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J. Math. Comput. Sci. 11 (2021), No. 2, 2075-2091

<https://doi.org/10.28919/jmcs/5450>

ISSN: 1927-5307

## A NOTE ON CONCIRCULAR CURVATURE TENSOR IN LORENTZIAN ALMOST PARA-CONTACT GEOMETRY

SATYENDRA PRATAP SINGH<sup>1</sup>, PUNIT KUMAR SINGH<sup>2,\*</sup>, VIRENDRA NATH PATHAK<sup>1</sup>

<sup>1</sup>Department of Mathematical and Statistical Sciences, Shri Ramswaroop Memorial University,  
Lucknow-Dewa Road 225003, U. P. India

<sup>2</sup>Department of Mathematics & Astronomy, University of Lucknow, Lucknow-226007, U. P. India

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**Abstract.** The paper deals with the notion of different classes of concircular curvature tensor on Lorentzian almost para-contact manifolds admitting a quarter-symmetric metric connection. In this paper we study Lorentzian almost para-contact manifolds with respect to the quarter-symmetric metric connection satisfying the curvature condition  $\bar{Z}\bar{S} = 0$ . We also investigate the properties of  $\xi$ -concircularly flat,  $\phi$ -concircularly flat and quasi-concircularly flat Lorentzian almost para-contact manifolds admitting a quarter-symmetric metric connection and it is found that in each of above cases the manifold is generalized  $\eta$ -Einstein manifold.

**Keywords:** Lorentzian almost para-contact manifolds; Lorentzian quarter-symmetric metric connection; concircular curvature tensor;  $\eta$ -Einstein manifold.

**2010 AMS Subject Classification:** 53C25, 53D10, 53D15.

### 1. INTRODUCTION

Differential Geometry is the most important and very interesting branch of mathematics and physics from ancient days. In differential geometry, there are various topics which have very important applications in mathematics and physics both.

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\*Corresponding author

E-mail address: [singhpunit1993@gmail.com](mailto:singhpunit1993@gmail.com)

Received January 18, 2021

A transformation of an  $n$ -dimensional Riemannian manifold  $M$ , which transforms every geodesic circle of  $M$  into a geodesic circle is called a *concircular transformation* ([10], [23]). A concircular transformation is always a conformal transformation [10]. Here geodesic circle means a curve in  $M$  whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [6]). An interesting invariant of a concircular transformation is the *concircular curvature tensor*  $Z$ . It is defined by ([23], [24])

$$(1.1) \quad Z = R - \frac{r}{n(n-1)}R_0,$$

where  $R$  is the curvature tensor,  $r$  is the scalar curvature and

$$(1.2) \quad R_0(U, V, W) = g(V, W)U - g(U, W)V, \text{ for } U, V, W \in TM.$$

Using above equation (1.2), we obtain

$$(1.3) \quad g(Z(U, V)W, X) = g(R(U, V)W, X) - \frac{r}{n(n-1)}[g(V, W)g(U, X) - g(U, W)g(V, X)],$$

where  $U, V, W, X \in TM$ .

The concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature because Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature.

In 1977, D. E. Blair stated that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying  $R(U, V)\xi = 0$  [5]. On the other hand, we know that on a manifold  $M$  equipped with a Sasakian structure  $(\eta, \xi, \psi, g)$ ,

$$(1.4) \quad R(U, V)\xi = \eta(V)U - \eta(U)V = R_0(U, V)\xi \text{ for all } U, V \in TM.$$

In 1989, K. Matsumoto introduced the concept of Lorentzian para- Sasakian manifolds [11]. Later, the same concept was independently introduced by I. Mihai and R. Rosca [13]. The Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [12], U. C. De and A. A. Shaikh [14] and several others such as ([15], [19], [20]). K. Matsumoto and I. Mihai obtained some interesting results for conformally recurrent and conformally symmetric

$P$ -Sasakian manifold in [1]. In 1924, the notion of semi-symmetric connection on a differentiable manifold was firstly introduced by Friedmann and Schouten [8]. A linear connection  $\bar{\nabla}$  on a differentiable manifold  $M$  is said to be a semi-symmetric connection if the torsion tensor  $T$  of the connection satisfies

$$(1.5) \quad T(U, V) = \eta(V)U - \eta(U)V,$$

where  $\eta$  is a 1-form and  $\xi$  is a vector field defined by  $\eta(U) = g(U, \xi)$ , for all vector fields  $U$  on  $\Gamma(TM)$ ,  $\Gamma(TM)$  is the set of all differentiable vector fields on  $M$ . A. Barman ([2], [3]) studied para-Sasakian manifold admitting semi-symmetric metric and non metric connection. On the other hand, in 1975, Golab [9] introduced and studied quarter-symmetric connection in differentiable manifolds along with affine connections.

