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GAUSSIAN QUADRATURE METHOD FOR SOLVING DIFFERENTIAL DIFFERENCE EQUATIONS HAVING BOUNDARY LAYERS

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Abstract: In this paper, the Gaussian quadrature method is described for the solution of differential difference problems having boundary layers. The given problem is replaced by an asymptotically equivalent first order differential equation with the perturbation parameter as deviating argument. Then, Gaussian two point quadrature is implemented to solve this first order differential equation with perturbation parameter as deviating parameter. Several numerical problems are illustrated to demonstrate the layer behaviour. Comparison of maximum errors in the solution of the problems is made with other methods available in the literature to demonstrate the applicability of the present method.

Keywords: singular perturbations; delay differential equations; reduction of order.

2010 AMS Subject Classification: 65L11, 65Q10

1. INTRODUCTION

Differential-difference equation models have stronger mathematical structure when associated with ODEs for the analysis of biosystem dynamics and they produce improved stability with the nature of the underlying processes and analytical results. Delay differential equations model the problems where there is after effect affecting at least one of the variables involved in the problem as compared to ordinary differential equations which model the problems

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in which variables react to current conditions. Due to these reasons, differential-difference equation models are more preferred to ordinary differential equations. The category of differential-difference equations which have delay/advance and singularly perturbed behaviour is recognized as singularly perturbed differential-difference equations. In general, a singularly perturbed differential-difference equation is an ordinary differential equation, where a small positive parameter multiplies the highest order derivative, including at least one delay and/or advance parameters. Solutions of these equations exhibit variety of interesting phenomenon like rapid oscillations, turning point behaviour, boundary and interior layers. These problems possess the boundary layer characteristics. *A boundary layer is an interval or region in which the solution changes rapidly.* In these layers the physical variables change extremely rapidly over small domains in space or short intervals of time. This type of differential equations occur in the modelling of numerous practical phenomena in bioscience, engineering, control theory, such as in variational problems in control theory, in describing the human pupil-light reflex, in a variety of models for physiological processes or diseases and first exit time problems in the modelling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites. Stein [12] was first to study of bistable devices. Derstin et al [2], and variational problems in control theory Glizer [5] where they provide the best and in many cases the only realistic simulation of the observed. Lange and Miura [9, 10] gave an asymptotic approach for a class of boundary-value problems for linear second-order differential-difference equations. Kadalbajoo and Sharma [6, 7, 8], presented a numerical approaches to solve singularly perturbed differential-difference equation, which contains negative shift in the either in the derivative term or the function but not in the derivative term. Analytical discussion on these problems is available in the books O'Malley [11] Elsgolts and Norkin [4] and Driver [3]. In this paper, the Gaussian quadrature method is described for the solution of differential difference problems having boundary layers. The given problem is replaced by an asymptotically equivalent first order differential equation with the perturbation parameter as deviating argument. Then, Gaussian two point quadrature is implemented to solve this first order differential equation with perturbation parameter as deviating parameter. Several numerical problems are illustrated to demonstrate the layer behaviour. Comparison of maximum errors in the solution of the problems is made with other methods available in the literature to demonstrate the applicability of the present method.

2. DESCRIPTION OF THE METHOD

Consider the problem:

$$\varepsilon w''(s) + p(s)w'(s) + q(s)w(s - \delta) + r(s)w(s) + t(s)w(s + \eta) = f(s) \quad (1)$$

on $(0, 1)$, under the boundary conditions

$$\begin{aligned} w(s) &= \varphi(s) & \text{on } -\delta \leq s \leq 0, \\ w(1) &= \gamma(s) & \text{on } 1 \leq s \leq 1 + \eta, \end{aligned} \quad (2)$$

where ε is small parameter, $0 < \varepsilon \ll 1$, $p(s), q(s), r(s), t(s), f(s), \varphi(s)$ and $\gamma(s)$ are smooth functions and $0 < \delta = o(\varepsilon), 0 < \eta = o(\varepsilon)$ are respectively the delay (negative shift) and the advance (positive shift) parameter.

The solution of Eqn. (1)-(2) exhibits; layer at the left end of the interval if $p(s) - \delta q(s) + \eta t(s) > 0$ and layer at the right end of the interval if $p(s) - \delta q(s) + \eta t(s) < 0$. If $p(s) = 0$, then one may have oscillatory solution or two layers (one at each end) depending upon the cases whether $q(s) + r(s) + t(s)$ is positive or negative. Using, Taylor series, we have,

$$w(s - \delta) \approx w(s) - \delta w'(s) \quad (3a)$$

$$w(s + \eta) \approx w(s) + \eta w'(s) \quad (3b)$$

Using Eqn. (3) in Eqn. (1), we obtain

$$\varepsilon w''(s) + a(s)w'(s) + b(s)w(s) = f(s) \quad (4)$$

Equation (4) is a second order singular perturbation problem.

