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SECOND ORDER ROTATABLE DESIGNS OF SECOND TYPE USING BALANCED INCOMPLETE BLOCK DESIGNS

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Abstract: In this paper second order rotatable designs of second type using balanced incomplete block designs is suggested. This design is compared with second order rotatable designs of first type using balanced incomplete block designs (cf. Das and Narasimham [6]) on the basis of efficiency.

Keywords: response surface methodology; balanced incomplete block design, rotatability; orthogonality, efficiency.

2010 AMS Subject Classification: 62K05.

1. INTRODUCTION

Response surface designs is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable or response. Box and Hunter [1] introduced designs having spherical variance function are called rotatable designs. Das and Narasimham [6] constructed rotatable designs using balanced incomplete block designs (BIBD). Raghavarao [21] constructed second order rotatable designs (SORD) using incomplete block designs. Draper and Guttman [7] suggested an index of

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rotatability. Khuri [14] introduced a measure of rotatability for response surface designs. Draper and Pukelshein [8] developed another look at rotatability. Park et al. [16] introduced new measure of rotatability for second order response surface designs. Das et al. [5] developed modified response surface designs. Kim [13] introduced extended central composite designs (CCD) with the axial points are indicated by two numbers. Victorbabu and Vasundharadevi [25] suggested modified second order response surface designs using BIBD. Victorbabu [23] constructed modified SORD and second order slope rotatable designs using a pair of BIBD. Victorbabu et al. [26] studied modified second order response surface designs using pairwise balanced designs (PBD). Victorbabu [22] suggested a review on SORD. Victorbabu et al. [27] suggested modified second order response surface designs using CCD. Victorbabu and Vasundharadevi [28] studied second order response surface designs using SUBA with two unequal block sizes. Victorbabu [24] constructed modified SORD using a pair of SUBA with two unequal block sizes. Park and Park [17] suggested the extension of CCD for second order response surface models. Victorbabu and Surekha [29] suggested measure of rotatability for second order response surface designs using incomplete block designs. Victorbabu and Surekha [30] developed measure of rotatability for second order response surface designs using BIBD. Victorbabu et al. [31-32] studied measure of rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes and a pair of BIBD. Jyostna et al. [9] suggested measure of rotatability for second degree polynomial using CCD. Jyostna and Victorbabu [10-12] studied measure of modified rotatability for second degree polynomials using BIBD, PBD and SUBA with two unequal block sizes. Chiranjeevi et al. [2] extended the work of Kim [13] and suggested SORD of second type using CCD for $9 \leq v \leq 17$ (v: number of factors). Chiranjeevi and Victorbabu [3-4] studied SORD of second type using SUBA with two unequal block sizes and PBD.

In this paper, second order rotatable designs of second type using balanced incomplete block designs is suggested. This design is compared with second order rotatable designs of first type using balanced incomplete block designs (Das and Narasimham [6]) on the basis of efficiency.

2. PRELIMINARIES

Stipulations and formulas for second order rotatable designs

Suppose we want use the second order polynomial response surface design D = ((x_{iu})) to fit the surface,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j}\sum b_{ij} x_{iu} x_{ju} + \varphi_u \quad (2.1)$$

where x_{iu} represents the level of i^{th} factor (i=1,2,…,v) in the u^{th} run (u=1,2,…,N) of the experiment and φ_u are uncorrelated random error with mean zero and variance σ^2. Then 'D' is said to be SORD if the variance of $Y_u(x_1, x_2, \ldots, x_v)$ with respect to each of independent variable (x_i) is only a function of the distance $(d^2 = \sum_{i=1}^{v} x_i^2)$ of the point $(x_1, x_2, \ldots, x_v)$ from the origin (center) of the design such a spherical variance function for estimation of responses in the second order polynomial model is achieved if the design points satisfy the following the following conditions (cf. Box and Hunter [1]).

All odd order moments are must be zero. In their words when at least one odd power x's equal to zero.

1. $\sum x_{iu} = 0, \sum x_{iu} x_{ju} = 0, \sum x_{iu} x_{ju}^2 = 0, \sum x_{iu} x_{ju} x_{ku} = 0,$
$\sum x_{iu}^3 = 0, \sum x_{iu} x_{ju}^3 = 0, \sum x_{iu} x_{ju} x_{ku}^2 = 0, \sum x_{iu} x_{ju} x_{ku} x_{lu} = 0.$

for i≠j≠k≠l; (2.2)

2. (i) $\sum x_{iu}^2 = \text{constant} = N\mu_2$ (ii) $\sum x_{iu}^4 = \text{constant} = cN\mu_4$ for all i (2.3)

3. $\sum x_{iu}^2 x_{ju}^2 = \text{constant} = N\mu_4; \text{for all } i \neq j$ (2.4)

4. $\frac{\mu_4}{\mu_2^2} > \frac{v}{(c+v-1)}$ (2.5)

5. $\sum x_{iu}^4 = c \sum x_{iu}^2 x_{ju}^2$ (2.6)

where c, μ_4 and μ_2 are constants.

