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EQUITABLE COLORING BY SPECTRAL CLUSTERING IN THE DISTRIBUTED WEB NETWORKS

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Abstract. The limited number of servers and more number of users in a distributed system is a great concern in load balancing. In this paper the efficient way of allocating tasks to available servers using equitable coloring of graphs is studied. For balancing the load, the complexity of graph is reduced by spectral clustering, which partitions the graph for further processes. The web model of graphs such as generalized structure of wheel and web graphs are investigated. The spectrum of these graphs are calculated for partitioning and then equitable coloring is applied. This approach results in the proper distribution of load in the web networks.

Keywords: equitable coloring; eigenvectors; spectrum of the graph.

2010 AMS Subject Classification: 05C15, 05C50, 05C78.

1. INTRODUCTION

The graph considered in this paper is an undirected web structure graphs namely generalized wheel graph and generalized web graph. Both these graphs starts from the hub, which imitates the connections of clients to the server. The clients are taken as the vertices at different layers of the cycles in these graphs. Likewise the distributed network may have few servers providing

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service to more number of clients in a complex network. The aim of load balancing is the equitable distribution of work among all the reliable servers in the network. This can be acheived by grouping and allocating users equally in the hotspot among the servers.

Our focus is to design and implement the load balancing with equitable coloring by spectral partitoning and then assigning colors. The spectrum is obtained by the eigenvalues of the adjacency matrix of $W_{(n,m)}$ and $WB_{(n,m)}$. For partitioning the eigenvector of the λ_{mn}^{th} eigenvalue is calibrated and the vertices are clustered according to the positive and negative elements of the eigenvector X_{mn}. Further the cut edge disconnects the clusters from the central point and adjacent layers of cycles. Later the coloring of the vertices is done based on the adjacency within the clusters, which satisfies the condition of equitablility. In 1973, Meyer[1] introduced the idea of equity in vertex coloring, this concept helps in balancing the load in a distributed system.

The graph spectra appears to be one of the interesting concept in computer science and network analysis, which helps in the partition of graphs using eigenvalues. The study on spectra of graphs can be found in many papers, see e.g.[2, 3]. The proposed method of spectral clustering is a new approach to reduce the complexity of graph. The idea of applying equitable coloring in load balancing technique is a novel perspective. In this paper, the spectral clustering and optimal coloring in an equitable way for generalized $W_{(n,m)}$ and $WB_{(n,m)}$ are studied and illustrated.

2. PRELIMINARIES

Definition 2.1. [4] *For any integer $m \geq 4$, the wheel graph W_m consists of m vertices is obtained by joining the center vertex v with every other $m-1$ vertices $\{v_1, v_2, \ldots, v_m\}$ of the cycle graph C_{m-1}.*

Definition 2.2. [5] *A double-wheel graph DW_m of size m consists of $2C_m + K_1$, which contains two cycles of size m, where all the vertices of two cycles are attached to a common hub.*

Definition 2.3. *A n-wheel graph $W_{(n,m)}$ of size $mn+1$ consists of $nC_m + K_1$, which contains n cycles of size m, where all the vertices on n cycles are connected to a common hub.*

Definition 2.4. [6] *The web graph $W_{(2,n)}$ is obtained by joining pendant vertices of helm H_n to form a cycle and then adding a pendant edge to each vertex of outer cycle.*

Definition 2.5. [7] *A generalized web graph $WB_{(n,m)}$, $n \geq 3$, $m \geq 2$, is a graph obtained by joining all vertices $v_{(i,m)}$, $1 \leq i \leq n$, of the generalized prism P_n^m to a further vertex w, called the centre. Thus $WB_{(n,m)}$ contains $mn+1$ vertices and $2mn$ edges.*

Definition 2.6. [8] *The spectrum of a graph G is defined as $\sigma(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the eigenvalues of the adjacency matrix of G.*