Here,

$$a(s) = p(s) - \delta q(s) + \eta t(s) \quad (5a)$$

$$b(s) = q(s) + r(s) + t(s) \quad (5b)$$

We solve the equation (4) subject to the boundary conditions equation (2) by using the Gaussian two point quadrature.

3. NUMERICAL SCHEME

3.1. Problem with Left-end Boundary Layer

Using Taylor's expansion about the point s , we have

$$w'(s - \varepsilon) \approx w'(s) - \varepsilon w''(s)$$

implies
$$\varepsilon w''(s) = w'(s) - w'(s - \varepsilon) \quad (6)$$

With these Eqn. (6) and Eqn. (4) we get first order equation with ε as deviating argument:

$$w'(s) = w'(s - \varepsilon) - a(s)w'(s) - b(s)y(s) + f(s) \quad (7)$$

The domain $[0, 1]$ is partitioned into N sub domains of mesh size $h = \frac{1-0}{N}$ so that $s_i = 0 + ih$, $i =$

$0, 1, \dots, N$ are the mesh points. By integrating Eqn. (7) with respect to s from s_i to s_{i+1} , we get

$$\int_{s_i}^{s_{i+1}} w'(s) ds = \int_{s_i}^{s_{i+1}} (w'(s - \varepsilon) - a(s)w'(s) - b(s)y(s) + f(s)) ds$$

$$w_{i+1} - w_i = \int_{s_i}^{s_{i+1}} w'(s - \varepsilon) ds - \int_{s_i}^{s_{i+1}} a(s)w'(s) ds - \int_{s_i}^{s_{i+1}} b(s)y(s) dx + \int_{s_i}^{s_{i+1}} f(s) ds \quad (8)$$

Using Gaussian two-point quadrature formula, we have

$$\int_{-1}^1 F(s) ds = F\left(\frac{1}{\sqrt{3}}\right) + F\left(\frac{-1}{\sqrt{3}}\right)$$

For any continuous and differentiable function $F(s)$ in an arbitrary interval $[s_i, s_{i+1}]$, the Gaussian two-point quadrature formula is given by:

$$\int_{s_i}^{s_{i+1}} F(s) ds = \frac{h}{2} (F(s_i + k) + F(s_{i+1} - k)) \quad (9)$$

where

$$k = \frac{h \left(1 - \frac{1}{\sqrt{3}}\right)}{2}.$$

Using Eqn. (9) in Eqn. (8), we have

$$\begin{aligned}
 w_{i+1} - w_i &= w(s_{i+1} - \varepsilon) - w(s_i - \varepsilon) - a(s_{i+1})w(s_{i+1}) + a(s_i)w(s_i) \\
 &\quad + \frac{h}{2} [a'(s_{i+1} - k)w(s_{i+1} - k) + a'(s_i + k)w(s_i + k)] \\
 &\quad - \frac{h}{2} [b(s_{i+1} - k)w(s_{i+1} - k) + b(s_i + k)w(s_i + k)] \\
 &\quad + \frac{h}{2} [f(s_{i+1} - k) + f(s_i + k)]
 \end{aligned} \tag{10}$$

Using the linear interpolation for $w(s_{i+1} - \varepsilon)$, $w(s_i - \varepsilon)$, $w(s_i - k)$ and $w(s_{i+1} - k)$, the Eqn. (10) reduces to

$$\begin{aligned}
 &\left\{ \frac{\varepsilon}{h} + a'(s_i + k) \frac{k}{2} - b(s_i + k) \frac{k}{2} \right\} w_{i-1} + \\
 &\left\{ \frac{-2\varepsilon}{h} - a(s_i) - a'(s_{i+1} - k) \frac{k}{2} - a'(s_i + k) \left(\frac{h+k}{2} \right) + b(s_{i+1} - k) \frac{k}{2} + b(s_i + k) \left(\frac{h+k}{2} \right) \right\} w_i \\
 &+ \left\{ \frac{\varepsilon}{h} - a'(s_{i+1} - k) \left(\frac{h-k}{2} \right) + a(s_{i+1}) + b(s_i + k) \left(\frac{h-k}{2} \right) \right\} w_{i+1} = \frac{h}{2} \{ f(s_{i+1} - k) + f(s_i + k) \}
 \end{aligned} \tag{11}$$