The variances and covariances of the estimated parameters are

$$V(\hat{b}_0)=\frac{\mu_4(c+v-1)\sigma^2}{N[\mu_4(c+v-1)-v\mu_2^2]},$$

$$V(\hat{b}_i)=\frac{\sigma^2}{N\mu_2},$$

$$V(\hat{b}_{ij})=\frac{\sigma^2}{N\mu_4},$$

$$V(\hat{b}_{ii})=\frac{\sigma^2}{(c-1)N\mu_4}\left[\frac{\mu_4(c+v-2)-(v-1)\mu_2^2}{\mu_4(c+v-1)-v\mu_2^2}\right],$$

$$Cov(\hat{b}_0,\hat{b}_{ii})=\frac{-\mu_2\sigma^2}{N[\mu_4(c+v-1)-v\mu_2^2]},$$

$$Cov(\hat{b}_{ii},\hat{b}_{jj})=\frac{(\mu_2^2-\mu_4)\sigma^2}{(c-1)N\mu_4[\mu_4(c+v-1)-v\mu_2^2]} \quad \text{and other covariances vanish.} \tag{2.7}$$

The variance of the estimated response at the point ($x_{10}, x_{20},\ldots, x_{v0}$) is

$$\begin{aligned} V(\hat{Y}_0)=V(\hat{b}_0)+\left[V(\hat{b}_i)+2Cov(\hat{b}_0,\hat{b}_{ii})\right]d^2+V(\hat{b}_{ii})d^4+ \\ \sum x_{i0}^2x_{j0}^2\left[V(\hat{b}_{ij})+2Cov(\hat{b}_{ii},\hat{b}_{jj})-2V(\hat{b}_{ii})\right] \end{aligned} \tag{2.8}$$

The coefficient of $\sum x_{i0}^2x_{j0}^2$ in the above equation (2.8) is simplified to $(c-3)\sigma^2/(c-1)N\mu_4$.

A second order response surface design D is said to be rotatable, if in this design c=3 and all the other conditions (2.2) to (2.7) are satisfied.

3. MAIN RESULTS

3.1. SORD of first type using balanced incomplete block designs (cf. Das and Narasimham [6])

Balanced incomplete block design:

The parameters of BIBD denote by (v, b, r, k, λ) is an arrangement of v- treatments in b blocks each block contains k (<v) treatments, if

(i) Every treatment occurs at most once in each block

(ii) Each treatment occurs in exactly r blocks and

(iii) Every pair of treatments occurs together in λ times.

Incidence matrix:

Let (v, b, r, k, λ) denote parameters of a BIBD. Associated with any design D is the incidence Matrix N $v \times b = ((\eta_{ij}))$, (i=1,2,...,v; j=1,2,...,b) where η_{ij} denote the number of times the j^{th} treatment occurs in the i^{th} block.

Where

$\eta_{ij}=1$ if the j^{th} the treatment occurs in the i^{th} block

=0, otherwise

For example the plan and incidence matrix of BIBD is given as following design (v=3, b=3, r=2, k=2, λ=1).

Plan of BIBD

1	2
2	3
1	3

Incidence matrix of BIBD

		Treatments		
		1	**2**	**3**
	I	1	1	0
Blocks	**II**	0	1	1
	III	1	0	1

Generation of design points

	x_1	x_2	x_3
1	1	1	0
2	1	-1	0
3	-1	1	0
4	-1	-1	0
5	0	1	1
6	0	1	-1
7	0	-1	1
8	0	-1	-1
9	1	0	1
10	1	0	-1
11	-1	0	1
12	-1	0	-1
13	a	0	0
14	-a	0	0
15	0	a	0
16	0	-a	0
17	0	0	a
18	0	0	-a
19	0	0	0

Here the total number of experimental points $N=b2^{t(k)}+2v+n_0$.

Let (v, b, r, k, λ) denote parameters of BIBD, $2^{t(k)}$ denote a fractional replicate of 2^k in +1 or -1 levels in which no interaction with less than five factors are confounded. $[1-(v, b, r, k, \lambda)]$ denote the design points generated from transpose of the incidence matrix of BIBD. $[1-(v,b,r,k,\lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from BIBD by "multiplication" (cf. Raghavarao [18]), pp 298-300). We use the additional set of points like $(\pm a,0,\ldots,0),(0,\pm a,0,\ldots,0),\ldots,(0,0,\ldots,\pm a)$ etc. Here $(a,0,0,\ldots,0)2^1$ denote the 2v axial points generated from $(a,0,0,\ldots,0)$ point set. Let U denote the union of the design points generated from different sets of points, (n_0) denote the number of central points. The method of SORD of first type using BIBD (cf. Das and Narasimham [6]) is given in the following result.