A liner connection  $\bar{\nabla}$  on an  $n$ -dimensional Riemannian manifold  $(M, g)$  is called a quarter-symmetric connection [9] if its torsion tensor  $T$  satisfies

$$(1.6) \quad T(U, V) = \eta(V)\phi U - \eta(U)\phi V,$$

where  $\phi$  is a (1,1)tensor field.

The quarter-symmetric connection generalizes the notion of the semi-symmetric connection because if we assume  $\phi U = U$  in the above equation, the quarter-symmetric connection reduces to the semi-symmetric connection [8].

Moreover, if a quarter-symmetric connection  $\bar{\nabla}$  satisfies the condition

$$(1.7) \quad (\bar{\nabla}_U g)(V, W) = 0,$$

for all  $U, V, W$  on  $\Gamma(TM)$ , then  $\bar{\nabla}$  is said to be a quarter-symmetric metric connection.

In 2008, Venkatesha and C.S. Bagewadi [22] obtain some interesting results on concircular  $\phi$ -recurrent Lorentzian para-Sasakian manifolds which generalize the concept of locally concircular  $\phi$ -symmetric Lorentzian para-Sasakian manifolds. If curvature tensor  $R$  of Riemannian manifold  $M$  satisfies  $\nabla R = 0$ , then  $M$  is called locally symmetric. Later, many geometers have considered semi-symmetric spaces as a generalization of locally symmetric spaces. A Riemannian manifold  $M$  is said to be semi-symmetric if its curvature tensor  $R$  satisfies  $R(U, V).R = 0$ , where  $R(U, V)$  acts on  $R$  as a derivation and also it is called Ricci-semisymmetric manifold if

the relation  $R(U, V).S = 0$  holds, where  $R(U, V)$  the curvature operator.

In 2005, D. E. Blair and the authors [7] started a study of concircular curvature tensor of contact metric manifolds. B. J. Papantoniou [16] and D. Perrone [17] included the studies of contact metric manifolds satisfying  $R(X, \xi).S = 0$ , where  $S$  is the Ricci tensor. Motivated by these studies, we study a concircular curvature tensor in Lorentzian almost para contact geometry.

In this paper, we study a type of quarter-symmetric metric connection on Lorentzian almost para-contact manifolds. The paper is organized as follows: After introduction the section two is equipped with some prerequisites of a Lorentzian para contact manifolds. In section three the curvature tensor of Lorentzian almost para contact manifold with respect to the quarter-symmetric metric connection is defined. The section four is the study of Lorentzian almost para-contact manifolds with respect to the quarter-symmetric metric connection satisfying the curvature condition  $\bar{Z}.\bar{S} = 0$ . The section five is devoted to study of  $\xi$ -concircularly flat in a Lorentzian almost para-Sasakian manifolds with respect to the quarter-symmetric metric connection. In the section six, we define  $\phi$ -concircularly flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection and in the last last section, we investigate the properties of quasi-concircularly flat Lorentzian almost para-Sasakian manifolds admitting a quarter-symmetric metric connection.

## 2. PRELIMINARIES

An  $n$ -dimensional differentiable manifold  $M$  is said to be an Lorentzian almost para-contact manifold, if it admits an almost para-contact structure  $(\phi, \xi, \eta, g)$  consisting of a  $(1, 1)$  tensor field  $\phi$ , vector field  $\xi$ , 1-form  $\eta$  and a Lorentzian metric  $g$  satisfying

$$(2.1) \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = -1, \quad g(U, \xi) = \eta(U),$$

$$(2.2) \quad \phi^2 U = U + \eta(U)\xi,$$

$$(2.3) \quad g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V),$$

$$(2.4) \quad (\nabla_U \eta)V = g(U, \phi V) = (\nabla_V \eta)U,$$

for any vector field  $U, V$  on  $M$ . Such a manifold  $M$  is termed as Lorentzian para-contact manifold and the structure  $(\phi, \xi, \eta, g)$  a Lorentzian para-contact structure [11].