Rearranging this equation, we have

$$\begin{aligned}
 &\frac{\varepsilon}{h^2} (w_{i-1} - 2w_i + w_{i+1}) + \left\{ \frac{a'(s_i + k) \left(\frac{k}{2} \right) - b(s_i + k) \left(\frac{k}{2} \right)}{h} \right\} w_{i-1} \\
 &+ \left\{ \frac{-a(s_i) - a'(s_{i+1} - k) \left(\frac{k}{2} \right) - a'(s_i + k) \left(\frac{h+k}{2} \right) + b(s_{i+1} - k) \left(\frac{k}{2} \right) + b(s_i + k) \left(\frac{h+k}{2} \right)}{h} \right\} w_i \\
 &+ \left\{ \frac{-a'(s_{i+1} - k) \left(\frac{h-k}{2} \right) + a(s_{i+1}) + b(s_{i+1} - k) \left(\frac{h-k}{2} \right)}{h} \right\} w_{i+1} = \left\{ \frac{f(s_{i+1} - k) + f(s_i + k)}{2} \right\}
 \end{aligned} \tag{12}$$

Eqn. (12) can be rewritten in a three term recurrence relation as follows:

$$A_i w_{i-1} + B_i w_i + C_i w_{i+1} = F_i ; \quad i = 1, 2, \dots, N-1 \tag{13}$$

where

$$\begin{aligned}
 A_i &= \frac{\varepsilon}{h} + a'(s_i + k) \frac{k}{2} - b(s_{i+1} + k) \frac{k}{2} \\
 B_i &= \frac{-2\varepsilon}{h} - a(s_i) - a'(s_{i+1} + k) \frac{k}{2} - a'(s_i + k) \left(\frac{h+k}{2} \right) + b(s_{i+1} + k) \left(\frac{k}{2} \right) + b(s_i + k) \left(\frac{h+k}{2} \right) \\
 C_i &= \frac{\varepsilon}{h} - a'(s_{i+1} - k) \left(\frac{h-k}{2} \right) + a(s_{i+1}) + b(s_{i+1} - k) \left(\frac{h-k}{2} \right) \\
 F_i &= \frac{h}{2} [f(s_{i+1} - k) + f(s_i + k)]
 \end{aligned}$$

The tri-diagonal system is solved efficiently by Thomas Algorithm Angel and Bellman[1].

3.2. Problem with Right - end Boundary Layer

Again by Taylor series expansion we have

$$w'(s + \varepsilon) \approx w'(s) + \varepsilon w''(s)$$

$$\text{implies} \quad \varepsilon w''(s) \approx w'(s + \varepsilon) - w'(s) \quad (14)$$

accordingly the Eqn. (4) is reduced to the first order equation with ε as the deviating argument:

$$w'(s) = w'(s + \varepsilon) + a(s)w'(s) + b(s)w(s) - f(s) \quad (15)$$

On integrating Eqn. (15) on $[s_{i-1}, s_i]$; we get

$$\int_{s_{i-1}}^{s_i} w'(s) ds = \int_{s_{i-1}}^{s_i} w'(s + \varepsilon) ds + \int_{s_{i-1}}^{s_i} a(s)w'(s) ds + \int_{s_{i-1}}^{s_i} b(s)w(s) ds - \int_{s_{i-1}}^{s_i} f(s) ds$$

Using Gaussian quadrature two-point formula for any continuous and differentiable function $F(s)$ in an arbitrary interval $[s_{i-1}, s_i]$, we get

$$\int_{s_{i-1}}^{s_i} F(s) ds = \frac{h}{2} (F(s_{i-1} - k) + F(s_i + k)) \quad (16)$$

Using Eqn. (16), from Eqn. (15) we get

$$\begin{aligned}
 w(s_i) - w(s_{i-1}) &= w(s_{i-1} + \varepsilon) - w(s_i + \varepsilon) + a(s_{i-1})y(s_{i-1}) - a(s_i)w(s_i) \\
 &\quad - \frac{h}{2} [a'(s_{i-1} - k)w(s_{i-1} - k) + a'(s_i + k)w(s_i + k)] \\
 &\quad + \frac{h}{2} [b(s_{i-1} - k)w(s_{i-1} - k) + b(s_i + k)w(s_i + k)] \\
 &\quad - \frac{h}{2} [f(s_{i-1} - k) + f(s_i + k)]
 \end{aligned} \tag{17}$$