Result: The design points, $[1\text{-}(v,b,r,k,\lambda)]2^{t(k)}U(a,0,\ldots,0)2^1U(n_0)$ will give a v-dimensional SORD of first type using BIBD in $N=b2^{t(k)}+2v+n_0$ design points, with

$$a^4=\frac{2^{t(k)}(3\lambda\text{-r})}{2}.$$

The condition for the design becomes an orthogonal design.

From equation 2 (i) of (2.3) and (3) of (2.4), we have

$$\sum x_{iu}^2=r2^{t(k)}+2a^2=N\mu_2$$

$$\sum x_{iu}^2 x_{ju}^2=\lambda 2^{t(k)}=N\mu_4$$

For the convenience N is replaced by M

By using the orthogonality condition we have

$$\mu_2^2=\mu_4$$

$$\left(\frac{r2^{t(k)}+2a^2}{M}\right)^2=\frac{\lambda 2^{t(k)}}{M}$$

then we can obtain

$$a^2=\left(\frac{\sqrt{\lambda 2^{t(k)}M}\text{-}r2^{t(k)}}{2}\right)(\text{for orthogonality}) \tag{3.1}$$

and the condition for the design become rotatability.

From equation 2 (ii) of (2.3) and 3 of (2.4), we have

$$\sum x_{iu}^4=r2^{t(k)}+2a^4=3N\mu_4$$

$$\sum x_{iu}^2 x_{ju}^2=\lambda 2^{t(k)}=N\mu_4$$

For the convenience N is replace by M

Then the rotatability condition equation (2.6), we have

$$\sum x_{iu}^4=c\sum x_{iu}^2 x_{ju}^2$$

$$\Rightarrow r2^{t(k)}+2a^4=3(\lambda 2^{t(k)})$$

$$\Rightarrow a^4=\frac{2^{t(k)}(3\lambda\text{-r})}{2}\ (\text{for rotatability}) \tag{3.2}$$

3.2. Proposed method of SORD of second type using balanced incomplete block design

Following methods construction of Das and Narasimham [6] and Kim [13], Here we studied SORD of second type using BIBD is given bellow.

In the same design (v=3, b=3, r=2, k=2, λ=1) generation of design points in BIBD

	X_1	X_2	X_3
1	1	1	0
2	1	-1	0
3	-1	1	0
4	-1	-1	0
5	0	1	1
6	0	1	-1
7	0	-1	1
8	0	-1	-1
9	1	0	1
10	1	0	-1
11	-1	0	1
12	-1	0	-1
13	a_1	0	0
14	$-a_1$	0	0
15	0	a_1	0
16	0	$-a_1$	0
17	0	0	a_1
18	0	0	$-a_1$
19	a_2	0	0
20	$-a_2$	0	0
21	0	a_2	0
22	0	$-a_2$	0
23	0	0	a_2
24	0	0	$-a_2$
25	0	0	0

Here number of central points is an integer are greater than or equal to 1 In BIBD the axial points are indicated by two numbers a_1 and a_2 and $a_2 \geq a_1 > 0$, the total number of experimental points are $N = b2^{t(k)} + 4v + n_0$

Let (v, b, r, k, λ) denote parameters of BIBD, $2^{t(k)}$ denote a fractional replicate of 2^k in +1 or -1 levels in which no interaction with less than five factors are confounded. [1- (v, b, r, k, λ)] denote the design points generated from transpose of the incidence matrix of BIBD. Let $[1\text{-}(v, b, r, k, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from BIBD by "multiplication" (cf. Raghavarao [18], pp 298-300). We use the additional set of points like $(\pm a_1, 0, \ldots, 0), (0, \pm a_1, 0, \ldots, 0), \ldots, (0, 0, \ldots, \pm a_1)$ and $(\pm a_2, 0, \ldots, 0), (0, \pm a_2, 0, \ldots, 0), \ldots, (0, 0, \ldots, \pm a_2)$ are two sets of axial points. Here $(a_1, 0, 0, \ldots, 0)2^1 U (a_2, 0, 0, \ldots, 0)2^1$ denote the 4v axial points generated from $(a_1, 0, 0, \ldots, 0)$ and $(a_2, 0, 0, \ldots, 0)$ point sets. Let U denote the union of the design points generated from different sets of points, and (n_0) denote the number of central points. The method of construction of SORD of second type (cf. Kim [13]) using BIBD is given in the following theorem.

