r\}$ *satisfies the condition*(1.14)*. Then*

$$P_n y=P_{n-1}y+\sum_{l=1}^{r}\sum_{k\in\mathbb{Z}}\left\langle y,\psi_{l,n-1,k}\right\rangle\psi_{l,n-1,k}\ .$$

Lemma 2: *Let* $\varphi(x)\in L_2(\mathbb{R})$ *be a refinable function, and the operator* P_n *is defined in(15). Then for any function* $y\in L_2(\mathbb{R})$

$$\lim_{n\to\infty}P_n y=y.$$

By using the previous two lemmas, we have the function representation by the tight wavelet frame functions in space $X(\psi)$ as follows:

$$P_n y = \sum_{l}^{r} \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle y, \psi_{l,n,k} \rangle \psi_{l,n,k} \tag{16}$$

Therefore, we will use the UEP applied to the B-spline functions in different orders to construct a symmetric framelets system. Considering the B-spline of order m , the wavelet masks is defined to be

$$h_l(\xi) = (-i)^l e^{-ij\frac{\xi}{2}} \sqrt{\binom{r}{l}} \sin^l\left(\frac{\xi}{2}\right) \cos^{r-l}\left(\frac{\xi}{2}\right), \quad l = 1, 2, \ldots, r \tag{17}$$

Then, the Fourier transform of the r wavelet functions are;

$$\psi_l := -i^l e^{-ij\frac{\xi}{2}} \sqrt{\binom{r}{l}} \frac{\sin^l\left(\frac{\xi}{4}\right) \cos^{r+l}\left(\frac{\xi}{4}\right)}{\left(\frac{\xi}{4}\right)^r} \tag{18}$$

Note that each framelets function ψ_l is a real valued symmetric supported in $\left[-(m+j)/2.(m+j)/2\right]$.

Examples 1: *Let* $\varphi_0(.)$ *be the B-spline function of order 2. Then* $h_0 = \frac{1}{4}\left(1+e^{-i\omega}\right)^2$, *the Fourier transform of the high pass filter* $h_1 = \frac{-1}{4}\left(1-e^{-i\omega}\right)^2$ *and* $h_2 = \frac{-\sqrt{2}}{4}\left(1-e^{-2i\omega}\right)$. *The corresponding system* $X(\psi) = \{\psi_1, \psi_2\}$ *is a framelets, where* $\psi_1 = \frac{-1}{2}\left(|x-2| - 2|2x-3| + 6|x-1| - 2|2x-1| + |x|\right)$ and $\psi_2 = \frac{1}{2}\left(|x-2| - |2x-3| + |2x-1| - |x|\right)$

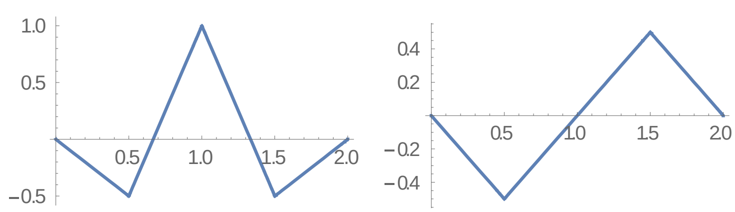


FIGURE 2. The framelets functions generated by UEP applied to B-spline function of order 2.

Example 2: *We consider the B-spline function of order 4. Then* $h_1 = \frac{1}{4}\left(1-e^{-i\omega}\right)^4$,

$h_2 = \frac{-1}{4}\left(1-e^{-i\omega}\right)^3\left(1+e^{-i\omega}\right)$ $h_3 = \frac{-\sqrt{6}}{16}\left(1-e^{-i\omega}\right)^2\left(1+e^{-i\omega}\right)^2$ *and* $h_4 = \frac{-1}{4}\left(1-e^{-i\omega}\right)\left(1+e^{-i\omega}\right)^3$ *. So the system* $X\left(\psi\right)=\left\{\psi_1,\psi_2,\psi_3,\psi_4\right\}$ *is a framelets*, where

$$\psi_1 = \frac{1}{3}\left(|x-4|^3 - |2x-1|^3 + 28|x-3|^3 - 7|2x-5|^3 + 70|x-2|^3 - 7|2x-3|^3 + 28|x-1|^3 - |2x-1|^3 + |x|^3\right)$$

$$\psi_2 = \frac{-1}{3}\left(|x-4|^3 - 6|x-3.5|^3 + 14|x-3|^3 - 14|x-2.5|^3 + 14|x-1.5|^3 - 14|x-1|^3 + 6|x-0.5|^3 - |x|^3\right)$$

$$\psi_3 = \frac{-1}{2\sqrt{6}}\left(|x-4|^3 - 4|x-3.5|^3 + 4|x-3|^3 + 4|x-2.5|^3 - 10|x-2|^3 + 4|x-1.5|^3 + 4|x-1|^3 - 4|x-0.5|^3 + |x|^3\right)$$

and

$$\psi_4 = \frac{1}{3}\left(|x-4|^3 - 2|x-3.5|^3 - 2|x-3|^3 + 6|x-2.5|^3 - 6|x-1.5|^3 + 2|x-1|^3 + 2|x-0.5|^3 - |x|^3\right)$$

