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ORDINARY LEAST SQUARES ESTIMATION OF PARAMETERS OF LINEAR MODEL

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Abstract: This research article primarily focuses on the method of ordinary least squares estimation of parameters of linear model. Here an innovative proof of Gauss-Markoff theorem for linear estimation has been presented. An extensive discussion in evaluating Best Linear Unbiased Estimator (BLUE) of a linear parametric function of the classical linear model is made by using the Gauss-Markoff theorem. Furthermore the importance of mean vector and covariance matrix of BLUE is discussed. Moreover generalized Gauss-Markoff theorem for linear estimation, properties of OLS estimators and problems of linear model by violating the assumptions are extensively discussed.

Keywords: heteroscedasticity; autocorrelation; mean vector; covariance matrix; positive definite symmetric matrix.

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1. INTRODUCTION

Regression analysis is a statistical method to establish the relationship between variables.

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Regression analysis has a wide number of applications in almost all fields of science, including Engineering, Physical and Chemical Sciences; Economics, Management, Social, Life and Biological Sciences. In fact, regression analysis may be the most frequently used statistical technique in practice.

Suppose that there exists a linear relationship between a dependent variable Y and an independent variable X. In the scatter diagram, if the points cluster around a straight line then the mathematical form of the linear model may be specified as

$$Y_i = \beta_0 + \beta_1 X_i, \quad i = 1, 2, \ldots n. \tag{1}$$

where β_0 is the intercept and β_1 is the slope.

Generally the data points in the scatter diagram do not fall exactly on a straight line, so equation (1) should be modified to account for this. Let the difference between the observed value of Y and the straight line $(\beta_0 + \beta_1 X)$ be an error ε. It is convenient to think of ε as a statistical error; that is, it is a random variable that accounts for the failure of the model to fit the data exactly. The error may be made up of the effects of other variables, measurement errors and so forth. Thus, a more plausible model may be specified as

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, i = 1, 2, \ldots, n. \tag{2}$$

Equation (2) is called a Linear Regression Model or Linear Statistical Model. Customarily X is called the independent variable and Y is called the dependent variable. However, this often causes confusion with the concept of statistical independence, so we refer to X as the Predictor or Regressor variable and Y as the Response variable. Since the equation (2) involves only one Regressor variable, it is called a ‘Simple Linear Regression Model’ or a ‘Two-Variable Linear Regression Model’.

A Three – variable Linear Regression Model may be written as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i, i = 1, 2, \ldots, n \tag{3}$$

This linear regression model contains two regressor variables. The term linear is used because eq. (3) is a linear function of the unknown parameters β_0, β_1 and β_2.

In general, the response variable Y may be related to k regressor or predictor variables. The model

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \ldots + \beta_k X_{ki} + \varepsilon_i, i = 1, 2, \ldots n \quad (4)$$

is called a 'Multiple Linear Regression Model' with k independent variables. The parameters β_j, j=0,1,2..,k are known as regression coefficients. This model describes a hyperplane in the k – dimensional space of the independent variables X_j's. The parameter β_j represents the expected change in the dependent variable Y per unit change in X_j, when all of the remaining predicted variables X_q's ($q \neq j$) are held constant. Thus, the parameters β_j, j = 1, 2,…, k are often known as 'Partial Regression Coefficients.

Multiple linear regression models are often used as empirical models or approximating functions. That is, the exact relationship between Y and $X_1, X_2, \ldots, X_k$ is unknown but over certain ranges of the independent variables, the linear regression model is an adequate approximation to the true unknown function.

In practice, certain nonlinear regression models such as cubic polynomial models and response surface models may often still be analyzed by multiple linear regression techniques. For instance, consider the cubic polynomial model

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \varepsilon_i, i = 1, 2, \ldots, n \quad (5)$$

Let $X_1 = X$, $X_2 = X^2$ and $X_3 = X^3$ then eq. (5) can be rewritten as

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i, i = 1, 2, \ldots, n \quad (6)$$

which is a multiple linear regression model with three independent variables.