Theorem (1): The design points,

$[1\text{-}(v, b, r, k, , \lambda)]2^{t(k)} U (a_1, 0, \ldots, 0)2^1 U (a_2, 0, \ldots, 0)2^1 U (n_0)$ will give a

v-dimensional SORD of second type using BIBD in $N = b2^{t(k)} + 4v + n_0$ design points, with

$$a_1^4 + a_2^4 = \frac{2^{t(k)}(3\lambda - r)}{2} \quad (\text{for rotatability})$$

Proof: For the design points generated from SORD of second type using BIBD, simple symmetry conditions (2.2) are true. Further, conditions (2.3) and (2.4) are true as follows:

$$\sum x_{iu}^2 = r2^{t(k)} + 2(a_1^2 + a_2^2) = N\mu_2 \tag{3.3}$$

$$\sum x_{iu}^4 = r2^{t(k)} + 2(a_1^4 + a_2^4) = cN\mu_4 \tag{3.4}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} = N\mu_4 \tag{3.5}$$

Solving the equations (3.4) and (3.5), we get $a_1^4 + a_2^4 = \frac{2^{t(k)}(3\lambda - r)}{2}$ (for rotatability).

Example 1: We illustrate the theorem (1) to obtain a SORD of second type using BIBD with parameters (v=3, b=3, r=2, k=2, λ=1). The design points $[1\text{-}(3,3,2,2,1)]2^{t(2)} U (a_1, 0, 0, \ldots, 0)2^1 U (a_2, 0, 0, \ldots, 0)2^1 U (n_0 = 1)$ will give v-dimensional SORD of second type using BIBD in N=25 design points with one central point. From (3.3), (3.4) and (3.5) we have

$$\sum x_{iu}^2 = 8 + 2(a_1^2 + a_2^2) = N\mu_2 \quad (3.6)$$

$$\sum x_{iu}^4 = 8 + 2(a_1^4 + a_2^4) = cN\mu_4 \quad (3.7)$$

$$\sum x_{iu}^2 x_{ju}^2 = 4 = N\mu_4 \quad (3.8)$$

From (3.7) and (3.8), we can obtain the rotatability value $a_1^4 + a_2^4 = 2$, here we assume an arbitrary value $a_1=1$, then we get $a_2=1$ and $c=3$ (for rotatability).

The Non singularity condition (2.5), we have

$$\frac{0.16}{0.16} > \frac{3}{3+3-1} \Rightarrow 1.0000 > 0.6$$

Hence the non singularity condition is also satisfied.

The variance and covariance of the estimated parameters are given as fallows

$$V(\hat{b}_0) = 0.1\sigma^2$$

$$V(\hat{b}_i) = 0.1\sigma^2$$

$$V(\hat{b}_{ij}) = 0.25\sigma^2$$

$$V(\hat{b}_{ii}) = 0.125\sigma^2$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = -0.05\sigma^2$$

$$Cov(\hat{b}_{ii}, \hat{b}_{jj}) = 0 \quad (3.9)$$

The variance of the estimated response at the point $(x_{10}, x_{20}, \ldots, x_{v0})$ is

$$V(\hat{Y}) = 0.1539\sigma^2 + d^2(0.2433\sigma^2) + d^4(0.0403\sigma^2) \quad (3.10)$$

Table 1 gives the values of the variance of the estimated responses for different factors of SORD of second type using BIBD.

Table1: The variance of estimated response for different factors for $3 \leq v \leq 15$ (by using theorem 1)

(v, b, r, k, λ)	n_0	N	$a_1^4+a_2^4$	a_1	a_2	$V(\hat{Y})$
(3,3,2,2,1)	1	25	2	1	1	$0.1\sigma^2 + d^4(0.125\sigma^2)$
(4,4,3,3,2)	1	49	12	1	1.8212	$0.0612\sigma^2 + d^4(0.0313\sigma^2)$
(5,5,4,4,3)	1	101	40	1	2.4989	$0.0346\sigma^2 + d^4(0.0104\sigma^2)$
(6,10,5,3,2)	1	105	4	1	1.3161	$0.0381\sigma^2 + d^4(0.0312\sigma^2)$
(7,7,4,4,2)	1	141	16	1	1.9679	$0.0319\sigma^2 + d^4(0.0156\sigma^2)$
(8,14,7,4,3)	6	262	16	1	1.9679	$0.0194\sigma^2 + d^2(-0.0001\sigma^2) + d^4(0.0104\sigma^2)$
(9,18,8,4,3)	18	342	8	1	1.6265	$0.0160\sigma^2 + d^4(0.0104\sigma^2)$
(10,18,9,5,4)	1	329	24	1	2.1899	$0.0183\sigma^2 + d^2(-0.0001\sigma^2) + d^4(0.0078\sigma^2)$
(11,11,5,5,2)	1	221	8	1	1.6265	$0.0294\sigma^2 + d^2(0.0001\sigma^2) + d^4(0.0156\sigma^2)$
(12,22,11,6,5)	23	775	64	1	2.8173	$0.0090\sigma^2 + d^4(0.0031\sigma^2)$
(13,26,12,6,5)	40	924	48	1	2.5607	$0.0081\sigma^2 + d^4(0.0031\sigma^2)$
(15,15,7,7,3)	26	1046	96	1	3.1219	$0.0081\sigma^2 + d^4(0.0026\sigma^2)$