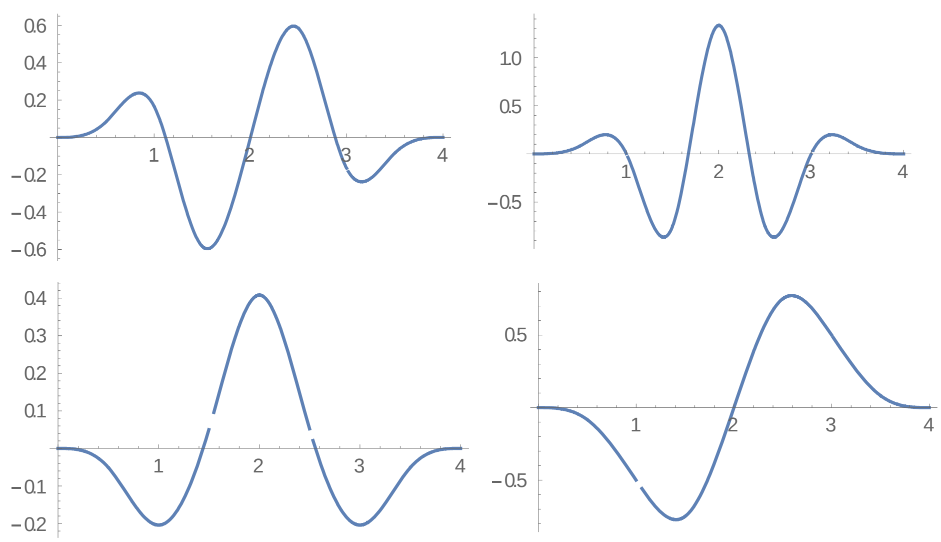


FIGURE 3: The framelets functions generated by UEP applied to B-spline function of order 4.

The UEP is a sequence of more general theorem on MRA based tight wavelet frames. In fact, there are several generalizations of UEP [18], the first generalization of the UEP is the OEP. The OEP is used to obtain framelets whose truncated wavelet system has high approximation order and whose generators have high order vanishing moments. The compactly supported wavelet function ψ is said to be have a vanishing moment property of order m if

$$\int x^m \psi(x)\,dx = 0, \quad 0 \le k \le m-1$$

Theorem 2 [20]: *Suppose that the refinable function φ and the masks $h_0, h_1, \dots, h_n$ satisfying the general setup, and there exists a $2\pi-$periodic function Θ that is nonnegative, essentially bounded, continuous at the origin with $\Theta(0)=1$. Assume that for $\zeta \in \sigma(V_0)$ and $\zeta+\pi \in \sigma(V_0)$, the following equalities*

$$\begin{aligned} &\left|h_0(\zeta)\right|^2 \Theta(2\zeta) + \sum_{l=1}^{n} \left|h_l(\zeta)\right|^2 = \Theta(\zeta) \\ &h_0(\zeta)\overline{h_0(\zeta+\pi)}\Theta(2\zeta) + \sum_{l=1}^{n} h_l(\zeta)\overline{h_l(\zeta+\pi)} = 0 \end{aligned} \tag{19}$$

hold. Then the wavelet system $X(\Psi)$ defined by $h_1, \dots, h_n$ is a tight wavelet frame.

The function Θ is called the fundamental function. For applying the OEP for B-spline function of order m to construct framelets of order $2l$, we should choose $\Theta(\omega) = 1 + \sum_{j=1}^{l} c_j \sin^{2j}(\omega/2)$ as a suitable approximation, at the origin, to $1/\left|\varphi\right|^2 = O\left(\left|.\right|^{2l}\right)$. In other words, Θ must approximate the function $1/\left|\varphi\right|^2$ at the origin to order l, If φ is the B-spline function of order m, then $\left|\varphi(\omega)\right|^2 = \left|\operatorname{sinc}(\omega/2)\right|^{2m}$. Thus, we should choose Θ as $2\pi-$periodic function which approximates the function

$\left|1/\operatorname{sinc}(\omega/2)\right|^{2m} = \left|\dfrac{\omega/2}{\sin(\omega/2)}\right|^{2m} = \dfrac{\arcsin(\sin(\omega/2))}{\sin(\omega/2)} = 1 + \sum_{j=0}^{\infty} \dfrac{(2j-1)!!}{(2j)!!(2j+1)} \sin^{2j}(\omega/2)$, and $\psi_l(\omega) = h_l(\omega/2)\varphi_l(\omega/2)$, where the number of vanishing moments for ψ_l is equal the order of zero at $\omega = 0$ for h_l.

Example 3: Take $h_0(\omega) = \left(1+e^{-i\omega}\right)^2/4$ and $\Theta(\omega) = (4-\cos\omega)/3$, where φ is the linear B-spline function. Then $\{\psi_1, \psi_2\}$ generate a framelets with vanishing moments of order 2, the corresponding high pass filters are;

$$h_1(\omega)=\frac{-1}{4}\left(1-e^{-i\omega}\right)^2, \text{ and } h_2(\omega)=\frac{-\sqrt{6}}{24}\left(1-e^{-i\omega}\right)^2\left(e^{-i\omega}+4e^{-i2\omega}+e^{-i3\omega}\right).$$

In particular,

$$\psi_1(\omega)=\frac{-1}{\omega^2}\left(1-e^{-i\omega/2}\right)^4 \text{ and } \psi_2(\omega)=\frac{-1}{\sqrt{6}\omega^2}\left(1-e^{-i\omega/2}\right)^4\left(e^{-i\omega/2}+4e^{-i\omega}+e^{-3i\omega/2}\right).$$