2. Ordinary Least Squares Estimation of Parameters of Linear Model

Consider the Classical Linear Regression model

$$Y_{nx1} = X_{nxk} \beta_{kx1} + \varepsilon_{nx1} \quad (7)$$

with usual assumptions such as

$$E(\varepsilon)=0,\ E(\varepsilon\varepsilon')=\sigma^2 \mathrm{I}_n \tag{8}$$

Write the residual sum of squares as

$$e'e=\left(Y-X\hat{\beta}\right)'\left(Y-X\hat{\beta}\right) \tag{9}$$

$$=\mathrm{Y}'\mathrm{Y}-\hat{\beta}'\mathrm{X}'\mathrm{Y}-\mathrm{Y}'\mathrm{X}\hat{\beta}+\hat{\beta}'\mathrm{X}'\mathrm{X}\hat{\beta}$$

$$\Rightarrow\ \mathrm{e}'\mathrm{e}=\mathrm{Y}'\mathrm{Y}-2\hat{\beta}'\mathrm{X}'\mathrm{Y}+\hat{\beta}'\mathrm{X}'\mathrm{X}\hat{\beta} \qquad \left[\because \mathrm{Y}'\mathrm{X}\hat{\beta}=\hat{\beta}'\mathrm{X}'\mathrm{Y}\right]$$

where $\hat{\beta}$ is the least squares estimator of β

By the least squares estimation method, $\hat{\beta}$ minimizes the residual sum of squares $\mathrm{e}'\mathrm{e}$.

First order condition: $\frac{\partial}{\partial\hat{\beta}}\left(\mathrm{e}'\mathrm{e}\right)=\mathrm{O}\Rightarrow -2\mathrm{X}'\mathrm{Y}+2\mathrm{X}'\mathrm{X}\hat{\beta}=\mathrm{O}$

$$\Rightarrow \mathrm{X}'\mathrm{X}\hat{\beta}=\mathrm{X}'\mathrm{Y} \tag{10}$$

The system (10) contains ‘n’ simultaneous linear equations, which is called the ‘System of Normal Equations’. Since, the system of normal equations is always consistent, these exists at least a non-zero solution of $\hat{\beta}$,which gives the ordinary least squares (OLS) estimator of β.

i.e., $$\hat{\beta}=\left(X'X\right)^{-1}X'Y \tag{11}$$

Further, consider the OLS residual vector

$$e=Y-X\hat{\beta} \tag{12}$$

$$=\mathrm{X}\beta+\varepsilon-\mathrm{X}\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'(\mathrm{X}\beta+\varepsilon)$$

$$=\varepsilon-\mathrm{X}\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\varepsilon$$

$$=\left(\mathrm{I}_{\mathrm{n}}-\mathrm{X}\left(\mathrm{X}'\mathrm{X}\right)^{-1}\mathrm{X}'\right)\varepsilon$$ [I_n is a unit matrix of order n]

$$\Rightarrow e=M\varepsilon \tag{13}$$

where $M=\left(I_n-X\left(X'X\right)^{-1}X'\right)$ is a symmetric idempotent matrix such that $\mathrm{M}'\mathrm{M}=\mathrm{M}$, $\mathrm{M}'=\mathrm{M}$ and MX=O.

Now, consider the OLS residual sum of squares

$$e'e = (M\varepsilon)'(M\varepsilon) = \varepsilon' M\varepsilon$$

$$\Rightarrow E(e'e) = E(\varepsilon' M\varepsilon)$$

$$= E(trace\, \varepsilon' M\varepsilon) \; [\because \varepsilon' M\varepsilon \; is\; a\; scalar]$$

$$= E(trace\, M\varepsilon\varepsilon')$$

$$= (trace\, M) E(\varepsilon\varepsilon')$$

$$= \sigma^2 trace\, M \qquad \left[\because E(\varepsilon\varepsilon') = \sigma^2 I_n\right]$$

$$= \sigma^2 \mathrm{trace}\left(\mathrm{I_n} - \mathrm{X}(\mathrm{X'X})^{-1}\mathrm{X'}\right)$$

$$= \sigma^2\left[\mathrm{trace\, I_n} - \mathrm{trace}(\mathrm{X'X})^{-1}(\mathrm{X'X})\right]$$

$$= \sigma^2\left[n - trace\, I_k\right]$$

$$\Rightarrow E(e'e) = \sigma^2(n-k)$$

$$or\; E\left(\frac{e'e}{n-k}\right) = \sigma^2$$

$$E(S^2) = \sigma^2$$

$S^2 = \dfrac{e'e}{n-k}$ is an unbiased estimator of σ^2.