**Definition 2.1.** A Lorentzian almost para contact manifold  $M$  is called Lorentzian para-Sasakian manifold or briefly  $LP$ -Sasakian manifold if  $(\phi, \xi, \eta, g)$  satisfies the conditions

$$(2.5) \quad d\eta = 0, \quad \nabla_U \xi = \phi U,$$

$$(2.6) \quad (\nabla_U \phi)V = g(U, V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi,$$

for  $U, V$  tangent to  $M$ , where  $\nabla$  denotes the covariant differentiation with respect to Lorentzian metric  $g$ .

Moreover, the curvature tensor  $R$ , the Ricci tensor  $S$  and the Ricci operator  $Q$  in a Lorentzian para-Sasakian manifold  $M$  with respect to the Levi-Civita connection  $\nabla$  satisfies the following relations [18]

$$(2.7) \quad \eta(R(U, V)W) = g(V, W)\eta(U) - g(U, W)\eta(V),$$

$$(2.8) \quad R(\xi, U)V = g(U, V)\xi - \eta(V)U,$$

$$(2.9) \quad R(\xi, U)\xi = -R(U, \xi)\xi = U + \eta(U)\xi,$$

$$(2.10) \quad R(U, V)\xi = \eta(V)U - \eta(U)V,$$

$$(2.11) \quad S(U, \xi) = (n-1)\eta(U), \quad Q\xi = (n-1)\xi$$

$$(2.12) \quad S(\phi U, \phi V) = S(U, V) + (n-1)\eta(U)\eta(V),$$

for all vector fields  $U, V, W \in \Gamma(TM)$ .

**Definition 2.2.** A Lorentzian para-Sasakian manifold  $M$  is said to be an  $\eta$ -Einstein manifold [18] if its Ricci tensor  $S$  of the Levi-Civita connection is of the form

$$(2.13) \quad S(U, V) = ag(U, V) + b\eta(U)\eta(V) \text{ for all } U, V \in \Gamma(TM)$$

where  $a$  and  $b$  are smooth functions on the manifold  $M$ .

**Definition 2.3.** A Lorentzian almost para contact manifold  $M$  is called Lorentzian para-Kenmotsu manifold if  $(\phi, \xi, \eta, g)$  satisfies the conditions

$$(2.14) \quad (\nabla_U \phi)V = -g(\phi U, V)\xi - \eta(V)\phi U,$$

for any vector field  $U, V$  on  $M$ .

In the Lorentzian para-Kenmotsu manifold, we have

$$(2.15) \quad \nabla_X \xi = -X - \eta(X)\xi,$$

$$(\nabla_U \eta)V = -g(U, V) - \eta(U)\eta(V),$$

where  $\nabla$  the operator of covariant differentiation with respect to the Lorentzian metric  $g$ .

*Example 1.* [21] Let  $M = \{(x^1, x^2, \dots, x^m, y^1, y^2, \dots, y^m, z) = (x^i, y^i, z) \in \mathbb{R}^{2m+1}, \text{ where } x^i, y^i, z \in \mathbb{R} \text{ and } i = 1, 2, \dots, m\}$  denote an  $n(= 2m + 1)$ -dimensional smooth manifold. Let us define the structure tensor  $\phi$  as:

$$\phi(\xi) = 0, \phi(X_i) = Y_i, \phi(Y_i) = X_i.$$

If  $g$  represents the Lorentzian metric of  $M$  defined by

$$g = -(\eta \otimes \eta) + e^{-2z} \sum_{i=1}^n (dx^i \otimes dx^i + dy^i \otimes dy^i),$$

then by linearity properties, we can easily show that the relations

$$\phi^2 X = X + \eta(X)\xi, g(X, \xi) = \eta(X)$$

hold for all vector fields  $X$  on  $R^{2m+1}$ . Thus,  $(M, \phi, \xi, \eta, g)$  forms a Lorentzian para-Kenmotsu manifold with the  $\phi$ -basis  $X_i = e^z \frac{\partial}{\partial x^i}, Y_i = e^z \frac{\partial}{\partial y^i}$  and  $\xi = \frac{\partial}{\partial z}$ , where  $i = 1, 2, \dots, m$ .