Then

$$\psi_1=\frac{-1}{2}\left(|x-2|-2|2x-3|+6|x-1|-2|2x-1|+|x|\right)$$
$$\psi_2=\frac{1}{2\sqrt{6}}\left(-|x-3|+9|x-2|-8|2x-3|+9|x-1|-|x|\right)$$

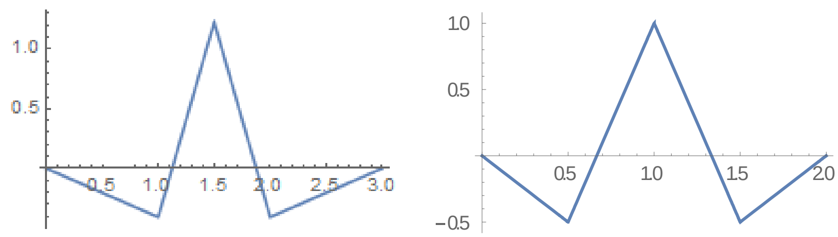


FIGURE 4: The graph of the symmetric wavelet functions ψ_1 and ψ_2 derived in example 3

Example 4: Let the refinable function φ be the B-spline of order 3. Take low pass filter to be $h_0(\omega)=\left(1+e^{-i\omega}\right)^4/16$, and $\Theta(\omega)=\frac{2452}{945}-\frac{1657}{840}\cos(\omega)+\frac{44}{105}\cos(2\omega)-\frac{311}{7560}\cos(3\omega)$. Then the high pass filter are:

$$h_1(\omega)=s_1\left(1-e^{-i\omega}\right)\left(1+8e^{-i\omega}+e^{-i2\omega}\right)$$

$$h_2(\omega)=s_2\left(1-e^{-i\omega}\right)^4\left(1+8e^{-i\omega}\left(\frac{7775}{4396}s-\frac{53854}{1099}\right)e^{-i2\omega}+8e^{-i3\omega}+e^{-i\omega}\right)$$

$$h_3(\omega)=s_3\left(1-e^{-i\omega}\right)\left(1+8e^{-i\omega}+\left(21+\frac{t}{8}\right)\left(e^{-i2\omega}+e^{-i4\omega}\right)+te^{-i3\omega}+8e^{-i5\omega}+e^{-i6\omega}\right)$$

where $s=\frac{317784}{7775}+\frac{56\sqrt{16323699891}}{2418025}$

$s_1=\frac{\sqrt{1111374578360-245493856965s}}{62697600}$, $s_2\ \frac{\sqrt{1543080-32655s}}{40320}$ and $s_3=\frac{\sqrt{32655}}{20160}$.

So,

$$\psi_1 = \frac{1}{3600}\sqrt{\frac{2460313}{1555}}(|x-5|^3 - 35|x-4|^3 + 20|2x-7|^3 - 350|x-3|^3 + 56|2x-5|^3 - 350\ |x-2|^3 + 20|2x-3|^3 - 35|x-1|^3 + |x|^3)$$

$$\psi_2 = \frac{-1}{143689213440}\sqrt{\frac{443}{15}}(199568352|x-6|^3 - 16955218889|x-5|^3 + 14161261961|2x-9|^3 - 361590873308|x-4|^3 + 86356459199|2x-7|^3 - 851590490870|x-3|^3 + 86356459199|2x-5|^3\ 361590873308|x-2|^3 + 14161261961|2x-3|^3 - 16955218889|x-1|^3 + 199568352|x|^3)$$

$$\psi_3 = \frac{1}{3600\sqrt{32655}}(7775|x-7|^3 - 76902|x-6|^3 + 405720|x-5|^3 - 14140|2x-9|^3 - 1657425|x-4|^3 + 358488|2x-7|^3 - 1657425|x-3|^3 - 14140|2x-5|^3 + 405720|x-2|^3 - 76902|x-1|^3 + 7775|x|^3)$$

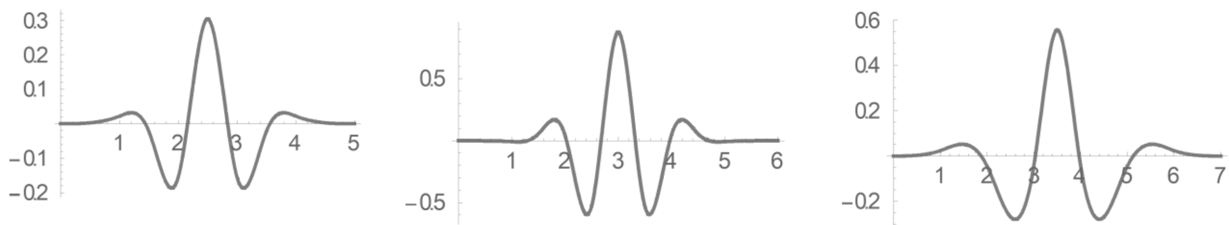


FIGURE 5: The graph of the symmetric wavelet functions ψ_1, ψ_2 and ψ_3 which derived in example 4.

In the next example, we will have a Framelets system with 2 vanishing moments, where we choose the function Θ in which the order of the zero of $1-\Theta|\varphi_0|^2$ at the origin is 2.