3. GAUSS-MARKOFF THEOREM FOR LINEAR ESTIMATION

This theorem is useful to find the Best Linear Unbiased Estimator (BLUE) of a linear parametric function of the classical linear regression model.

Statement: In the Gauss-Markoff linear model, $Y = X\beta + \varepsilon$ with usual assumptions; the BLUE of a linear parametric function $C'\beta$ is given by $C'\hat{\beta}$, where $\hat{\beta}$ is the ordinary least squares estimator of β. Here, C is a (kx1) vector of known coefficients.

Proof: Consider the Gauss-Markoff linear model

$$Y_{nx1} = X_{nxk}\beta_{kx1} + \varepsilon_{nx1} \tag{14}$$

such that, $E(\varepsilon) = O, E(\varepsilon\varepsilon') = \sigma^2 I_n$ and $\varepsilon \sim N(O, \sigma^2 I_n)$

Suppose that the linear parametric function $C'\beta$ is estimable. Then, there exists a linear function of observations vector $P'Y$ such that

$E(P'Y) = C'\beta$, P is a vector of unknown coefficients.

$$\Rightarrow P'X\beta = C'\beta \left[\because E(Y) = X\beta\right]$$

$$or\ X'P = C \tag{15}$$

One may have $Var(P'Y) = (P'P)Var(Y)$

$$= \sigma^2 (P'P)$$

The BLUE of $C'\beta$ can be obtained by minimizing the $Var(P'Y)$ with respect to P subject to the restriction $X'P = C$

Write the constrained minimization function as

$$\phi = P'P - 2\lambda'(X'P - C) \tag{16}$$

where λ is a (kx1) vector of unknown Lagrangian multipliers.

First order condition: $\frac{\partial \phi}{\partial P} = 0 \Rightarrow 2P - 2X\lambda = 0$ or $P = X\lambda$ (17)

From (15) and (17), one may obtain

$$X'X\lambda = C \tag{18}$$

The BLUE of $C'\beta$ is given by

$$P'Y = \lambda' X'Y \tag{19}$$

It can be shown that $\lambda' X'Y = C'\hat{\beta}$.

Here, $\hat{\beta} = (X'X)^{-1} X'Y$ is the OLS estimator of β.

From the ordinary least squares estimation method, one may write the system of normal equations as

$$X'X\hat{\beta} = X'Y$$

One may obtain, $\lambda'X'Y = \lambda'X'X\hat{\beta} = C'\hat{\beta}\left[\because X'XC = C\right]$

Hence, the BLUE of linear parametric function $C'\beta$ is given by $C'\hat{\beta}$, where $\hat{\beta} = \left(X'X\right)^{-1} X'Y$.

4. MEAN VECTOR AND COVARIANCE MATRIX OF BLUE

Consider the Gauss-Markoff linear model

$$Y_{nx1} = X_{nxk}\beta_{kx1} + \varepsilon_{nx1} \tag{20}$$

such that $E\left(Y\right) = X\beta$ and $Var\left(Y\right) = \sigma^2 I_n$. Suppose that a linear parametric function $C'\beta$ is estimable. Then, there exists a linear function of observation vector $\lambda'Y$ such that

$$\begin{aligned} &E\left(\lambda'Y\right) = C'\beta \quad \text{or} \quad \lambda'X\beta = C'\beta \\ &\text{or} \quad X'\lambda = C \end{aligned} \tag{21}$$

By the condition for the existence of BLUE, one may have

$$X'XP = C \tag{22}$$

$$\text{or} \quad \left.\begin{aligned} &\left(X'X\right)^{-1} C = P, \qquad \text{if} \quad \rho\left(X\right) = k \\ &\left(X'X\right)^{g} C = P, \qquad \text{if} \quad \rho\left(X\right) < k \end{aligned}\right\} \tag{23}$$

Further, one may write the BLUE for $C'\beta$ as

$$C'\hat{\beta} = \lambda'Y = P'X'Y \tag{24}$$

Consider, $Var\left(\lambda'Y\right) = \left(\lambda'\lambda\right) Var\left(Y\right) = \sigma^2\left(\lambda'\lambda\right) \quad \left[\because \lambda = XP\right]$

$$= \sigma^2\left(P'X'XP\right) \tag{25}$$

(i) $Var\left(\lambda'Y\right) = \sigma^2\left[C'\left(X'X\right)^{-1}\left(X'X\right)\left(X'X\right)^{-1} C\right] \quad \text{if} \quad \rho\left(X\right) = k$

$$\Rightarrow Var\left(\lambda'Y\right) = \sigma^2\left[C'\left(X'X\right)^{-1} C\right] \quad \text{if} \ \rho\left(X\right) = k \tag{26}$$