Using linear interpolation for the terms $w(s_{i-1} + \varepsilon)$, $w(s_i + \varepsilon)$, $w(s_{i-1} - k)$ and $w(s_i + k)$ in Eqn. (17) we get;

$$\begin{aligned}
 &\left\{ \frac{\varepsilon}{h} - a(s_{i-1}) - a'(s_{i-1} + k) \left(\frac{h-k}{2} \right) + b(s_{i-1} + k) \left(\frac{h-k}{2} \right) \right\} w_{i-1} + \\
 &\left\{ \frac{-2\varepsilon}{h} + a(s_i) - a'(s_i - k) \left(\frac{h+k}{2} \right) - a'(s_{i-1} + k) \left(\frac{k}{2} \right) + b(s_i - k) \left(\frac{h+k}{2} \right) + b(s_{i-1} + k) \left(\frac{k}{2} \right) \right\} w_i \\
 &+ \left\{ \frac{\varepsilon}{h} + a'(s_i - k) \left(\frac{k}{2} \right) - b(s_i - k) \left(\frac{k}{2} \right) \right\} w_{i+1} = \frac{h}{2} \{ f(s_i - k) + f(s_{i-1} + k) \}
 \end{aligned} \tag{18}$$

Rearranging this Eqn. (18), we have

$$\begin{aligned}
 &\frac{\varepsilon}{h^2} (w_{i-1} - 2w_i + w_{i+1}) + \left\{ \frac{-a(s_{i+1}) - a'(s_{i-1} + k) \left(\frac{h-k}{2} \right) + b(s_{i-1} + k) \left(\frac{h-k}{2} \right)}{h} \right\} w_{i-1} \\
 &+ \left\{ \frac{a(s_i) - a'(s_i - k) \left(\frac{h+k}{2} \right) - a'(s_{i-1} + k) \left(\frac{k}{2} \right) + b(s_i - k) \left(\frac{h+k}{2} \right) + b(s_{i-1} + k) \left(\frac{k}{2} \right)}{h} \right\} w_i \\
 &+ \left\{ \frac{a'(s_i - k) \left(\frac{k}{2} \right) - b(s_i - k) \left(\frac{k}{2} \right)}{h} \right\} w_{i+1} = \left\{ \frac{f(s_i - k) + f(s_{i-1} + k)}{2} \right\}
 \end{aligned}$$

We arrange this as the three term recurrence relation:

$$A_i w_{i-1} + B_i w_i + C_i w_{i+1} = F_i \quad \text{for } i = 1, 2, \dots, N-1 \tag{20}$$

where

$$\begin{aligned}
 A_i &= \left\{ \frac{\varepsilon}{h} - a(s_{i-1}) - a'(s_{i-1} + k) \frac{h-k}{2} + b(s_{i-1} + k) \frac{h-k}{2} \right\}, \\
 B_i &= \left\{ \frac{-2\varepsilon}{h} + a(s_i) - a'(s_i + k) \frac{h+k}{2} - a'(s_{i-1} + k) \left(\frac{k}{2} \right) + b(s_i - k) \left(\frac{h+k}{2} \right) + b(s_{i-1} + k) \left(\frac{k}{2} \right) \right\} \\
 C_i &= \left\{ \frac{\varepsilon}{h} + a'(s_i - k) \left(\frac{k}{2} \right) - b(s_i - k) \left(\frac{k}{2} \right) \right\} \\
 F_i &= \frac{h}{2} [f(s_i - k) + f(s_{i-1} + k)]
 \end{aligned}$$

The system of Eqn. (20) is solved by using Thomas algorithm given in Angel and Bellman [1].

4. NUMERICAL EXPERIMENTS

We have solved several problems from the model problem and solution is compared with exact or solution by other methods available in literature. The exact solution of the model boundary value problem

$$\varepsilon w''(s) + p(s)w'(s) + q(s)y(s - \delta) + r(s)w(s) + t(s)w(s + \eta) = f(s)$$

under the boundary conditions

$$\begin{aligned}
 w(s) &= \varphi(s) \quad \text{on } -\delta \leq s \leq 0, \\
 w(1) &= \gamma(s) \quad \text{on } 1 \leq s \leq 1 + \eta,
 \end{aligned}$$

is given by $w(s) = c_1 e^{m_1 s} + c_2 e^{m_2 s} + \frac{f}{c}$

where

$$c = q + r + t$$

$$c_1 = \frac{(-f + \gamma c + e^{m_2} (f - \phi c))}{(c(e^{m_1} - e^{m_2}))},$$

$$c_2 = \frac{(f - \gamma c + e^{m_1} (-f + \phi c))}{(c(e^{m_1} - e^{m_2}))}$$

$$m_1 = \frac{\left[-(p - q\delta + t\eta) + \sqrt{(p - q\delta + t\eta)^2 - 4c\varepsilon} \right]}{2\varepsilon}$$

$$m_2 = \frac{\left[-(p - q\delta + t\eta) - \sqrt{(p - q\delta + t\eta)^2 - 4c\varepsilon} \right]}{2\varepsilon}$$

Problem 1. Consider the problem with layer at left end

$$\varepsilon w''(s) + w'(s) + 2w(s - \delta) - 3w(s) = 0$$

$$\text{with } w(s) = 1, \quad -\delta \leq s \leq 0, \quad w(s) = 1, \quad 1 \leq s \leq 1 + \eta$$