Example 5: Let the refinable function φ be the quadratic B-spline. Take low pass filter to be $h_0(\omega) = \left(1+e^{-i\omega}\right)^3/8$, and $\Theta(\omega) = \frac{3-\cos\omega}{2}$. Then the high pass filter are:

$$h_1(\omega) = \frac{-\sqrt{2}}{24}\left(1-e^{-i\omega}\right)^3, h_2(\omega) = \frac{-1}{24}\left(1-e^{-i\omega}\right)^3\left(1+6e^{-i\omega}+e^{-i2\omega}\right)$$

$$h_3(\omega) = \frac{-\sqrt{13}}{48}\left(1-e^{-i\omega}\right)^2\left(1+5e^{-i\omega}+5e^{-i2\omega}+e^{-i3\omega}\right).$$

The $X(\Psi) = X(\psi_1, \psi_2, \psi_3)$ is defined as follows;

$$\psi_1 = h_1(\omega/2)\varphi_0(\omega/2) = \frac{-\sqrt{2}}{24}\left(1-e^{-i\omega/2}\right)^3\left(\frac{1-e^{-i\omega/2}}{\omega/2}\right)^4,$$

$$\psi_1 = \frac{-1}{9\sqrt{2}}\left(-|2t-7|^3+7|t-3|^3-21|2t-5|^3+35|t-2|^3-35|2t-3|^3+21|t-1|^3-7|2t-1|^3+|t|^3\right)$$

$$\psi_2 = h_2(\omega/2)\varphi_0(\omega/2) = \frac{-1}{24}\left(1-e^{-i\omega/2}\right)^3\left(1+6e^{-i\omega/2}+e^{-i\omega}\right)\left(\frac{1-e^{-i\omega/2}}{\omega/2}\right)^4,$$

$$\psi_2 = \frac{-1}{8}(-|-9t+2|^3+|t-4|^3+20|2t-7|^3-84|t-3|^3+154|2t-5|^3-154|t-2|^3$$
$$+84|2t-3|^3-20|t-1|^3-|2t-1|^3+|t|^3)$$

$$\psi_3 = h_3(\omega/2)\varphi_0(\omega/2) = \frac{-\sqrt{13}}{48}\left(1-e^{-i\omega/2}\right)^2\left(1+5e^{-i\omega/2}+5e^{-i\omega}+e^{-i3\omega/2}\right)\left(\frac{1-e^{-i\omega/2}}{\omega/2}\right)^4,$$

$$\psi_3 = \frac{-2\sqrt{13}}{57}(|2t-9|^3-|t-4|^3-10|2t-7|^3+26|t-3|^3\text{-}2|2t-5|^3$$
$$-16|t-2|^3+26|2t-3|^3-10|t-1|^3-|2t-1|^3+|t|^3)$$

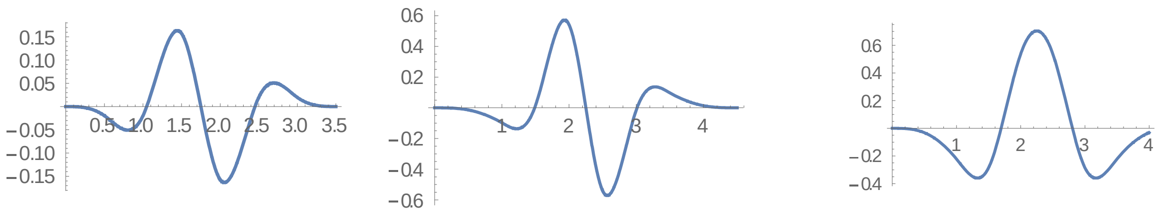


FIGURE 6: The graph of the symmetric wavelet functions ψ_1, ψ_2 and ψ_3 derived from example 5

For using framelets in function approximation, signal and image processing, it is better to use a framelets system that is shift-invariant. The framelets system X is called a τ - shift-invariant, if $\varphi \in X$, then we have $\varphi(.-\tau k) \in X$ for any $k \in \mathbb{Z}$. Whereas, the framelets $X(\psi)$ generated by UEP and OEP are not shift-invariant, and to convert the framelets to a shift invariant system we need to over-sample the affine system $X(\psi)$ below level 0. This over-sampled is called the quasi-affine system [19], which is defined as follows;

Definition 5: *Let* $\Psi = \{\psi_i, i = 1,\ldots,r\}$ *be a set of functions. A quasi-affine system from level L s defined to be*

$$X^L(\psi) = \{\psi_{l,n,k} : 1 \le l \le r; n,k \in \mathbb{Z}\},$$

where

$$\psi_{l,n,k} = \begin{cases} D^n T_k \psi_l, & n \geq L \\ 2^{\frac{n-l}{2}} T_{2^{-L}k} D^n \psi_{l,} & n \leq L \end{cases}$$

In fact, in [19] shown that a wavelet system $X(\psi)$ is a tight frame for $L_2(\mathbb{R})$ if its corresponding quasi-affine system $X^L(\Psi)$ is a tight frame for $L_2(\mathbb{R})$, and their approximation order is similar. In particular, for the framelets system $X(\psi)$, the approximation order is m for all $f \in W_2^m(\mathbb{R})$ if

$$\| f - P_n \|_{L_2(\mathbb{R})} = O\left(2^{-nm}\right)$$

where $W_2^m(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \middle| \| f \|_{W_2^m} = \sqrt{2\pi} \left\| \left(1 + |\ .\ |^m\right) f \right\|_{L^2(\mathbb{R})} < \infty \right\}$ is the Sobolev space. For simplicity, we consider the quasi-affine tight framelets at level 0 for IAE solving.