(ii) $\mathrm{Var}(\lambda'Y) = \sigma^2\left[C'(X'X)^g(X'X)(X'X)^g C\right]$ if $\rho(X) < k$

$$\Rightarrow \mathrm{Var}(\lambda'Y) = \sigma^2\left[C'(X'X)^g C\right] \quad \text{if} \quad \rho(X) < k \tag{27}$$

Thus, the mean vector and covariance matrix of BLUE $C'\hat{\beta}$ are given by

$$\text{(i) } E\left(C'\hat{\beta}\right) = C'\beta \tag{28}$$

and

$$\left.\begin{aligned} &\text{(ii) } \mathrm{Var}\left(C'\hat{\beta}\right) = \sigma^2\left[C'(X'X)^{-1}C\right] \quad \text{if } \rho(X) = k \\ &\text{and } \mathrm{Var}\left(C'\hat{\beta}\right) = \sigma^2\left[C'(X'X)^g C\right] \quad \text{if } \rho(X) < k \end{aligned}\right\} \tag{29}$$

Remarks: By taking C as a (kx1) vector of one's one may obtain,

$$\text{(i) } E\left(\hat{\beta}\right) = \beta \tag{30}$$

$$\left.\begin{aligned} &\text{and (ii) } \mathrm{Var}\left(\hat{\beta}\right) = \sigma^2(X'X)^{-1} \quad \text{if } \rho(X) = k \\ &\text{and } \mathrm{Var}\left(\hat{\beta}\right) = \sigma^2(X'X)^g \quad \text{if } \rho(X) < k \end{aligned}\right\} \tag{31}$$

5. GENERALIZED GAUSS-MARKOFF THEOREM FOR LINEAR ESTIMATION

One may obtain the Generalized Gauss-Markoff linear model by violating the assumption $E(\varepsilon\varepsilon') = \sigma^2 I_n$, in the Gauss- Markoff Linear model.

Consider the linear regression model,

$$Y_{n\times 1} = X_{n\times k}\,\beta_{k\times 1} + \varepsilon_{n\times 1} \tag{32}$$

such that

$$\left.\begin{aligned} &E(\varepsilon) = O \quad \text{or} \quad E(Y) = X\beta \quad \text{and} \quad E(\varepsilon\varepsilon') = \sigma^2\Omega \\ &\text{or } \mathrm{Var}(Y) = \sigma^2\Omega \end{aligned}\right\} \tag{33}$$

where σ^2 is known and Ω is a known positive definite symmetric matrix. The linear regression model (32) along with assumptions (33) is known as the Generalized Gauss- Markoff linear model, which was first given by Aitken in year 1932.

Statement: In the Generalized Gauss-Markoff linear model $Y_{nx1} = X_{nxk}\beta_{kx1} + \varepsilon_{nx1}$ such that

$E(\varepsilon) = O$, $E(\varepsilon\varepsilon') = \sigma^2\Omega$; the BLUE for a linear parametric function $C'\beta$ is given by $C'\tilde{\beta}$, where $\tilde{\beta}$ is the unique Generalized Least Squares (GLS) estimator for β, which can be obtained by solving a system of Generalized Normal Equations $\left(X'\Omega^{-1}X\right)\tilde{\beta} = \left(X'\Omega^{-1}Y\right)$. Also an unbiased estimator of σ^2 is given by

$\tilde{\sigma}^2 = \dfrac{e'\Omega^{-1}e}{n-r}$, where $e = Y - \hat{Y}$ is OLS residual vector and $\rho(X) = r$.

Here Ω is a known positive definite symmetric matrix.