Definition 2.7. [9] *A cut vertex of a graph G is a vertex whose removal increases the number of connected components of the graph. A cut edge of a graph is defined in a similar manner as an edge whose removal increases the number of connected components of the graph. For a connected graph of order n has atmost $n-2$ cut vertices and atmost $n-1$ cut edges.*

Definition 2.8. [10] *If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, ..., V_k$ such that each V_i is an independent set and the condition $||Vi| - |Vj|| \leq 1$ holds for every pair (i, j), then G is said to be equitably k-colorable. The smallest integer k for which G is equitably k-colorable is known as the equitable chromatic number of G and denoted by $\chi_=(G)$.*

3. Equitable Coloring on Spectral Clustering of Generalized Wheel and Web Graphs

Theorem 3.1. *The equitable coloring of n-wheel graph $W_{(n,m)}$ with n partitioned clusters is*

$$\chi_=(W_{(n,m)}) = \begin{cases} 3, \ if \ m \text{ is odd} \\ 2, \ if \ m \text{ is even.} \end{cases}$$

Proof. The n-wheel graph consists of $mn+1$ vertices and $2mn$ edges. Its adjacency matrix is formed by

$$A(W_{(n,m)}) = \begin{cases} 1, \ if \ i \text{ and } j \text{ are adjacent} \\ 0, \ if \ i \text{ and } j \text{ are non- adjacent.} \end{cases}$$

The adjacency situations of n-wheel graph are with 1's on

$i = 1, j = 2, 3, \ldots, mn+1$

$i = 2, 3, \ldots, mn+1, j = 1$

$i = (n-1)m+2, j = mn+1,$

$i = mn+1, j = (n-1)m+2,$

$(n-1)m+2 \leq i \leq mn, j=i+1,$

$(n-1)m+3 \leq i \leq mn+1, j=i-1.$

The non-adjacency situations are with 0's on $i=j$ and elsewhere. Here n represents the number of cycles and m is the number of vertices on each cycle. Therefore the adjacency matrix is constructed as follows.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \\ v_{m+1} \\ v_{m+2} \\ v_{m+3} \\ \vdots \\ v_{mn} \\ v_{mn+1} \end{array}
\begin{array}{c} \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & \cdots & v_m & v_{m+1} & v_{m+2} & v_{m+3} & \cdots & v_{mn} & v_{mn+1} \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\
1 & 0 & 1 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\
1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\
1 & 1 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\
1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\
1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0
\end{pmatrix} \end{array}$$

Set $det\,(A(nW_m)-\lambda I)=0$. The characteristic equation of this adjacency matrix with order $mn+1$ is of the form $(-\lambda)^{mn+1}+tr(-\lambda)^{mn}+\ldots+det(A)=0$ which has exactly $mn+1$ roots. Therfore there should be $mn+1$ eigen values.

i.e, $\lambda_1, \lambda_2, \ldots, \lambda_{mn+1}$.

By determining the eigen vectors of these eigen values and choosing the eigenvector of λ_{mn}, the clustering is done by grouping the positive and negative entries of this eigen vector and fixing the median value as zero. Hence the entire graph can be partitioned into $n+1$ clusters. The links within the clusters remains and the edge cut set S removes the links between the vertices of the clusters.

$S=\{v_1v_2, v_1v_3, v_1v_4, \ldots, v_1v_{mn+1}\}$

The clusters are

$C_1=\{v_1\}$

$C_2 = \{v_2, v_3, \dots, v_{m+1}\}$

$C_3 = \{v_{m+2}, v_{m+3}, \dots, v_{2m+1}\}$

$\vdots$

$C_{n+1} = \{v_{m(n-1)+2}, v_{m(n-1)+3}, \dots, v_{mn+1}\}$

For the vertex v_1 in the cluster C_1, assign the color 1 in both the following cases.