4. Study of Orthogonality in SORD of Second Type Using BIBD

An orthogonal design is one in which the terms in the fitted model are uncorrelated with one another and thus the parameter estimates are uncorrelated. In this case, the variance of the predicated response at any point x in the experimental region, is expressible as a weighted sum of the variance of the parameter estimates in the model. For second order moments $\sum x_{iu}^2$ and $\sum x_{iu}^2 x_{ju}^2$ is difficult to obtain. This is because the moments $\sum x_{iu}^2$ and $\sum x_{iu}^2 x_{ju}^2$ are necessarily positive. Hence, we consider the model with the pure quadratic terms correlated for their means. In regard to orthogonality, this model is often used for the sake of simplicity in the calculation. A design is said to be orthogonal we shall investigate the restriction $\left(\sum x_{iu}^2\right)^2 = N\sum x_{iu}^2 x_{ju}^2$ i.e $\left(N\mu_2^2\right)^2 = N\left(N\mu_4\right)$ i.e $\mu_2^2 = \mu_4$ to get SORD of second type using BIBD.

$$a_1^2+a_2^2=\frac{\sqrt{N(\lambda 2^{t(k)})}-r2^{t(k)}}{2} \tag{4.1}$$

It must be established the equation (4.1) makes SORD of second type using BIBD an orthogonal system. However $N=b2^{t(k)}+4v+n_0$, the value of (4.1) depends on factors (v), n_0 and the design points of SORD of second type using BIBD. The following table 2 gives the values of orthogonality of second order response surface methodology using various parameters of SORD of second type using BIBD and n_0, the value of '$a_1^2+a_2^2$' makes orthogonal second order response surface designs by using SORD of second type using BIBD.

Let (v, b, r, k, λ) denote parameters of BIBD, $2^{t(k)}$ denote a fractional replicate of 2^k in +1 or -1 levels in which no interaction with less than five factors are confounded. [1- (v, b, r, k, λ)] denote the design points generated from transpose of the incidence matrix of BIBD. Let $[1-(v, b, r, k, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from BIBD by "multiplication" (cf. Raghavarao [18], pp 298-300). We use the additional set of points like $(\pm a_1,0,...,0),(0,\pm a_1,0,...,0),...,(0,0,...,\pm a_1)$ and $(\pm a_2,0,...,0),(0,\pm a_2,0,...,0),...,(0,0,...,\pm a_2)$ are two sets of axial points. Here $(a_1,0,0,...,0)2^1U(a_2,0,0,...,0)2^1$ denote the 4v axial points generated from $(a_1,0,0,...,0)$ and $(a_2,0,0,...,0)$ point sets. Let U denote the union of the design points generated from different sets of points, and (n_0) denote the number of central points. The method of study of orthogonality of a SORD of second type using BIBD is given in the following theorem.

Theorem(2): The design points,

$[1-(v,b,r,k,,\lambda)]2^{t(k)}U(a_1,0,...,0)2^1U(a_2,0,...,0)2^1U(n_0)$ will give a

v-dimensional SORD of second type using BIBD in $N=b2^{t(k)}+4v+n_0$ design points, with

$$a_1^2+a_2^2=\frac{\sqrt{N(\lambda 2^{t(k)})}-r2^{t(k)}}{2} \text{(for orthogonality).}$$

Proof: For the design points generated from second order rotatable designs of second type using BIBD, simple symmetry conditions (2.2) are true. Further, conditions (2.3) and (2.4) are true as follows

$$\sum x_{iu}^2 = r2^{t(k)} + 2(a_1^2 + a_2^2) = N\mu_2 \tag{4.2}$$

$$\sum x_{iu}^4 = r2^{t(k)} + 2(a_1^4 + a_2^4) = cN\mu_4 \tag{4.3}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{t(k)} = N\mu_4 \tag{4.4}$$