Proof: Consider the Generalized Gauss-Markoff linear regression model,

$$Y_{nx1} = X_{nxk}\beta_{kx1} + \varepsilon_{nx1} \tag{34}$$

Such that $E(\varepsilon) = O$ and $E(\varepsilon\varepsilon') = \sigma^2\Omega$. Since, Ω is known positive definite symmetric matrix, these exists a nonsingular matrix M such that $MM' = \Omega$ or $\left(M'\Omega^{-1}M\right) = I$ (35)

Pre-multiplying on both of sides (34) by M^{-1} gives

$M^{-1}Y = M^{-1}X\beta + M^{-1}\varepsilon$

or $Y^* = X^*\beta + \varepsilon^*$ (36)

where $Y^* = M^{-1}Y$, $X^* = M^{-1}X$ and $\varepsilon^* = M^{-1}\varepsilon$

Consider (i) $E\left(\varepsilon^*\right) = E\left(M^{-1}\varepsilon\right) = M^{-1}E(\varepsilon) = O$ (37)

$$\text{(ii)}\quad E\left(\varepsilon^*\varepsilon^{*'}\right) = E\left[\left(M^{-1}\varepsilon\right)\left(M^{-1}\varepsilon\right)'\right]$$

$$= \left[M^{-1}E(\varepsilon\varepsilon')M^{-1'}\right]$$

$$= \sigma^2\left(M^{-1}\Omega M'^{-1}\right)$$

$$= \sigma^2\left(M'\Omega^{-1}M\right)^{-1}$$

$$\Rightarrow E\left(\varepsilon^{*}\varepsilon^{*'}\right) = \sigma^2 I \tag{38}$$

Thus, the Generalized Gauss-Markoff linear regression model reduces to an ordinary Gauss-Markoff linear model given in (36), (37) and (38).

Now, the unique GLS estimator of β can be obtained by solving the system of normal equations,

$$X^{*'}X^{*}\tilde{\beta} = X^{*'}Y^{*}$$

$$\Rightarrow \left(M^{-1}X\right)'\left(M^{-1}X\right)\tilde{\beta} = \left(M^{-1}X\right)'\left(M^{-1}Y\right)$$

$$\Rightarrow \left[X'\left(MM'\right)^{-1}X\right]\tilde{\beta} = X'\left(MM'\right)^{-1}Y$$

$$\Rightarrow \left(X'\Omega^{-1}X\right)\tilde{\beta} = X'\Omega^{-1}Y \tag{39}$$

One BLUE of $C'\beta$ is given by $C'\tilde{\beta}$

where $\tilde{\beta} = \left(X'\Omega^{-1}X\right)^{-1}\left(X'\Omega^{-1}Y\right)$, if $\rho(X) = k$

or $\tilde{\beta} = \left(X'\Omega^{-1}X\right)^{g}\left(X'\Omega^{-1}Y\right)$, if $\rho(X) < k$

Also an unbiased estimator of error variance σ^2 is given by

$$\tilde{\sigma}^2 = \frac{e^{*'}e^{*}}{n-r}, \quad \text{where } e^{*} = Y^{*} - X^{*}\hat{\beta}$$

$$\text{or } \tilde{\sigma}^2 = \frac{\left(Y - X\tilde{\beta}\right)'\left(MM'\right)^{-1}\left(Y - X\beta\right)}{n-r} = \frac{e'\Omega^{-1}e}{n-r}, \quad \text{where } r = \rho(X) \tag{40}$$

6. Properties of OLS Estimators

Following are some important properties of the OLS estimators $\hat{\beta}$ and $\hat{\sigma}^2$:

(i) The OLS estimator $\hat{\beta}$ is the BLUE for β. The mean vector and the covariance matrix are respectively given by β and $\sigma^2\left(X'X\right)^{-1}$.

(ii) The OLS estimator $\hat{\beta}$ is the maximum likelihood estimator for β and hence, it is consistent.

(iii) The OLS estimator $\hat{\beta}$ follows multivariate normal distribution with mean vector β and the covariance matrix $\sigma^2\left(X'X\right)^{-1}$.

(iv) One OLS estimator $\hat{\sigma}^2 = \frac{e'e}{n-k}$ is an unbiased and consistent estimator of σ^2.

(v) $\left[\frac{e'e}{\sigma^2}\right]$ or $\left[\frac{(n-k)\hat{\sigma}^2}{\sigma^2}\right]$ follows χ^2-distribution with (n-k) degrees of freedom.