Problem 2. Consider the problem with layer at left end

$$\varepsilon w''(s) + w'(s) - 3w(s) + 2w(s + \eta) = 0$$

$$\text{with } w(s) = 1, \quad -\delta \leq s \leq 0, \quad w(s) = 1, \quad 1 \leq s \leq 1 + \eta$$

Problem 3. Consider the problem with layer at left end

$$\varepsilon w''(s) + w'(s) - 2w(s - \delta) - 5w(s) + w(s + \eta) = 0$$

$$\text{with } w(s) = 1, \quad -\delta \leq s \leq 0, \quad w(s) = 1, \quad 1 \leq s \leq 1 + \eta$$

Problem 4. Consider the non-homogeneous problem with layer at left end

$$\varepsilon w''(s) + w'(s) - 2w(s - \delta) + w(s) - w(s + \eta) = -1$$

$$\text{with } w(s) = 1, \quad -\delta \leq s \leq 0, \quad w(s) = 1, \quad 1 \leq s \leq 1 + \eta$$

Problem 5. Consider the problem with layer at right end

$$\varepsilon w''(x) - w'(s) - 2w(s - \delta) + w(s) = 0$$

$$\text{with } w(s) = 1, \quad -\delta \leq s \leq 0, \quad w(s) = -1, \quad 1 \leq s \leq 1 + \eta$$

Problem 6. Consider the problem with right end layer

$$\varepsilon w''(s) - w'(s) + w(s) - 2w(s + \eta) = 0$$

$$\text{with } w(s) = 1, \quad -\delta \leq s \leq 0, \quad w(s) = -1, \quad 1 \leq s \leq 1 + \eta$$

Problem 7. Consider the problem with layer at right end

$$\varepsilon w''(s) - w'(s) - 2w(s - \delta) + w(s) - 2w(s + \eta) = 0$$

$$\text{with } w(s) = 1, \quad -\delta \leq s \leq 0, \quad w(s) = -1, \quad 1 \leq s \leq 1 + \eta$$

5. DISCUSSIONS AND CONCLUSION

The Gaussian quadrature two-point scheme is applied for the solution of differential difference equations having boundary layers. An equivalent first order differential equation with the perturbation parameter as deviating argument is deduced from the given boundary value problem. Then, Gaussian two-point quadrature technique is implemented to solve the first order equation with deviating parameter. Several numerical problems are illustrated to demonstrate the method. Comparison of maximum errors in the solution of the problems with other methods is tabulated to justify the method. From the tables, we noticed that, the proposed scheme produced accurate results to the given problem.

Table 1. Maximum errors in Problem 1 with $\varepsilon = 0.1$

$N \rightarrow$	8	32	128	512
$\delta \downarrow$	Proposed method			
0.00	1.1834e-02	5537e-03	1770e-04	2.3113e-04
0.05	1.3305e-02	8619e-03	9278e-04	2.4977e-04
0.09	1.4512e-02	4.1191e-03	1.0557e-03	2.6536e-04
$\delta \downarrow$	Results by Kadalbajoo and Sharma [84]			
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09659609	0.03640566	0.00924661	0.00202998
0.09	0.09277401	0.03556652	0.00895172	0.00192488

Table 2. Maximum errors in Problem 2 with $\varepsilon = 0.1$

$N \rightarrow$	8	32	128	512
$\eta \downarrow$ Proposed method				
0.00	0037e-03	2.8330e-03	7.4107e-04	1.8720e-04
0.05	8.1037e-03	2.6115e-03	6.8579e-04	1.7341e-04
0.09	7.4391e-03	2.4443e-03	6.4520e-04	1.6328e-04
$\eta \downarrow$ Results by Kadalbajoo and Sharma [7-8]				
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09977501	0.03727087	0.00979659	0.00224472
0.09	0.10031348	0.03723863	0.00996284	0.00458698

Table 3 Maximum error for Problem 3 with $\delta = \eta = 0.5\varepsilon$

$\varepsilon \downarrow N \rightarrow$	8	16	32	64	128	256
Proposed method						
10^{-1}	6.5260e-02	3.179e-02	1.7272e-02	8.7909e-03	4.4321e-03	2.2250e-03
10^{-2}	4.5711e-02	2.6776e-02	1.5201e-02	1.0011e-02	5.9380e-03	2.104e-03
10^{-3}	4.7675e-02	2.8249e-02	1.5423e-02	8.0250e-03	4.0492e-03	2.0555e-03
Results by Kadalbajoo and Sharma [7-8]						
10^{-1}	0.12011566	0.07181396	0.04482982	0.02694612	0.01516093	0.00775036
10^{-2}	0.18727108	0.10697821	0.05904116	0.03079689	0.01567964	0.00799076
10^{-3}	0.20429729	0.11915028	0.06879232	0.03655236	0.01893849	0.00963304