Case 1. If m is odd, the vertices of the remaining clusters can be grouped as

$$C_i = \{v_k/(i-2)m+2 \leq k \leq (i-1)m+1; 2 \leq i \leq n+1\}.$$

(i)If $m \equiv 0 \bmod 3$ and $m \equiv 2 \bmod 3$, the colors assigned for its vertices are

$$v_k = \begin{cases} 1, & k \equiv 2 \bmod 3 \\ 2, & k \equiv 0 \bmod 3 \\ 3, & k \equiv 1 \bmod 3, (i-2)m+2 \leq k \leq (i-1)m+1; 2 \leq i \leq n+1 \end{cases}$$

(ii)If $m \equiv 1 \bmod 3$, the colors assigned are

$$v_k = \begin{cases} 1, & k \equiv 2 \bmod 3 \\ 2, & k \equiv 0 \bmod 3 \\ 3, & k \equiv 1 \bmod 3, (i-2)m+2 \leq k \leq (i-1)m; 2 \leq i \leq n+1 \end{cases}$$

For the vertices $v_{(i-1)m+1}, 2 \leq i \leq n+1$, the colors assigned are

$$v_p = \begin{cases} 1, & p \equiv 1 \bmod 3 \\ 2, & p \equiv 2 \bmod 3 \\ 3, & p \equiv 0 \bmod 3, \text{where } p = (i-1)m+1 \end{cases}$$

Case 2. If m is even

For the vertices $v_k(2 \leq k \leq mn+1)$ in the clusters $C_2, C_3, \dots, C_{n+1}$, the colors are assigned as

$$v_k = \begin{cases} 1, & k \equiv 1 \bmod 2 \\ 2, & k \equiv 0 \bmod 2. \end{cases}$$

As the graph is sub-divided, note that the coloring is done based on the adjacency within the clusters such that the equitable condition $||V_i| - |V_j|| \leq 1, i \neq j$, is attained. It is clear that

$$\chi_{=}(W_{(n,m)}) = \begin{cases} 3, \ if \ m \text{ is odd} \\ 2, \ if \ m \text{ is even.} \end{cases} \qquad \square$$

Illustration:

Consider the generalized wheel graph of order $W_{(4,4)}$ with 4 cycles and having 4 vertices on each cycle. The eigen values of the adjacency matrix for this graph are

$$\begin{array}{ccc} \lambda_1 = -3.1231 & \lambda_2 = -2 & \lambda_3 = -2 \\ \lambda_4 = -2 & \lambda_5 = -2 & \lambda_6 = -4.0731e-15 \\ \lambda_7 = -1.3430e-15 & \lambda_8 = -1.3298e-15 & \lambda_9 = -1.1311e-15 \\ \lambda_{10} = -8.1937e-16 & \lambda_{11} = -4.5415e-16 & \lambda_{12} = 9.2461e-16 \\ \lambda_{13} = 2.2519e-15 & \lambda_{14} = 2 & \lambda_{15} = 2 \\ \lambda_{16} = 2 & \lambda_{17} = 5.1231 & \end{array}$$

The eigenvector X corresponding to λ_{16} is

$$\begin{aligned} X_{16} = [& -8.391188888453039e-18 \quad -0.0688123852947728 \quad -0.0688123852947728 \\ & -0.0688123852947728 \quad -0.0688123852947727 \quad 0.382901015740260 \\ & 0.382901015740260 \quad 0.382901015740260 \quad 0.382901015740260 \\ & -0.314088630445487 \quad -0.314088630445487 \quad -0.314088630445487 \\ & -0.314088630445487 \quad -2.96604146969705e-17 \quad 5.51453879638563e-17 \\ & 3.80516035815009e-17 \quad -2.12946529880272e-34]^T \end{aligned}$$

Now the vertices are clustered as $C_1 = \{v_1\}$, $C_2 = \{v_2, v_3, v_4, v_5\}$, $C_3 = \{v_6, v_7, v_8, v_9\}$, $C_4 = \{v_{10}, v_{11}, v_{12}, v_{13}\}$, $C_5 = \{v_{14}, v_{15}, v_{16}, v_{17}\}$, afterwards the coloring is allocated and found to be equitable as shown in figure 1.