(vi) The variance of $\hat{\sigma}^2$ is given by $\operatorname{Var}\left(\hat{\sigma}^2\right) = \frac{2\sigma^4}{n-k}$ (41)

(vii) The OLS estimators $\hat{\beta}$ and $\hat{\sigma}^2$ are the efficient estimators of β and σ^2 respectively.

(viii) The OLS estimators $\hat{\beta}$ and $(n-k)\hat{\sigma}^2$ are two joint sufficient statistics of β and σ^2.

(ix) $\left(\frac{n-k}{n}\right)\hat{\sigma}^2$ is the maximum likelihood estimator for σ^2.

(x) The Rao – Cramer lower bounds for the variances of $\hat{\beta}$ and $\hat{\sigma}^2$ are respectively given by $\sigma^2\left(X'X\right)^{-1}$ and $\frac{2\sigma^4}{n}$.

(xi) The OLS estimators $\hat{\beta}$ and $\hat{\sigma}^2$ are asymptotically efficient estimators of β and σ^2 respectively.

(xii) $\sqrt{n}\left(\hat{\beta}-\beta\right) \overset{asy}{\sim} N\left[O, \sigma^2\left(\underset{n\to\infty}{Lim}\left(\frac{X'X}{n}\right)^{-1}\right)\right]$

and $\sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \overset{asy}{\sim} N\left[O, 2\sigma^4\right]$.

7. PROBLEMS OF LINEAR MODEL BY VIOLATING THE ASSUMPTIONS

Several problems can be arised by violating the crucial assumptions about the linear regression model such as:

(i) Problems of biased and Inconsistent estimators for β will be arised if $E(\varepsilon) \neq 0$;

(ii) Problems of heteroscedasticity and autocorrelation will be arised if $E(\varepsilon\varepsilon') \neq \sigma^2 I_n$;

(iii) Problem of multicollinearlity will be arised if $\rho(X) \neq k$;

(iv) Problem of stochastic regressors will be arised if the data matrix X is a stochastic matrix;

(v) Problem of errors in variables will be arised if there are errors in the independent variables;

(vi) Problems of non-normal errors and non-parametric linear regression analysis will be arised if ε does not follow multivariate normal distribution;

(vii) Problem of Random Coefficient Regression (RCR) models will arise if the regression coefficients governed by some probability distribution.

8. CONCLUSION AND FUTURE RESEARCH

In the above talk an innovative proof of Gauss-Markoff theorem for linear estimation has been presented. An extensive discussion in evaluating Best Linear Unbiased Estimator (BLUE) of a linear parametric function of the classical linear model is made by using the Gauss-Markoff theorem. Furthermore the importance of mean vector and covariance matrix of BLUE is discussed. Moreover generalized Gauss-Markoff theorem for linear estimation, properties of OLS estimators and problems of linear model by violating the assumptions are extensively discussed. In the context of future research one can extend these ideas to nonlinear regression models and can derive some useful results in estimating the parameters of nonlinear statistical models.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] J.P. Praveen, R. Donthi, S.V. Prasad, B. Mahaboob, B. Venkateswarlu, A glance on the estimation of Cobb-Douglas production functional model, AIP Conf. Proc. 2177 (2019), 020067.

[2] J.P. Praveen, B. Mahaboob, R. Donthi, S.V. Prasad, B. Venkateswarlu, On stochastic linear regression model selection, AIP Conf. Proc. 2177 (2019), 020068.

[3] B. Mahaboob, J.P. Praveen, R. Donthi, S.V. Prasad, B. Venkateswarlu, Criteria for selection of stochastic linear model selection, AIP Conf. Proc. 2177 (2019), 020041.

[4] R. Donthi, S.V. Prasad, B. Mahaboob, J.P. Praveen, B. Venkateswarlu, Estimation methods of nonlinear regression models, AIP Conf. Proc. 2177 (2019), 020081.

[5] R. Donthi, J.P. Praveen, S.V. Prasad, B. Mahaboob, B. Venkateswarlu, Numerical techniques of nonlinear regression model estimation, AIP Conf. Proc. 2177 (2019), 020082.

[6] B. Mahaboob, S.V. Prasad, J.P. Praveen, R. Donthi, B. Venkateswarlu, On misspecification tests for stochastic linear regression model, AIP Conf. Proc. 2177 (2019), 020039.

[7] B. Mahaboob, K.A. Ajmath, B. Venkateswarlu, C. Narayana, J.P. Praveen, On Cobb-Douglas production function model, AIP Conf. Proc. 2177 (2019), 020040.