Solving equations (4.2) and (4.4) using $\mu_2^2 = \mu_4$

$$\left(\frac{r2^{t(k)} + 2(a_1^2 + a_2^2)}{N}\right)^2 = \frac{\lambda 2^{t(k)}}{N}$$

then we can obtain

$$a_1^2 + a_2^2 = \frac{\sqrt{N(\lambda 2^{t(k)})} - r2^{t(k)}}{2} \text{(for orthogonality).}$$

Example 2. We illustrate the theorem 2 second order rotatable designs of second type using BIBD with parameters $(v=3, b=3, r=2, k=2, \lambda=1)$. The design points $[1-(3,3,2,2,1)]2^{t(2)}U(a_1,0,0,\ldots,0)2^1U(a_2,0,0,\ldots,0)2^1U(n_0=1)$ will give a v-dimensional SORD of second type using BIBD in N=25 design points with one central point. From (3.3), (3.4) and (3.5) we have

$$\sum x_{iu}^2 = 8 + 2(a_1^2 + a_2^2) = N\mu_2 \tag{4.5}$$

$$\sum x_{iu}^4 = 8 + 2(a_1^4 + a_2^4) = cN\mu_4 \tag{4.6}$$

$$\sum x_{iu}^2 x_{ju}^2 = 4 = N\mu_4 \tag{4.7}$$

From (4.5) and (4.7), using $\mu_2^2 = \mu_4$, we can obtain the orthogonality value

$$\Rightarrow a_1^2 + a_2^2 = \frac{\sqrt{(25)?\ 4)} - 8}{2}$$

$$= 1.$$

Table 2: values of orthogonality of SORD of second type using BIBD (using theorem 2)

(3,3,2,2,1)			(4,4,3,3,2)			(5,5,4,4,3)			(6,10,5,3,2)		
(n_0=1)	N	$a_1^2+a_2^2$	(n_0=1)	N	$a_1^2+a_2^2$	(n_0=1)	N	$a_1^2+a_2^2$	(n_0=1)	N	$a_1^2+a_2^2$
n_0	25	1	n_0	49	2	n_0	101	2.8138	n_0	105	0.4939
n_0+1	26	1.099	n_0+1	50	2.1421	n_0+1	102	2.9857	n_0+1	106	0.5913
n_0+2	27	1.1962	n_0+2	51	2.2829	n_0+2	103	3.1568	n_0+2	107	0.6882
n_0+3	28	1.2915	n_0+3	52	2.4222	n_0+3	104	3.327	n_0+3	108	0.7846
n_0+4	29	1.3852	n_0+4	53	2.5602	n_0+4	105	3.4965	n_0+4	109	0.8806

(7,7,4,4,2)			(8,14,7,4,3)			(9,18,8,4,3)			(10,18,9,5,4)		
(n_0=1)	N	$a_1^2+a_2^2$	(n_0=16)	N	$a_1^2+a_2^2$	(n_0=18)	N	$a_1^2+a_2^2$	(n_0=1)	N	$a_1^2+a_2^2$
n_0	141	1.5857	n_0	262	0.1714	n_0	342	0.0625	n_0	329	0.5534
n_0+1	142	1.7046	n_0+1	263	0.1783	n_0+1	343	.1561	n_0+1	330	0.6636
n_0+2	143	1.8231	n_0+2	264	0.285	n_0+2	344	0.2495	n_0+2	331	0.7736
n_0+3	143	1.9411	n_0+3	265	0.3915	n_0+3	345	0.3428	n_0+3	332	0.8835
n_0+4	144	2.0588	n_0+4	266	0.4978	n_0+4	346	0.436	n_0+4	333	0.9932

(11,11,5,5,2)			(12,22,11,6,5)			(13,26,12,6,5)			(15,15,7,7,3)		
(n_0=1)	N	$a_1^2+a_2^2$	(n_0=2)	N	$a_1^2+a_2^2$	(n_0=4)	N	$a_1^2+a_2^2$	(n_0=1)	N	$a_1^2+a_2^2$
n_0	221	2.0476	n_0	775	0.0682	n_0	924	0.2498	n_0	1046	0.0714
n_0+1	222	2.1426	n_0+1	776	0.1817	n_0+1	925	0.3538	n_0+1	1047	0.1785
n_0+2	223	2.2374	n_0+2	777	0.2952	n_0+2	926	0.4578	n_0+2	1048	0.2855
n_0+3	224	2.332	n_0+3	778	0.4086	n_0+3	927	0.5617	n_0+3	1049	0.3925
n_0+4	225	2.4264	n_0+4	779	0.522	n_0+4	928	0.6655	n_0+4	1050	0.4994

5. Efficiency Comparison for SORD of Second Type Using BIBD with SORD of First Type Using BIBD

In this section, SORD of second type using BIBD is used as the basis for estimating specific coefficient in the response surface model, SORD of second type using BIBD is compared with SORD of first type using BIBD. This comparison criterion is based on the precision at which the coefficient is estimated. It is consider that the numbers of experimental plots are required at same way.