**3. CURVATURE TENSOR OF LORENTZIAN ALMOST PARA CONTACT MANIFOLD WITH RESPECT TO THE QUARTER-SYMMETRIC METRIC CONNECTION**

A relation between the quarter-symmetric metric connection  $\bar{\nabla}$  and the Levi-Civita connection  $\nabla$  in an  $n$ -dimensional Lorentzian para-Sasakian manifold  $M$  is given by [20]

$$(3.1) \quad \bar{\nabla}_U V = \nabla_U V + \eta(V)\phi U - g(\phi U, V)\xi.$$

The curvature tensor  $\bar{R}$  of a Lorentzian para-Sasakian manifold  $M$  with respect to the quarter-symmetric metric connection  $\bar{\nabla}$  is defined by

$$(3.2) \quad \bar{R}(U, V)W = \bar{\nabla}_U \bar{\nabla}_V W - \bar{\nabla}_V \bar{\nabla}_U W - \bar{\nabla}_{[U, V]}W.$$

From the equations (2.1) – (2.6), (3.1) and (3.2), we obtain

$$(3.3) \quad \begin{aligned} \bar{R}(U, V)W &= R(U, V)W + [g(\phi U, W)\phi V - g(\phi V, W)\phi U] \\ &\quad + [g(V, W)\eta(U) - g(U, W)\eta(V)]\xi \\ &\quad + \eta(W)[\eta(V)U - \eta(U)V]. \end{aligned}$$

where  $U, V, W \in \Gamma(TM)$  and  $R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]}W$  is the Riemannian curvature tensor with respect to the Levi-Civita connection  $\nabla$ .

The Ricci tensor  $\bar{S}$  and the Scalar curvature  $\bar{r}$  in a Lorentzian para-Sasakian manifold  $M$  with respect to the quarter-symmetric metric connection  $\bar{\nabla}$  are defined by

$$(3.4) \quad \bar{S}(V, W) = \sum_{i=1}^n \varepsilon_i g(\bar{R}(e_i, V)W, e_i),$$

$$(3.5) \quad \bar{r} = \sum_{i=1}^n \varepsilon_i \bar{S}(e_i, e_i)$$

where  $\{e_1, e_2, \dots, e_{n-1}, e_n = \xi\}$  be a local orthonormal basis of vector fields in  $M$  and  $\varepsilon_i = g(e_i, e_i)$ .

Now contracting  $U$  in (3.3), we get

$$(3.6) \quad \bar{S}(V, W) = S(V, W) + (n - 1)\eta(V)\eta(W) - (\text{trace}\phi)g(\phi V, W).$$

Again contracting  $V$  and  $W$  in (3.6), we get

$$(3.7) \quad \bar{r} = r - (n - 1) - (\text{trace}\phi)^2.$$

From equation (3.3) and (3.6), we have

$$(3.8) \quad \bar{R}(U, V)\xi = \bar{R}(\xi, U)V = 0,$$

$$(3.9) \quad \bar{S}(V, \xi) = 0,$$

$$(3.10) \quad \bar{S}(\phi U, \phi V) = \bar{S}(U, V).$$

#### 4. LORENTZIAN ALMOST PARA CONTACT MANIFOLD SATISFYING $\bar{Z} \cdot \bar{S} = 0$ WITH RESPECT TO A QUARTER-SYMMETRIC METRIC CONNECTION

We consider Lorentzian almost para contact manifolds (LP-Sasakian manifolds) with respect to a quarter-symmetric metric connection  $\bar{\nabla}$  satisfying the curvature condition  $\bar{Z} \cdot \bar{S} = 0$ . Then

$$(\bar{Z}(U, V) \cdot \bar{S})(W, X) = 0.$$

So,

$$(4.1) \quad \bar{S}(\bar{Z}(U, V)W, X) + \bar{S}(W, \bar{Z}(U, V)X) = 0.$$

Putting  $U = \xi$  in (4.1), we get

$$(4.2) \quad \bar{S}(\bar{Z}(\xi, V)W, X) + \bar{S}(W, \bar{Z}(\xi, V)X) = 0.$$

Now from equations (1.1), (1.2) and (3.8), we have

$$(4.3) \quad \bar{Z}(\xi, V)W = -\frac{\bar{r}}{n(n-1)}[g(V, W)\xi - \eta(W)V].$$

Using (4.3) in (4.2) and putting  $W = \xi$  and using (3.9), we obtain

$$\bar{r}\bar{S}(V, X) = 0.$$

This implies that  $\bar{r} = 0$ .