[8] B. Mahaboob, K.A. Azmath, B. Venkateswarlu, C. Narayana, B.M. Rao, An evaluation in generalized LSE of linearized stochastic statistical model with non-spherical errors, AIP Conf. Proc. 2177 (2019), 020038.

[9] J.P. Praveen, B.N. Rao, Y. Haranadh, B.V.A. Rao, C. Narayana, B. Mahaboob, Existence of Solution of the Nonlinear Differential Equation in the Modeling of Eardrum by Using Homotopy Perturbation Method, Adv. Math.: Sci. J. 9 (2020), 4853–4862

[10] B. Mahaboob, J.P. Praveen, B.V.A. Rao, Y. Haranadh, C. Narayana, G.B. Prakash, A Study On Multiple Linear

Regression Using Matrix Calculus, Adv. Math.: Sci. J. 9 (2020), 4863–4872.

[11] J.P. Praveen, B.N. Rao, B. Mahaboob, C. Rajaiah, Y. Harnath, C. Narayana, Series Decomposition Method For Asymmetric Nonlinear Oscillations, Adv. Math.: Sci. J. 9 (2020), 8069–8076.

[12] J.P. Praveen, B.N. Rao, B. Mahaboob, B.V.A. Rao, An Application of Laplace Transform, Inverse Laplace Transform and Pade's Approximant in the Periodic Solution of Duffing Equation of Motion, Int. J. Emerg. Trends Eng. Res. 8(9) (2020), 5979-5983.

[13] J.P. Praveen, B.N. Rao, B. Mahaboob, M. Rajaiah, Y. Harnath, C. Narayana, On The Simulation of Beck Column through a Simple Xiong-Wang-Tabarrok Experimental Model of Centripetally Loaded Column, Int. J. Emerg. Trends Eng. Res. 8(9) (2020), 5100-5103.

[14] B. Mahaboob, C. Narayana, P.S. Reddy, Ch. Suresh, G.B. Prakash, Y.H. Krishna, Methods And Applications of Linear Regression Models, Eur. J. Mol. Clinic. Med. 7 (3) (2020), 4873-4881.

[15] B. Venkateswarlu, B. Mahaboob, K.A. Ajmath, C. Narayana, Application of DEA in Super Efficiency Estimation, Int. J. Sci. Technol. Res. 9(2) (2020), 4496-4499.

[16] B. Venkateswarlu, B. Mahaboob, C.S. Reddy, C. Narayana, Evaluating Different Types of Efficiency Stability Regions and Their Infeasibility in DEA, Int. J. Sci. Technol. Res. 9(2) (2020), 3944-3949.

[17] B. Venkateswarlu, B. Mahaboob, C. Subbaramireddy, C. Narayana, Multi-Criteria Optimization Techniques in DEA: Methods &Applications, Int. J. Sci. Technol. Res. 9(2) (2020), 509-515.

[18] B. Mahaboob, B. Venkateswarlu, K.A. Azmath, C. Narayana, J.P. Praveen, On OLS Estimation of Stochastic Linear Regression Model, Int. J. Eng. Adv. Technol. 8(6) (2019), 1953-1955.

[19] B. Mahaboob, B. Venkateswarlu, C. Narayana, M. Sivaiah, Bias in the Maximum Likelihood Estimation of Parameters of Nonlinear Regression Models, Int. J. Sci. Technol. Res. 8(11), (2019), 1252-1255.

[20] B. Venkateswarlu, B. Mahaboob, C. Narayana, C. Subbarami Reddy, Evaluation of Slack Based Efficiency of a Decision Making Unit, Int. J. Sci. Technol. Res. 8(11) (2019), 1178-1182

[21] B. Venkateswarlu, B. Mahaboob, K.A. Azmath, C. Narayana, C. Muralidaran, An Application of Linear Programming in the Estimation of Technical Efficiency of DMU, Int. J. Eng. Adv. Technol. 8(6) (2019), 1956-1959.

[22] B. Venkateswarlu, B. Mahaboob, C. Subbarami Reddy, C. Narayana, New Results Production Theory by

Applying Goal Programming, Int. J. Sci. Technol. Res. 8(12) (2019), 1918-1923.