4. Matrix Formulation Using Framelets

In this section, the approximation solution for the unknown functions $y(t)$ and $z(t)$. The solution is obtained by truncating the quasi-affine framelets of $y(t)$ and $z(t)$. The approximation solution is defined as follows;

$$P_n(y) = \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \alpha_{l,j,k} \psi_{l,j,k} \tag{1.20}$$

$$P_n(z) = \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{l,j,k} \psi_{l,j,k}, \tag{1.21}$$

The approximation solution will be obtained by computing the unknown coefficients $\alpha_{l,j,k}$ and $\beta_{l,j,k}$. Now, by inserting equations(1.20) and(1.21) into the system(1.3) we get the system

$$\begin{gathered} \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \alpha_{l,j,k} \psi_{l,j,k}(t) - \rho(t) \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{l,j,k} \psi_{l,j,k}(t) = f(t) + \\ \int_0^t \left(k_{11}(t,s) \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \alpha_{l,j,k} \psi_{l,j,k}(s) + k_{12}(t,s) \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{l,j,k} \psi_{l,j,k}(s) \right) ds \\ 0 = g(t) + \int_0^t \left(k_{21}(t,s) \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \alpha_{l,j,k} \psi_{l,j,k}(s) + k_{22}(t,s) \sum_{l=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \beta_{l,j,k} \psi_{l,j,k}(s) \right) ds \end{gathered} \tag{1.22}$$

System(1.22) could be written in the form

$$\begin{aligned}&\sum_{l=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\alpha_{l,j,k}\left[\psi_{l,j,k}(t)-\int_0^t k_{11}(t,s)\psi_{l,j,k}(s)ds\right]\right)-\\&\rho(t)\sum_{l=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\beta_{l,j,k}\left[\psi_{l,j,k}(t)-\int_0^t k_{12}(t,s)\psi_{l,j,k}(s)ds\right]=f(t)\\&\sum_{l=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\alpha_{l,j,k}\left[-\int_0^t k_{21}(t,s)\psi_{l,j,k}(s)ds\right]\right)-\sum_{l=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\beta_{l,j,k}\left[\int_0^t k_{22}(t,s)\psi_{l,j,k}(s)ds\right]=g(t)\end{aligned}\tag{1.23}$$

Note that, if the framelets system $X(\Psi)=\{\psi_{l,j,k}, l=1,\ldots,r\}_{j,k\in\mathbb{Z}}$ based in the refinable function φ in the subspace $V_n\subseteq L^2(\mathbb{R})$, then the values for j run from $-n$ to n, and $k=-2^n,\ldots,2^n-1$. Hence, the system (1.23) requires $4\,n\,r\left(2^{n+1}-1\right)$ of unknowns $\{\alpha_{l,j,k}\}$ and $\{\beta_{l,j,k}\}$. Considering the $\{t_n, n=1,\ldots,2\,n\,r\left(2^n-1\right)\}$ different collocation points in the interval $[a,b]$, and substituting these points in the system(1.23), we get $4\,n\,r\left(2^n-1\right)$ linear equations;

$$\begin{aligned}&\sum_{l=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\alpha_{l,j,k}\left[\psi_{l,j,k}(t_i)-\int_0^{t_i} k_{11}(t_i,s)\psi_{l,j,k}(s)ds\right]\right)-\\&\rho(t_i)\sum_{l=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\beta_{l,j,k}\left[\psi_{l,j,k}(t_i)-\int_0^{t_i} k_{12}(t_i,s)\psi_{l,j,k}(s)ds\right]=f(t_i)\\&\sum_{l=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\alpha_{l,j,k}\left[-\int_0^{t_i} k_{21}(t_i,s)\psi_{l,j,k}(s)ds\right]\right)-\sum_{l=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\beta_{l,j,k}\left[\int_0^{t_i} k_{22}(t_i,s)\psi_{l,j,k}(s)ds\right]=g(t_i)\end{aligned}\tag{1.24}$$

The unknowns $\{\alpha_{l,j,k}\}$ and $\{\beta_{l,j,k}\}$ are determined by solving the resulting system of equation(1.24). This can be formulated as a matrix equation

$$\begin{pmatrix}[\mathrm{K}_{11}] & [\mathrm{K}_{12}]\\ [\mathrm{K}_{21}] & [\mathrm{K}_{22}]\end{pmatrix}\begin{pmatrix}[F]\\ [G]\end{pmatrix}=\begin{pmatrix}[\alpha]\\ [\beta]\end{pmatrix}\tag{1.25}$$

Where,

$$[\mathrm{K}_{ij}]=\begin{pmatrix}k_{i,j}\left(1,-n,-2^n\right) & \cdots & k_{i,j}\left(1,n,2^n-1\right)\\ \vdots & \ddots & \vdots\\ k_{i,j}\left(2nr\left(2^{n+1}-1\right),-n,-2^n\right) & \cdots & k_{i,j}\left(2nr\left(2^{n+1}-1\right),n,2^n-1\right)\end{pmatrix}, i,j=1,2$$

$k_{1,1}=\psi_{l,i,j}(t_1)-\int_0^{t_l}k_{11}(t_l,s)\psi_{l,i,j}(s)ds$, $k_{1,2}(l,j,k)=\rho(t_l)\left(\psi_{l,j,k}(t_l)-\int_0^{t_l}k_{12}(t_l,s)\psi_{l,j,k}(s)ds\right)$

$k_{2,1}=-\int_0^{t_l}k_{21}(t_l,s)\psi_{l,i,j}(s)ds$ and $k_{2,2}=-\int_0^{t_l}k_{22}(t_l,s)\psi_{l,i,j}(s)ds$.