Hence we can state following:



**Theorem 4.1.** *If Lorentzian para-Sasakian manifolds satisfying  $\bar{Z} \cdot \bar{S} = 0$  with respect to the quarter-symmetric metric connection, then the manifold is flat with respect to the quarter-symmetric metric connection.*

*Example 2.* Example of a LP-Sasakian manifold with respect to Quarter-symmetric metric connection. Taking a 3-dimensional manifold  $M = \{(x, y, v) \in R^3\}$ , where  $(x, y, v)$  are standard coordinates of  $R^3$ . Let  $e_1, e_2, e_3$  are vector fields on  $M$ , given by

$$e_1 = -e^v \frac{\partial}{\partial x}, \quad e_2 = -e^{v-x} \frac{\partial}{\partial y}, \quad e_3 = -\frac{\partial}{\partial v} = \xi,$$

Clearly,  $\{e_1, e_2, e_3\}$  is linearly independent set of vectors on  $M$ . So it forms a basis of  $\chi(M)$ . The Lorentzian metric  $g$  is defined by

$$g(e_i, e_j) = 0, \text{ for } i \neq j \text{ and } 1 \leq i, j \leq 3$$

$$\text{and } g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1.$$

Let  $\eta$  be a 1-form on  $M$  defined as  $\eta(U) = g(U, e_3) = g(U, \xi)$ , for all  $U \in \chi(M)$ , and let  $\phi$  be a  $(1, 1)$  tensor field on  $M$  defined as

$$\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0.$$

By applying linearity of  $\phi$  and  $g$ , we have

$$\eta(e_3) = -1, \quad \phi^2(U) = U + \eta(U)\xi,$$

and

$$g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V) \text{ for all } U, V \in \chi(M).$$

Let  $\nabla$  be a Levi-Civita connection with respect to the Riemannian metric  $g$ , we have

$$[e_1, e_2] = -e^v e_2, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = -e_1,$$

The Riemannian connection  $\nabla$  of the metric  $g$  is given by

$$2g(\nabla_U V, W) = Ug(V, W) + Vg(W, U) - Wg(U, V) \\ -g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]),$$

which is known as Koszul's formula, we can easily calculate

$$\begin{aligned} \nabla_{e_1} e_1 &= e_3, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= -e_2, \\ \nabla_{e_2} e_1 &= -e^v e_2, & \nabla_{e_2} e_2 &= -e_3 - e^v e_1, & \nabla_{e_2} e_3 &= -e_2, \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= 0, \end{aligned}$$

From the above it follows that the manifold satisfies  $\nabla_U \xi = \phi U$ , for  $\xi = e_3$  and  $(\nabla_U \phi)V = g(U, V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi$ . Hence the manifold is *LP*-Sasakian manifold.

Using (3.1), we have

$$\begin{aligned} \bar{\nabla}_{e_1} e_1 &= 0, & \bar{\nabla}_{e_1} e_2 &= 0, & \bar{\nabla}_{e_1} e_3 &= 0, \\ \bar{\nabla}_{e_2} e_1 &= -e^v e_2, & \bar{\nabla}_{e_2} e_2 &= -e^v e_1, & \bar{\nabla}_{e_2} e_3 &= 0, \\ \bar{\nabla}_{e_3} e_1 &= 0, & \bar{\nabla}_{e_3} e_2 &= 0, & \bar{\nabla}_{e_3} e_3 &= 0, \end{aligned}$$