For example in terms of estimating mixed quadratic coefficient b_{ij} ($i \neq j$), two experimental designs, lets try to compare D_1 and D_2. The number of experimental plots required in D_1 and D_2 are M and N respectively. The relative efficiency of D_1 and D_2 is given by the following equation (see Myers [15], section 7.2).

$$E\left(\frac{D_1}{D_2}\right)=\frac{\{Var(b_{ij})\,in\,D_2\}N}{\{Var(b_{ij})\,in\,D_1\}M} \tag{5.1}$$

Where $N=b2^{t(k)}+4v+n_0$ (Design points in SORD of second type using BIBD)

$M=b2^{t(k)}+2v+m_0$ (Design points in SORD of first type using BIBD)

In this case, in order to compare fairly, the experimental system should make the second product equal to value of $\frac{\sum x_{iu}^2}{N}$. It must be scaled and for this the following scaling criteria is used.

$$\text{(i)}\ \frac{1}{N}\sum x_{iu}=0$$

$$\text{(ii)}\ \frac{1}{N}\sum x_{iu}^2=1,\ (i=1,2,...,v) \qquad (\text{taking } \mu_2=1) \tag{5.2}$$

5.1 Comparison in mixed quadratic coefficient b_{ij} $(i \neq j)$

According to equation (2.7) $V(\hat{b}_{ij})=\frac{\sigma^2}{N\mu_4}$, but this before scaling equation (5.2) will be write in SORD of second type using BIBD. From (ii) of (5.2) $\frac{(r2^{t(k)}+2a_1^2+2a_2^2)}{N}$ each time making equation (5.2) will be is equal to 1, i.e. (ii) =1, then the scaling factor g

$$g=\left\{\left(\frac{b2^{t(K)}+4v+n_0}{r2^{t(k)}+2a_1^2+2a_2^2}\right)\right\}^{1/2} \tag{5.3}$$

However the $V(b_{ij})$ is multiplied by with the scaling factor 'g' than the $V(b_{ij})=\frac{\sigma^2}{N\mu_4}.\frac{1}{g^4}$ that is

$$V(b_{ij})=\frac{\sigma^2}{N\mu_4}\left\{\left[\frac{r2^{t(k)}+2a_1^2+2a_2^2}{b2^{t(k)}+4v+n_0}\right]\right\}^2$$

According to equation (5.1) the relative efficiency SORD of second type using BIBD versus SORD of first type using BIBD in the mixed quadratic coefficient of b_{ij} is obtained as follows

$$E\left(\frac{\text{SORD of second type using BIBD}}{\text{SORD of first type using BIBD}}\right)=\frac{\frac{\sigma^2}{N\mu_4}\left(\frac{r2^{t(k)}+2a^2}{b2^{t(k)}+2v+m_0}\right)^2(b2^{t(k)}+2v+m_0)}{\frac{\sigma^2}{N\mu_4}\left(\frac{r2^{t(k)}+2a_1^2+2a_2^2}{b2^{t(k)}+4v+n_0}\right)^2(b2^{t(k)}+4v+n_0)}$$

$$=\frac{(r2^{t(k)}+2a^2)^2(b2^{t(k)}+4v+n_0)}{(r2^{t(k)}+2a_1^2+2a_2^2)^2(b2^{t(k)}+2v+m_0)} \tag{5.4}$$

From equation (5.4) the condition that $E\left(\frac{\text{SORD of second type using BIBD}}{\text{SORD of first type using BIBD}}\right)>1$, than the SORD of second type using BIBD is more efficient than SORD of first type using BIBD

$$a_1^2+a_2^2<\frac{1}{2}\left\{(r2^{t(k)}+2a^2)\sqrt{\frac{b2^{t(k)}+4v+n_0}{b2^{t(k)}+2v+m_0}}-r2^{t(k)}\right\} \tag{5.5}$$

From the values of (3.1) and (4.1) substitute in (5.4) and then we get the value of greater than 1. From this orthogonal SORD of second type using BIBD has the same degree of efficiency as orthogonal SORD of first type using BIBD, and consider the efficiency of SORD of second type using BIBD is giving the better efficiency than SORD of first type using BIBD. Now, the efficiency comparison of SORD of second type using BIBD versus SORD of first type using BIBD with rotatability. Substituting the values of (3.2) into (5.5) and we evaluated that the SORD of second type using BIBD will be more efficient than the SORD of first type using BIBD with rotatability.