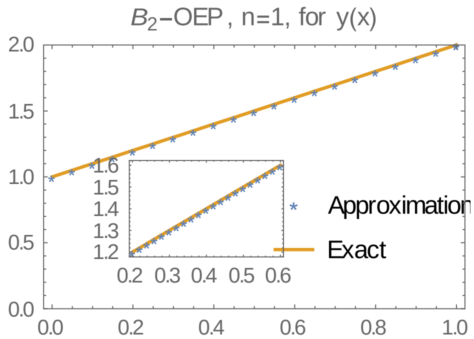
5. NUMERICAL EXAMPLES

Based on the method presented in this article, we solve equation(1.3) using the quasi-affine tight framelets constructed in Section 4. To validate the accuracy of our method, the examples are chosen from different articles to compare our method with others.

Example 6: [21] Consider equation (1.3) with $k_{11}(t,s)=ts+1$, $k_{1,2}(t,s)=st^2$, $k_{21}(t,s)=st^2+1$, $k_{22}(t,s)=ts+5$ and $\rho(t)=t+2$, the functions $f(t)$, $g(t)$ are chosen so that the exact solutions are $y(t)=t+1$ and $z(t)=t+2$. This example exists in [7]

The following tables contains the results of our method and the results of method in [7]

	OEP, B2, $n=1$		Results in [21]	
x	$\lvert y-P_n y\rvert$	$\lvert z-P_n z\rvert$	$\lvert y-P_n y\rvert$	$\lvert z-P_n z\rvert$
0.4	3.55271×10^{-15}	4.44089×10^{-16}	0.2×10^{-7}	0.1×10^{-8}
0.6	7.54952×10^{-15}	1.33227×10^{-15}	0.47×10^{-7}	0.2×10^{-7}
0.8	1.37668×10^{-15}	8.88178×10^{-16}	0.82×10^{-7}	0.33×10^{-7}
1	2.66454×10^{-15}	1.77636×10^{-15}	0.22×10^{-6}	0.75×10^{-7}



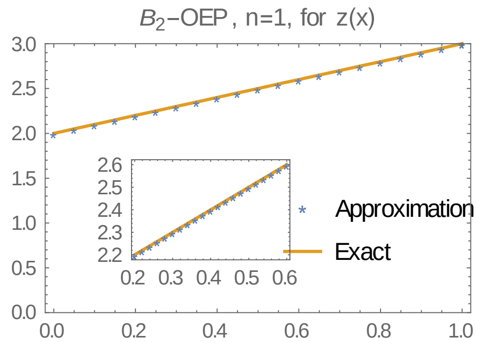


FIGURE 7: The graph of the exact solutions $y(x)$, $z(x)$ and the approximations $u_n(x)$ and $\tilde{z}_n(x)$ respectively. for example 6, $n=1$ by using framelets generated by OEP

Example 7: [8]Consider equation (1.3) with $k_{11}(t,s)=\cos(s+t)+3$, $k_{1,2}(t,s)=\dfrac{s}{1+t}$, $k_{21}(t,s)=se^{2t}+2$, $k_{22}(t,s)=ts+1$ and $\rho(t)=e^{t}+1$, the functions $f(t)$, $g(t)$ are chosen so that the exact solutions are $y(t)=t\cos(t)+2$ and $z(t)=\dfrac{1+t}{e^{t}}$. This example exists in [8]

The following tables contains the results of our method and the results of method in [8]

	UEP, B4, $n=1$		OEP, B2, $n=1$		Results in [8]	
x	$\lvert y-P_n y\rvert$	$\lvert z-P_n z\rvert$	$\lvert y-P_n y\rvert$	$\lvert z-P_n z\rvert$	$\lvert y-P_n y\rvert$	$\lvert z-P_n z\rvert$
0.2	2.32314×10^{-8}	1.15116×10^{-7}	1.003×10^{-3}	1.112×10^{-3}	0.277×10^{-3}	0.120×10^{-3}
0.4	3.89323×10^{-8}	1.01582×10^{-7}	4.281×10^{-3}	3.731×10^{-3}	0.543×10^{-3}	0.210×10^{-3}
0.6	9.83326×10^{-8}	8.4253×10^{-8}	2.647×10^{-3}	4.001×10^{-3}	0.757×10^{-3}	0.260×10^{-3}
0.8	1.32262×10^{-7}	7.20591×10^{-8}	1.926×10^{-3}	8.671×10^{-4}	0.894×10^{-3}	0.270×10^{-3}
1	2.47117×10^{-8}	1.94719×10^{-8}	2.224×10^{-3}	5.6897×10^{-3}	0.931×10^{-3}	0.244×10^{-3}