Using (1.6), the torsion tensor  $\bar{T}$ , with respect to quarter symmetric metric connection  $\bar{\nabla}$  as follows :

$$\begin{aligned} \bar{T}(e_i, e_i) &= 0, \quad \forall i = 1, 2, 3, \\ \bar{T}(e_1, e_2) &= 0, \quad \bar{T}(e_1, e_3) = e_3, \quad \bar{T}(e_2, e_3) = e_2, \end{aligned}$$

Also,

$$(\bar{\nabla}_{e_1} g)(e_2, e_3) = 0, \quad (\bar{\nabla}_{e_2} g)(e_3, e_1) = 0, \quad (\bar{\nabla}_{e_3} g)(e_1, e_2) = 0,$$

Thus  $M$  is a Lorentzian para-Sasakian manifold with quarter-symmetric metric connection  $\bar{\nabla}$ .

**5.  $\xi$ -CONCIRCULARLY FLAT LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH RESPECT TO THE QUARTER-SYMMETRIC METRIC CONNECTION**

**Definition 5.1.** Concircular curvature tensor  $\bar{Z}$  of Lorentzian para-Sasakian manifold  $M$  with respect to the quarter-symmetric metric connection is defined by

$$(5.1) \quad \bar{Z}(U, V)W = \bar{R}(U, V)W - \frac{\bar{r}}{n(n-1)}[g(V, W)U - g(U, W)V]$$

for all  $U, V, W \in \Gamma(TM)$  where  $\bar{R}$  is the curvature tensor and  $\bar{r}$  is the scalar curvature of  $M$  with respect to the quarter-symmetric metric connection  $\bar{\nabla}$

**Definition 5.2.** A Lorentzian para-Sasakian manifold is said to be  $\xi$ -concircularly flat [4] with respect to the quarter-symmetric metric connection  $\bar{\nabla}$  if

$$(5.2) \quad \bar{Z}(U, V)\xi = 0$$

for all  $U, V \in \Gamma(TM)$ .

Putting  $W = \xi$  in (5.1) and using (3.8) and (5.2), we have

$$(5.3) \quad \bar{r}[\eta(V)U - \eta(U)V] = 0.$$

Putting  $U = \xi$  in (5.3) and using (2.1), we have

$$(5.4) \quad \bar{r}[V + \eta(V)\xi] = 0.$$

Taking inner product of (5.4) with  $W$  and replacing  $V$  by  $QV$ , we have

$$(5.5) \quad \bar{r}[g(QV, W) + \eta(QV)\eta(W)] = 0.$$

Using  $S(V, W) = g(QV, W)$  and equations (2.11) and (3.7) in (5.5), we have

$$(5.6) \quad [r - (n - 1) - (\text{trace}\phi)^2][S(V, W) + (n - 1)\eta(V)\eta(W)] = 0$$

Equation (5.6) implies that either  $r = (n - 1) + (\text{trace}\phi)^2$  or  $S(V, W) = -(n - 1)\eta(V)\eta(W)$ .

Thus we can state the following:

**Theorem 5.3.** *If a Lorentzian para-Sasakian manifold  $M$  admitting a quarter-symmetric metric connection is  $\xi$ -concircularly flat with respect to the quarter-symmetric metric connection, then either scalar curvature of  $M$  is  $(n-1) + (\text{trace}\phi)^2$  or the manifold  $M$  is a special type of  $\eta$ -Einstein manifold with respect to the Levi-Civita connection.*

## 6. $\phi$ -CONCIRCULARLY FLAT LORENTZIAN PARA-KENMOTSU MANIFOLDS WITH RESPECT TO THE QUARTER-SYMMETRIC METRIC CONNECTION

Analogous to the equation (5.1), the concircular curvature tensor  $\bar{Z}$  with respect to the quarter-symmetric metric connection is defined by

$$(6.1) \quad \bar{Z}(U, V)W = \bar{R}(U, V)W - \frac{\bar{r}}{n(n-1)}[g(V, W)U - g(U, W)V]$$

where  $\bar{R}$  and  $\bar{r}$  are the Riemannian curvature tensor and the scalar curvature with respect to the connection  $\bar{\nabla}$ , respectively on  $M$ .

**Definition 6.1.** A Lorentzian para-Kenmotsu manifold is said to be  $\phi$ -concircularly flat with respect to the quarter-symmetric metric connection if

$$(6.2) \quad \phi^2 \bar{Z}(\phi U, \phi V)\phi W = 0,$$

for all  $U, V, W \in \Gamma(TM)$ .