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Table 4. Maximum error for Problem 3 with $\varepsilon = 0.1$

$N \rightarrow$	8	32	128	512
Proposed method				
$\delta \downarrow$	$\eta = 0.05$			
0.00	6.4039e-02	1.7890e-02	4.5754e-03	1.1494e-03
0.05	6.0475e-02	1.7272e-02	4.4321e-03	1.1147e-03
0.09	5.7591e-02	1.6748e-02	4.3186e-03	1.0870e-03
$\eta \downarrow$	$\delta = 0.05$			
0.00	6.2267e-02	1.7587e-02	4.5038e-03	1.1321e-03
0.05	6.0475e-02	1.7272e-02	4.4321e-03	1.1147e-03
0.09	5.9033e-02	1.7013e-02	4.3752e-03	1.1008e-03
Results by Kadalbajoo and Sharma [7-8]				
$\delta \downarrow$	$\eta = 0.05$			
0.00	0.09190267	0.03453494	0.01164358	0.00300463
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.11018870	0.04110846	0.01400144	0.00362925
$\eta \downarrow$	$\delta = 0.05$			
0.00	0.09720079	0.03640446	0.01229476	0.00317786
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.10632014	0.03965833	0.01348348	0.00349050

Table 5. Maximum error for Problem 4 for $\delta = \eta = 0.5\varepsilon$

$\varepsilon \downarrow N \rightarrow$	8	16	32	64	128
Proposed method					
10^{-1}	1.2449e-02	7.0229e-03	6934e-03	1.8905e-03	5553e-04
10^{-2}	6.9143e-03	4230e-03	0039e-03	2.0620e-03	1.1975e-03
10^{-3}	6498e-03	5.0433e-03	2.4786e-03	1.1279e-03	4.3665e-04
Results by Kadalbajoo and Sharma [7-8]					
10^{-1}	0.08579690	0.05129568	0.03202130	0.01924723	0.01098354
10^{-2}	0.13376506	0.07641301	0.04217226	0.02199778	0.01119974
10^{-3}	0.14592663	0.08510734	0.04913737	0.02610883	0.01352749

Table 6. Maximum errors for Problem 5 with $\varepsilon = 0.1$

$N \rightarrow$	8	32	128	512
	Proposed method			
$\delta \downarrow$				
0.00	4573e-02	3435e-03	2.4536e-03	6.2174e-04
0.05	7012e-02	1.0039e-02	2.6180e-03	6.6231e-04
0.09	8931e-02	1.0571e-02	2.7569e-03	6.9686e-04
	Results by Kadalbajoo and Sharma [7-8]			
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.09222560	0.03828329	0.01487799	0.00380679
0.09	0.10509460	0.03149275	0.01299340	0.00331935

Table 7. Maximum errors in Problem 6 with $\varepsilon = 0.1$

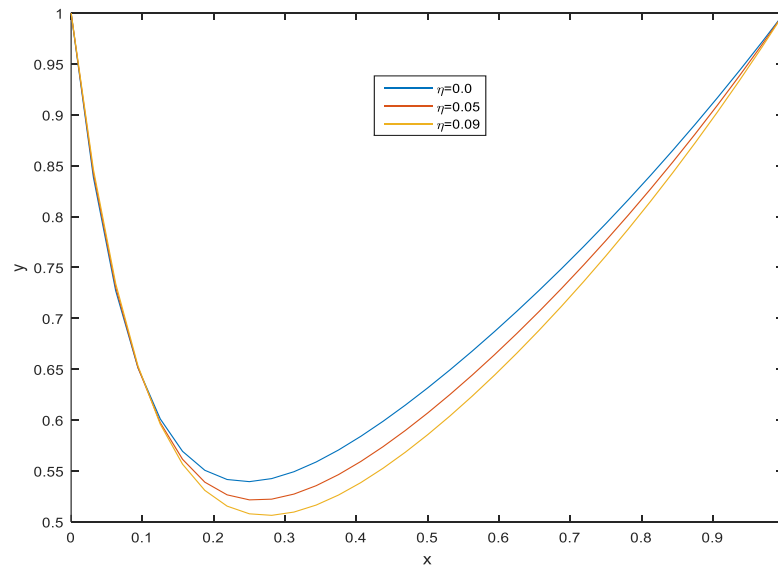
$N \rightarrow$	8	32	128	512
	Proposed method			
$\eta \downarrow$				
0.00	4573e-02	3435e-03	2.4536e-03	6.2174e-04
0.05	2177e-02	8.7029e-03	2.3021e-03	5.8424e-04
0.09	0332e-02	8.2900e-03	2.1895e-03	5.5647e-04
	Results by Kadalbajoo and Sharma [7-8]			
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.06834579	0.05516436	0.01972508	0.00506769
0.09	0.08328237	0.06168267	0.02169662	0.00558451