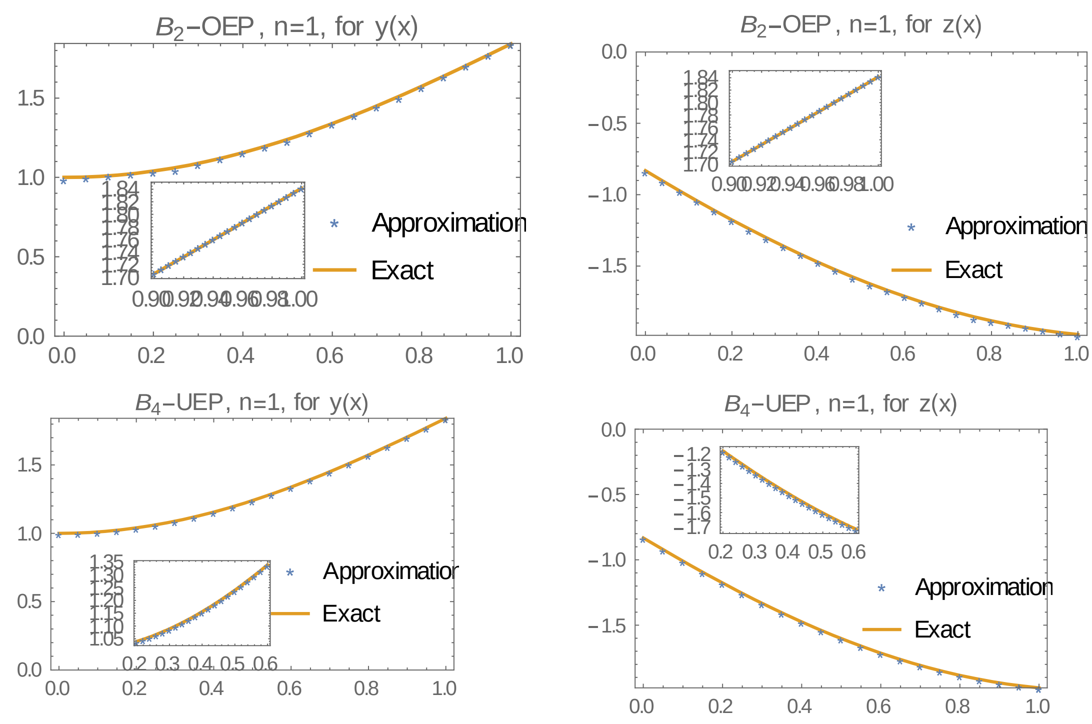


FIGURE 8: The graph of the exact solutions $y(x), z(x)$ and the approximations $u_n(x)$ and $\tilde{z}_n(x)$ respectively. for example 6, $n=1$ by using framelets generated by OEP and UEP

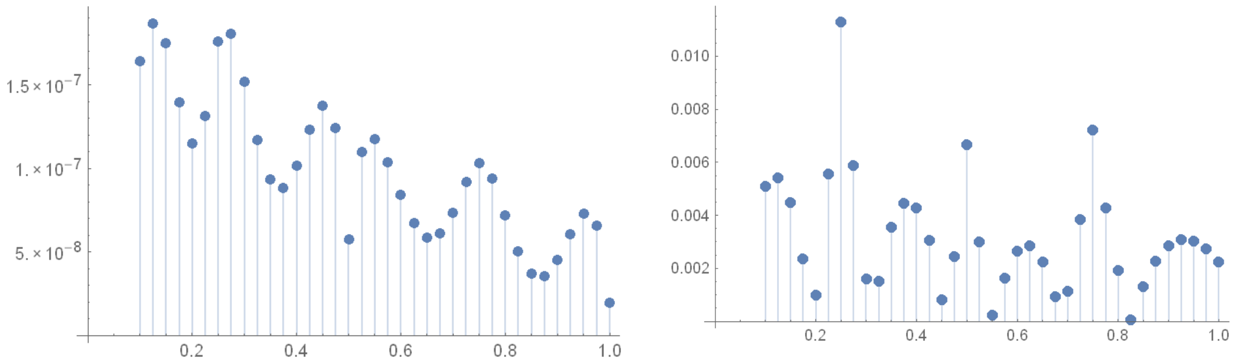


FIGURE 9: The error graphs for example 7 for the variable *y* using framelets generated by UEP and OEP, respectively.

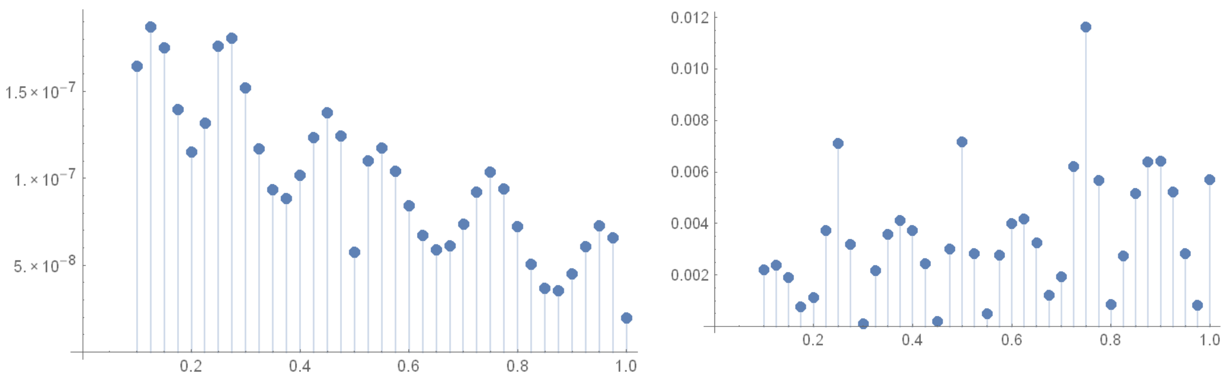


FIGURE 10 : The error graphs for example 7 for the variable *z* using framelets generated by UEP and OEP, respectively.