Theorem 3.2. *The equitable coloring of generalized web graph $WB_{(n,m)}$ with n partitioned clusters is* $\chi_{=}(WB_{(n,m)}) = \begin{cases} 3, \ if \ m \text{ is odd} \\ 2, \ if \ m \text{ is even.} \end{cases}$

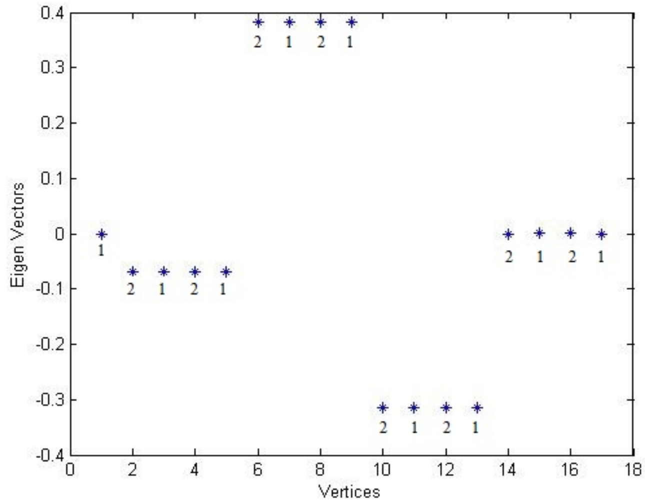

FIGURE 1. Clustering the vertices of $W_{(4,4)}$.

Proof. The generalized web graph consists of $m(n+1)+1$ vertices and $m(2n+1)$ edges. Its adjacency matrix is formed by

$$A(WB_{(n,m)}) = \begin{cases} 1, \ if \ i \text{ and } j \text{ are adjacent} \\ 0, \ if \ i \text{ and } j \text{ are non- adjacent.} \end{cases}$$

The adjacency matrix of generalized web graph is with 1's on

$i = 1, j = 2, 3, \ldots, m+1,$

$j = 1, i = 2, 3, \ldots, m+1,$

if $1 \leq k \leq m$

$(k-1)n+2 \leq i \leq kn+1, j = i+n,$

$(k-1)n+2 \leq j \leq kn+1, i = j+n,$

for $k \leq m-1$

$i = (k-1)n+2, j = kn+1,$

$i = kn+1, j = (k-1)n+2,$

$(k-1)n+2 \leq i \leq kn, j = i+1,$

$(k-1)n+3 \leq i \leq kn+1, j = i-1$

The non-adjacency situations are with 0's on $i = j$ and elsewhere. Here n represents the number of cycles and m is the number of vertices on each cycle.

$$\begin{array}{cc} & \begin{array}{ccccccccccc} v_1 & v_2 & v_3 & \cdots & v_m & v_{m+1} & v_{m+2} & v_{m+3} & \cdots & v_{m(n+1)} & v_{m(n+1)+1} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \\ v_{m+1} \\ v_{m+2} \\ v_{m+3} \\ \vdots \\ v_{m(n+1)} \\ v_{m(n+1)+1} \end{array} & \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}$$

By finding the characteristic equation of this adjacency matrix and determining its eigenvalues and eigenvectors, the vertices are clustered, subsequently followed by equitable coloring. The remaining proof is similar to theorem 3.1. □

Illustration:

Consider the generalized web graph of order $WB_{(4,7)}$ with 4 cycles and having 7 vertices on each cycle. The eigenvalues of the adjacency matrix for this graph are

$$\begin{array}{llll} \lambda_1 = -3.3028 & \lambda_2 = -3.3028 & \lambda_3 = -2.0399 & \lambda_4 = -2.0399 \\ \lambda_5 = -2.0143 & \lambda_6 = -2.0143 & \lambda_7 = -1.9205 & \lambda_8 = -0.9197 \\ \lambda_9 = -0.9197 & \lambda_{10} = -0.7514 & \lambda_{11} = -0.7514 & \lambda_{12} = -0.6099 \\ \lambda_{13} = -0.6099 & \lambda_{14} = -0.4788 & \lambda_{15} = 0.1718 & \lambda_{16} = 0.1718 \\ \lambda_{17} = 0.3453 & \lambda_{18} = 0.3453 & \lambda_{19} = 0.5468 & \lambda_{20} = 0.5468 \\ \lambda_{21} = 1.2208 & \lambda_{22} = 1.2536 & \lambda_{23} = 1.2536 & \lambda_{24} = 1.5531 \\ \lambda_{25} = 1.5531 & \lambda_{26} = 2.7674 & \lambda_{27} = 2.7674 & \lambda_{28} = 2.9467 \\ \lambda_{29} = 4.2319 & & & \end{array}$$

The eigenvector X corresponding to λ_{28} is

$$X_{28} = [0.2836 \quad 0.1194 \quad 0.1194 \quad 0.1194 \quad 0.1194$$
$$0.1194 \quad 0.1194 \quad 0.1194 \quad -0.1706$$
$$-0.1706 \quad -0.1706 \quad -0.1706 \quad -0.1706 \quad -0.1706$$
$$-0.1706 \quad -0.2809 \quad -0.2809 \quad -0.2809 \quad -0.2809$$
$$-0.2809 \quad -0.2809 \quad -0.2809 \quad -0.0953 \quad -0.0953$$
$$-0.0953 \quad -0.0953 \quad -0.0953 \quad -0.0953 \quad -0.0953]^T$$

Based on the eigenvectors the spectral clustering of vertices are $C_1 = \{v_1\}$, $C_2 = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$, $C_3 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$, $C_4 = \{v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}\}$, $C_5 = \{v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}\}$, later the coloring is assigned and which is equitable as shown in figure 2.

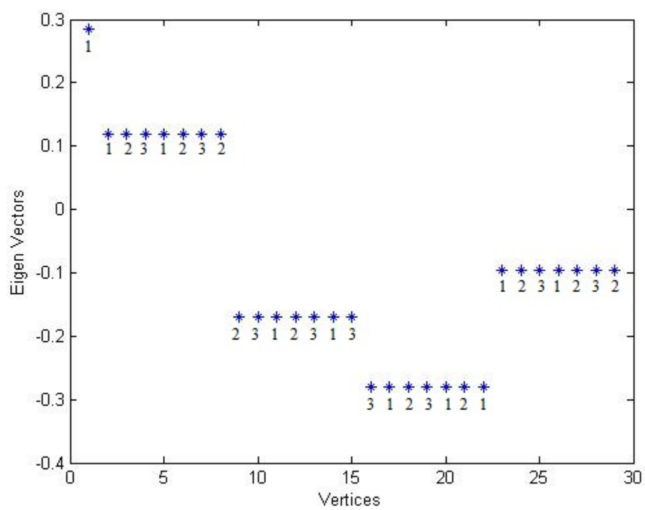

FIGURE 2. Clustering the vertices of $WB_{(4,7)}$.

4. CONCLUSION

The spectral clustering has extensive applications in the field of chemical compounds of molecular graphs and protein structures etc. It is hard to cluster the the natural complex structure of graphs. By relating the spectrum, the graph is partitioned and graph coloring is applied which enables the idea of sharing of resources to the craving users. This paper suggests the method of spectral partitioning and equitable coloring of graphs. The model of generalized graphs $W_{(n,m)}$ and $WB_{(n,m)}$ are investigated and illustrated. Moreover this procedure can be executed for various model of complex structured graphs.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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