Let  $M$  be an  $n$ -dimensional  $\phi$ -concircularly flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then from equation (6.2), it follows that

$$(6.3) \quad g(\bar{Z}(\phi U, \phi V)\phi W, \phi X) = 0.$$

From the equations (6.1) and (6.3), we have

$$(6.4) \quad g(\bar{R}(\phi U, \phi V)\phi W, \phi X) = \frac{\bar{r}}{n(n-1)}[g(\phi V, \phi W)g(\phi U, \phi X) - g(\phi U, \phi W)g(\phi V, \phi X)].$$

As we know that the curvature tensor  $\bar{R}$  of Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection  $\bar{\nabla}$  is defined by

$$(6.5) \quad \begin{aligned} \bar{R}(U, V)W &= R(U, V)W + g(V, W)\phi U - g(U, W)\phi V + g(\phi V, W)U \\ &\quad - g(\phi U, W)V + g(\phi V, W)\phi U - g(\phi U, W)\phi V. \end{aligned}$$

In view of (6.5), (6.4) takes the form

$$(6.6) \quad \begin{aligned} &g(R(\phi U, \phi V)\phi W, \phi X) - g(\phi U, \phi W)g(V, \phi X) + g(\phi V, \phi W)g(U, \phi X) \\ &+ g(\phi U, \phi X)g(V, \phi W) - g(\phi V, \phi X)g(U, \phi W) + g(V, \phi W)g(U, \phi X) \\ -g(U, \phi W)g(V, \phi X) &= \frac{\bar{r}}{n(n-1)} [g(\phi V, \phi W)g(\phi U, \phi X) - g(\phi U, \phi W)g(\phi V, \phi X)]. \end{aligned}$$

Let  $\{e_1, e_2, \dots, e_{n-1}, \xi\}$  be a local orthonormal basis of vector fields in  $M$ , then

$\{\phi e_1, \phi e_2, \dots, \phi e_{n-1}, \xi\}$  is also a local orthonormal basis in  $M$ . Putting  $U = X = e_i$  in (6.6)

and summing over  $i = 1$  to  $n - 1$ , we obtain

$$(6.7) \quad \sum_{i=1}^{n-1} g(R(\phi e_i, \phi V)\phi W, \phi e_i) = \frac{\bar{r}}{n(n-1)} \sum_{i=1}^{n-1} [g(\phi V, \phi W)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi W)g(\phi V, \phi e_i)],$$

For  $\phi$ -conformally flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection, it can be easily verify that

$$(6.8) \quad \sum_{i=1}^{n-1} g(R(\phi e_i, \phi V)\phi W, \phi e_i) = S(\phi V, \phi W) - g(\phi V, \phi W),$$

$$(6.9) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi W)g(\phi V, \phi e_i) = g(\phi V, \phi W),$$

$$(6.10) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi W)g(V, \phi e_i) = g(V, \phi W),$$

$$(6.11) \quad \sum_{i=1}^{n-1} g(e_i, \phi W)g(V, \phi e_i) = g(V, W) + \eta(V)\eta(W),$$

$$(6.12) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1,$$

$$(6.13) \quad \sum_{i=1}^{n-1} g(e_i, \phi e_i) = \psi.$$

So by virtue of (6.8) – (6.13), the equation (6.7) takes the form

$$(6.14) \quad S(\phi V, \phi W) = \left[ \frac{\bar{r}(n-2)}{n(n-1)} - \phi + 2 \right] g(\phi V, \phi W) - (\psi + n - 3)g(V, \phi W).$$

In view of (2.2), (2.3) and (2.12), (6.14) takes the form

$$(6.15) \quad S(V, W) = ag(V, W) + b\eta(V)\eta(W) - cg(V, \phi W),$$

where  $a = \left[ \frac{(n-2)[\bar{r} + (2\psi-1)(n-1) + \psi^2] - n(n-1)(\psi-2)}{n(n-1)} \right]$ ,  $b = \left[ \frac{(n-2)[\bar{r} + (2\psi-1)(n-1) + \psi^2] - n(n-1)(n+\psi-3)}{n(n-1)} \right]$  and  $c = (n + \psi - 3)$ .