The properties of redundancy, sparse approximation and the existence of fast decomposition and reconstruction algorithms make framelets very important and affective tool for solving many problems such as image compression and signal denoising. Although many compression applications of wavelets use wavelet bases, other types of applications work better with redundant wavelet families, of which framelets are the easiest to use.

The vanishing moments plays an important role in the image/signal compression [22], such that for a given function $f \in L_2(\mathbb{R})$ and a set of framelets $\{\psi_{l,j,k}\ \ l=,\dots,r\}_{j,k\in\mathbb{Z}}$ for $L_2(\mathbb{R})$ where ψ has large vanishing moments. Then, only few relatively coefficients of the function representation $f=\sum_{l=1}^{r}\sum_{j,k\in\mathbb{Z}}\langle f,\psi_{l,j,k}\rangle\psi_{l,j,k}$ will be large, so by throwing the rest we will obtained an efficient compression for the function f .

Conclusion

In this paper, we established a new, efficient method for solving third kind integral algebraic equation(1.3). It turns out the proposed method is easy to implement, and shows highly accurate results and the performance of the present method is reliable, efficient and converges to the exact solution. Furthermore, we compared our method with two different methods, where our method is easiest to use than others, and give more accurate results.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] G. Bal, Inverse problems for homogeneous transport equations: I. The one-dimensional case, Inverse Probl. 16 (2000), 997–1011.

[2] D. Shulaia, On one Fredholm integral equation of third kind, Georgian Math. J. 4 (1997), 464-476.

[3] D.A. Shulaia, E.I. Gugushvili, Inverse problem of spectral analysis of linear multigroup neutron transport theory in plane geometry, Transport Theory Stat. Phys. 29 (2000), 711–721.

[4] J.-P. Kauthen, The numerical solution of integral-algebraic equations of index 1 by polynomial spline collocation methods, Math. Comput. 70 (2000), 1503–1515.

[5] Q.D. Katatbeh, Y. Al-Jarrah, E.B. Lin, Numerical solution of integral-algebraic equations of index one using discretized method, Nonlinear Stud. 21 (4) (2014), 1-11.

[6] A. Al-Habahbeh, Y. Al-jarrah, Sinc collocation solution for integral algebraic equations of index-1, Adv. Theor. Appl. Math. 12 (2) (2017), 99-112.

[7] A. Bellour, E.A. Rawashdeh, Numerical solution of third kind integral-algebraic equations, Mat. Vesnik. 63 (2011), 223-233.

[8] A. Bellour, E.A. Rawashdeh, A. Zidna, Continuous collocation method for third kind, Math. Computers Simul. 137 (2017), 148-158.

[9] E.-B. Lin, Y. Aljarrah, Wavelet based methods for numerical solutions of two dimensional integral equations, Math. Aeterna, 4 (8) (2014), 839-853.

[10] Y. Al-Jarrah, E.-B. Lin, Wavelet interpolation method for solving singular integral equations, Appl. Math. 4 (2013), 1-4.

[11] E.-B. Lin, Y. Al-Jarrah, A Wavelet based method for the solution of Fredholm integral equations, Amer. J. Comput. Math. 2 (2012), 114-117.

[12] Y. Al-Jarrah, E.-B. Lin, Numerical solutions of Fredholm integral equations of the first kind by using Coiflets, J. Mod. Math. Front. 1 (4) (2012), 1-4.

[13] M. Mutaz, T. Alexander, C. Carlo, An Efficient Method Based on Framelets for Solving Fractional Volterra Integral Equations, Entropy, 22 (8) (2020), 824.

[14] M. Mohammad, A Numerical Solution of Fredholm Integral Equations of the Second Kind Based on Tight Framelets Generated by the Oblique Extension Principle, Symmetry, 11(7) (2019), 854.

[15] M. Mohammad, Biorthogonal-Wavelet-Based Method for Numerical, Entropy, 21 (11) (2019), 1098.

[16] R.J. Duffin, A.C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (2) (1952), 341-366.

[17] I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmonic Anal. 14 (2003) 1–46.

[18] A. Ron, Z. Shen, Affine Systems in$L_2(R^d)$: The Analysis of the Analysis Operator, J. Funct. Anal. 148 (1997), 408–447.

[19] B. Dong, Z. Shen, MRA-Based Wavelet Frames and Application, IAS Lecture Note Series, 2013.

[20] O. Christensen, An Introduction to Frames and Riesz Bases, Springer, New York, 2002.

[21] A. Bellour, E. A. Rawashdeh, Rawashdeh, Numerical solution of third kind integralalgebraic equations, Mat. Vesnik, 63 (2011), 223-233.

[22] B. Han, Q. Mo, Z. Zhao, X. Zhuang, Directional Compactly Supported Tensor Product Complex Tight Framelets with Applications to Image Denoising and Inpainting, J. Imag. Sci. 12 (4) (2019), 1739-1771.