$$a_1^2+a_2^2<\frac{1}{2}\left\{\left(r2^{t(k)}+2\sqrt{\frac{(3\lambda-r)2^{t(k)}}{2}}\right)\sqrt{\frac{b2^{t(k)}+4v+n_0}{b2^{t(k)}+2v+m_0}}-r2^{t(k)}\right\} \tag{5.6}$$

For example, in SORD of first type using BIBD when $(v=3,b=3,r=2,k=2,\lambda=1)$ and $m_0=1$ then we get M =19 then a=1.1892 and $n_0=1$, then the equation (5.6)

$$a_1^2+a_2^2<3.7216 \tag{5.7}$$

Among the rotatability of SORD of second type using BIBD, it is easy to find an experimental plan that satisfies the equation (5.7). For example in SORD of second type using BIBD with $a_1=1$, $a_2=1$ satisfy the rotatability property equation (3.7), and $a_1^2+a_2^2=2$ as it satisfies the equation (5.7) as well, it is more efficient than the rotatability SORD of first type using BIBD. Then the relative efficiency of SORD of second type using BIBD versus SORD of first type using BIBD equation (5.4) is as follows.

$$\frac{\left(8+2(1.4142)\right)^2(12+12+1)}{\left(8+2(2)\right)^2(12+6+1)}=1.0714$$

5.2 Comparison in the pure quadratic coefficient b_{ii}

Now this time in terms of estimating the pure quadratic coefficient b_{ii}, the efficiency of SORD of second type using BIBD is comparing with SORD of first type using BIBD, here the scaling factor the equation (5.2) is applied. The relative efficiency SORD of second type using BIBD versus SORD of first type using BIBD is as follows based on the equation (5.1)

$$E\left(\frac{\text{SORD of second type using BIBD}}{\text{SORD of first type using BIBD}}\right)=\frac{\sigma^2 e_1\left(\dfrac{r2^{t(k)}+2a^2}{b2^{t(k)}+2v+m_0}\right)^2\left(b2^{t(k)}+2v+m_0\right)}{\sigma^2 e_2\left(\dfrac{r2^{t(k)}+2a_1^2+2a_2^2}{b2^{t(k)}+4v+n_0}\right)^2\left(b2^{t(k)}+4v+n_0\right)}$$

$$=\frac{e_1\left(r2^{t(k)}+2a^2\right)^2\left(b2^{t(k)}+4v+n_0\right)}{e_2\left(r2^{t(k)}+2a_1^2+2a_2^2\right)^2\left(b2^{t(k)}+2v+m_0\right)} \tag{5.8}$$

Where,

$e_1=v(b_{ii})$ in SORD of first type using BIBD and $e_2=v(b_{ii})$ in SORD of second type using BIBD.

For example, in SORD of second type using BIBD of design $(v=3,b=3,r=2,k=2,\lambda=1)$, $n_o=1$, $a_1=1$, $a_2=1$, $e_2=0.125$ and in SORD of first type using BIBD $m_0=1$, $a=1.1892$, $e_1= 0.1947$, lets us compare the relative efficiency of SORD of second type using BIBD versus SORD of first type using BIBD, if you get the equation (5.8) as 1.6688, then we conclude that the SORD of second type using BIBD is more efficient than SORD of first type using BIBD.

5.3 Comparison in terms estimating the first order coefficient b_i

It can be developed in the same process of the $V(b_i)$ by multiplying the scaling factor then $V(b_i)=\dfrac{\sigma^2}{(r2^{t(k)}+2a_1^2+2a_2^2)}$ is to be multiplied by $1/g^2$ then we get, $V(b_i)=\dfrac{\sigma^2}{(r2^{t(k)}4v+n_0)}$ similarly the $V(b_i)$ is multiplied by scaling factor in SORD of first type using BIBD the we obtained $V(b_i)=\dfrac{\sigma^2}{(r2^{t(k)}+2v+n_0)}$ so finally we compare the relative efficiency of $E\left(\dfrac{\text{SORD of second type using BIBD}}{\text{SORD of first type using BIBD}}\right)$ and it obtained 1, then the efficiency of SORD of second type using BIBD is more efficient than SORD of first type using BIBD.

6. CONCLUSION

In this paper, SORD of second type using BIBD is developed. The variance covariance of the estimated parameters are studied and we evaluated for the SORD of second type using BIBD is most orthogonal for second order response surface designs and the results of the orthogonality are also provided in the paper.

The comparison between the SORD of second type using BIBD versus SORD of first type using BIBD for different coefficients are studied then we conclude that the SORD of second type using BIBD is more efficient than SORD of first type using BIBD. It is convenient to use the practical situations and give the more efficiency when compared to SORD of first type using BIBD.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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