[23] B. Mahaboob, B. Venkateswarlu, C. Narayana, J.R. Sankar, P. Balasiddamuni, A Treatise on Ordinary Least Squares Estimation of Parameters of Linear model, Int. J. Eng. Technol. 7 (2018), 518-522.

[24] B. Mahaboob, B. Venkateswarlu, C. Narayana, J.R. Sankar, P. Balasiddamuni, A Monograph on Nonlinear Regression Models, Int. J. Eng. Technol. 7 (2018), 543-546.

[25] B. Venkateswarlu, B. Mahaboob, J.R. Sankar, C. Narayana, B.M. Rao, An Application of Goal Programming in Data Envelopment Analysis, Int. J. Eng. Technol. 7 (2018), 523-525.

[26] C. Narayana, B. Mahaboob, B. Venkateswarlu, J.R. Sankar, A Memoir on Model Selection Criterion between Two Nested and Non-Nested Stochastic Nonlinear egression Models, Int. J. Eng. Technol. 7 (2018), 529-531.

[27] C. Narayana, B. Mahaboob, B. Venkateswarlu, J.R. Sankar, P. Balasiddamuni, A Study on Misspecification and Predictive Accuracy of Stochastic Linear Regression Models, Int. J. Eng. Technol. 7 (2018), 532-535.

[28] C. Narayana, B. Mahaboob, B. Venkateswarlu, J.R. Sankar, P. Balasiddamuni, A Discourse on Modified Likelihood Ratio (LR), Wald and Lagrange Multipliers (LM) Tests for Testing General Linear Hypothesis in Stochastic Linear Regression Model, Int. J. Eng. Technol. 7 (2018), 536-538.

[29] C. Narayana, B. Mahaboob, B. Venkateswarlu, J.R. Sankar, P. Balasiddamuni, A Treatise on Testing General Linear Hypothesis in Stochastic Linear Regression Model, Int. J. Eng. Technol. 7 (2018), 539-542.

[30] B. Mahaboob, B. Venkateswarlu, J.R. Sankar, Estimation of parameters of constant elasticity of substitution production functional model, IOP Conf. Ser.: Mater. Sci. Eng. 263 (2017), 042121.

[31] B. Mahaboob, B. Venkateswarlu, J.R. Sankar, P. Balasiddamuni, Computation of nonlinear least squares estimator and maximum likelihood using principles in matrix calculus, IOP Conf. Ser.: Mater. Sci. Eng. 263 (2017), 042125.

[32] B. Venkateswarlu, B. Mahaboob, C.S. Reddy, J.R. Sankar, A study on technical efficiency of a DMU (review of literature), IOP Conf. Ser.: Mater. Sci. Eng. 263 (2017), 042124.

[33] B. Mahaboob, B. Venkateswarlu, G. Mokeshrayalu, P. Balasiddamuni, New methods of testing nonlinear hypothesis using iterative NLLS estimator, IOP Conf. Ser.: Mater. Sci. Eng. 263 (2017), 042126.

[34] B. Venkateswarlu, B. Mahaboob, C.S. Reddy, B.M. Rao, Fitting of Full Cobb-Douglas and Full VRTS Cost Frontiers by solving Goal Programming Problem, IOP Conf. Ser.: Mater. Sci. Eng. 263 (2017), 042127.

[35] B. Mahaboob, B. Venkateswarlu, G. Mokeshrayalu, P. Balasiddamuni, A Different Approach to Estimate Nonlinear Regression Model using Numerical Methods, IOP Conf. Ser.: Mater. Sci. Eng. 263 (2017), 042122.

[36] M. Harish, S.S. Rao, B.N. Rao, B. Mahaboob, J.P. Praveen, A.I. Reddy, Specific optimal AWJM process parameters for Ti-6Al-4V alloy employing the modified Taguchi approach, J. Math. Comput. Sci. 11 (2021), 292-311.

[37] J.P. Praveen, B.N. Rao, Uncertainties in the periodic solution of a Truly nonlinear oscillator Differential Equation using MDTM, Int. J. Appl. Comput. Math. 5 (2019), 119.

[38] P.P. Jakkana, N.R. Boggarapu, M. Bodanapu, et al. Revisited the critical load assessment of Huang et al. on Willems Tested Beck column, Pertanika J. Sci. Tecchnol. 29 (2021), 251-262.