Contracting (6.15) over  $V$  and  $W$  gives

$$(6.16) \quad \bar{r} = n(n - \psi) - (\psi - 1)^2.$$

By using the value of  $\bar{r}$  in (6.15), we get

$$S(V, W) = \left( n - \frac{\psi}{n-1} \right) g(V, W) + \left( 1 - \frac{\psi}{n-1} \right) \eta(V)\eta(W) - (n + \psi - 3)g(V, \phi W)$$

From which it follows that the manifold is an  $\eta$ -Einstein manifold with respect to the quarter-symmetric metric connection.

Hence we can state following theorem:

**Theorem 6.2.** *An  $n$ -dimensional  $\phi$ -concurcularly flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized  $\eta$ -Einstein manifold with the scalar curvature given by (6.16).*

## 7. QUASI-CONCIRCULARLY FLAT LORENTZIAN PARA-SASAKIAN MANIFOLD WITH RESPECT TO THE QUARTER-SYMMETRIC METRIC CONNECTION

**Definition 7.1.** A Lorentzian para-Sasakian manifold  $M$  is said to be quasi-concurcularly flat with respect to the quarter-symmetric metric connection if

$$(7.1) \quad g(\bar{Z}(\phi U, V)W, \phi X) = 0$$

where  $U, V, W, X \in \Gamma(TM)$ .

From equation (5.1), we have

$$(7.2) \quad \begin{aligned} g(\bar{Z}(\phi U, V)W, \phi X) &= g(\bar{R}(\phi U, V)W, \phi X) - \frac{\bar{r}}{n(n-1)} [g(V, W)g(\phi U, \phi X) \\ &\quad - g(\phi U, W)g(V, \phi X)]. \end{aligned}$$

Using (7.1) in (7.2), we have

$$(7.3) \quad g(\bar{R}(\phi U, V)W, \phi X) = \frac{\bar{r}}{n(n-1)} [g(V, W)g(\phi U, \phi X) - g(\phi U, W)g(V, \phi X)].$$

Let  $\{e_1, e_2, \dots, e_{n-1}, \xi\}$  be a local orthonormal basis of vector fields in  $M$ , then  $\{\phi e_1, \phi e_2, \dots, \phi e_{n-1}, \xi\}$  is also a local orthonormal basis. Putting  $U = Z = e_i$  in (7.3) and summing over  $i = 1$  to  $n - 1$ , we obtain

$$(7.4) \quad \sum_{i=1}^{n-1} g(\bar{R}(\phi e_i, V)W, \phi e_i) = \frac{\bar{r}}{n(n-1)} \sum_{i=1}^{n-1} [g(V, W)g(\phi e_i, \phi e_i) - g(\phi e_i, W)g(V, \phi e_i)],$$

On LP-Sasakian manifold it can be verify that

$$(7.5) \quad \sum_{i=1}^{n-1} g(\bar{R}(\phi e_i, V)W, \phi e_i) = \bar{S}(V, W),$$

$$(7.6) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1,$$

$$(7.7) \quad \sum_{i=1}^{n-1} g(\phi e_i, W)g(V, \phi e_i) = g(V, W) + \eta(V)\eta(W).$$

So by virtue of (7.5), (7.6) and (7.7), the equation (7.4) takes the form

$$\bar{S}(V, W) = \left[ \frac{\bar{r}(n-2)}{n(n-1)} \right] g(V, W) - \left[ \frac{\bar{r}}{n(n-1)} \right] \eta(V)\eta(W).$$

or

$$\bar{S}(V, W) = ag(V, W) + b\eta(V)\eta(W),$$

where  $a = \left[ \frac{\bar{r}(n-2)}{n(n-1)} \right]$  and  $b = - \left[ \frac{\bar{r}}{n(n-1)} \right]$ .

From which it follows that the manifold is an  $\eta$ -Einstein manifold with respect to the quarter-symmetric metric connection.

Hence we can state the following theorem:

**Theorem 7.2.** *If a Lorentzian para-Sasakian manifold admitting a quarter-symmetric metric connection is quasi-concircularly flat, then the manifold with respect to the quarter-symmetric metric connection is an  $\eta$ -Einstein manifold.*

### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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