Table 8. Maximum errors in Problem 7 with $\varepsilon = 0.1$

$N \rightarrow$	8	32	128	512
$\delta \downarrow$	$\eta = 0.05$ Proposed method			
0.00	6.1682e-02	1.6643e-02	4.3793e-03	1.1118e-03
0.05	6.3459e-02	1.6949e-02	4.4649e-03	1.1320e-03
0.09	6.4697e-02	1.7233e-02	4.5266e-03	1.1463e-03
$\eta \downarrow$	$\delta = 0.05$			
0.00	6.4977e-02	1.7317e-02	4.5402e-03	1.1496e-03
0.05	6.3459e-02	1.6949e-02	4.4649e-03	1.1320e-03
0.09	6.2055e-02	1.6711e-02	4.3982e-03	1.1160e-03

Results by Kadalbajoo and Sharma [7-8]

$\delta \downarrow$	$\eta = 0.05$			
0.00	0.09930002	0.03685072	0.01331683	0.00342882
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.10044578	0.02850398	0.01038902	0.00266379
$\eta \downarrow$	$\delta = 0.05$			
0.00	0.10055269	0.02759534	0.01007834	0.00258299
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.09944067	0.03591410	0.01297367	0.00334044

**Fig. 1.** Numerical solution of Problem 1 with $\varepsilon = 0.1$.

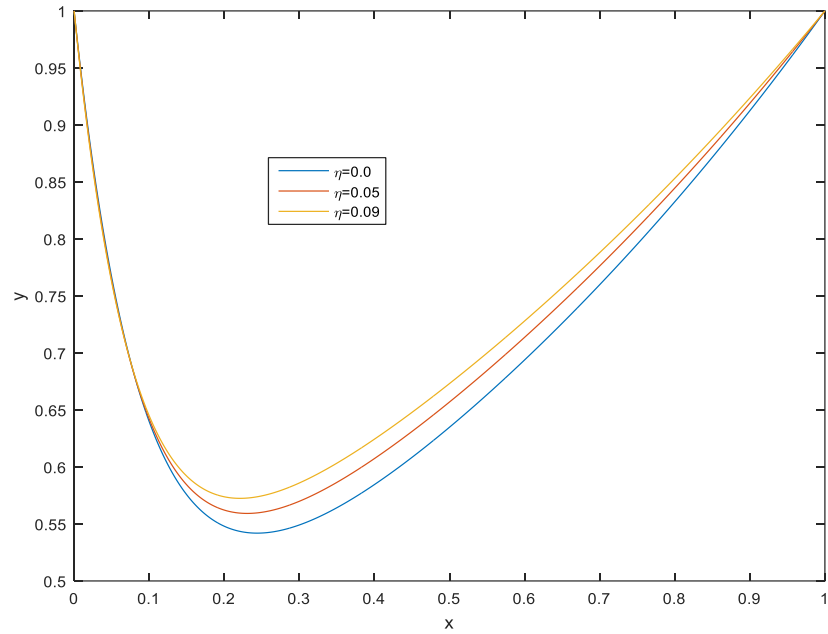


Fig. 2. Numerical solution of Problem 2 with $\varepsilon = 0.1$

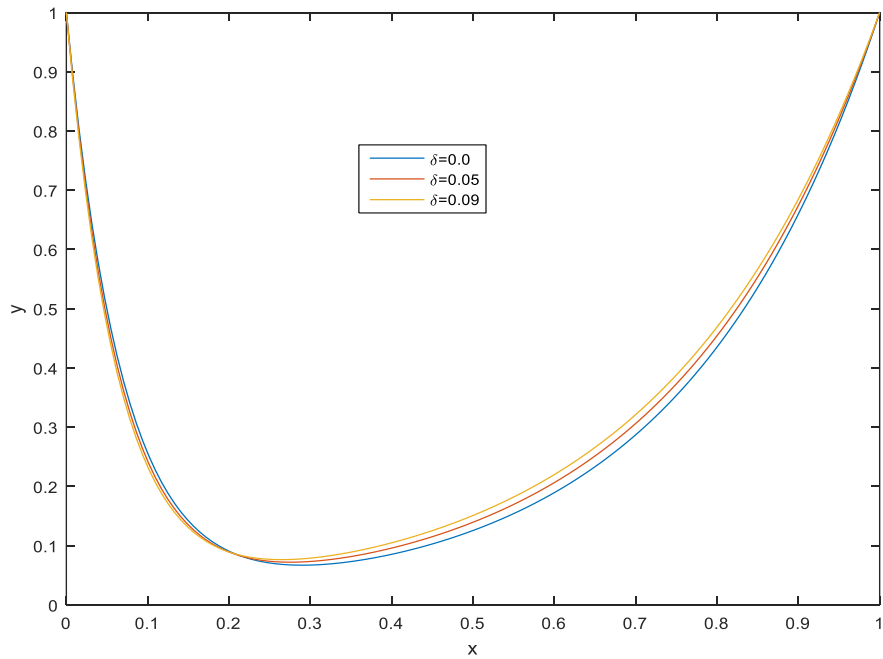


Fig. 3. Numerical solution of Problem 3 with $\varepsilon = 0.1$ and $\eta = 0.05$.

GAUSSIAN QUADRATURE METHOD

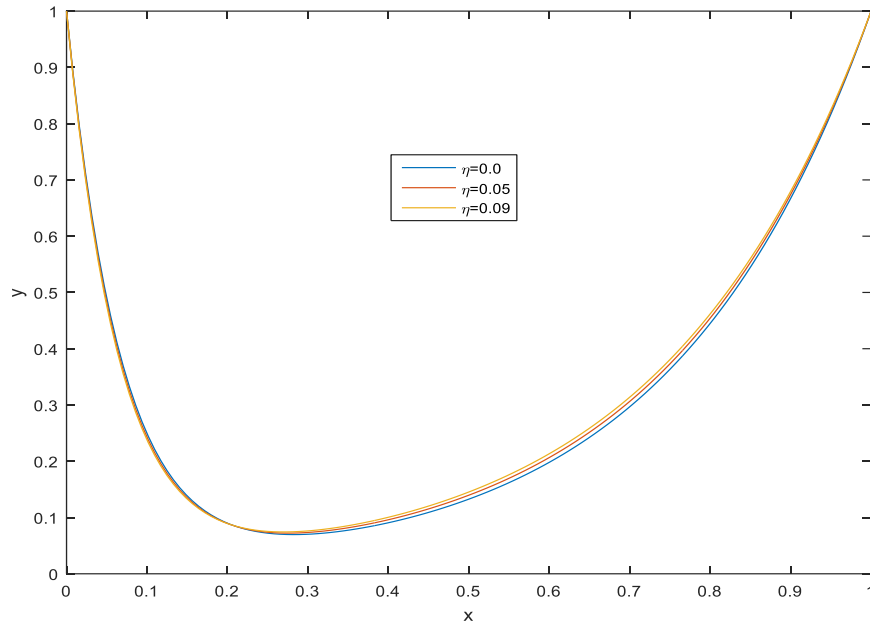


Fig. 4. Numerical solution of Problem 3 with $\varepsilon = 0.1$ and $\delta = 0.05$

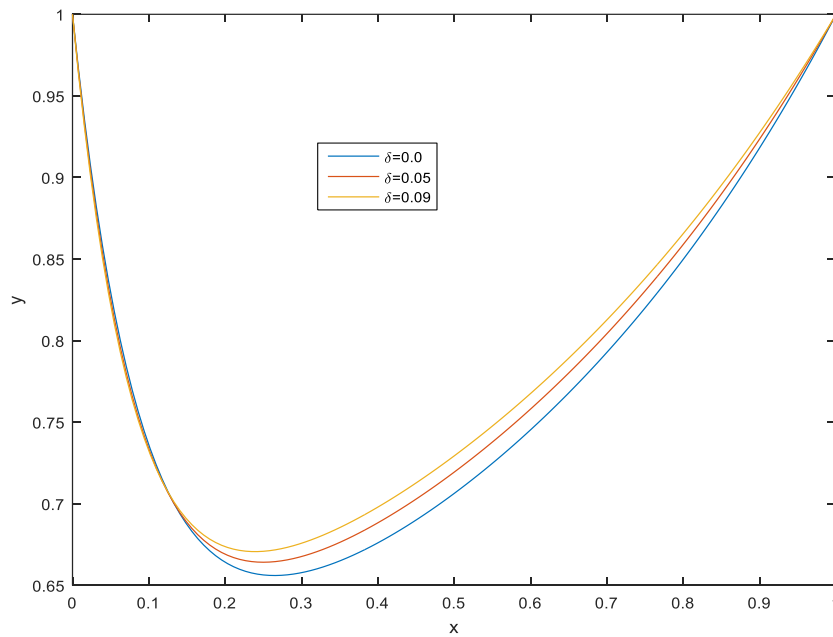


Fig. 5. Numerical solution of Problem 4 with $\varepsilon = 0.1$ and $\eta = 0.05$.

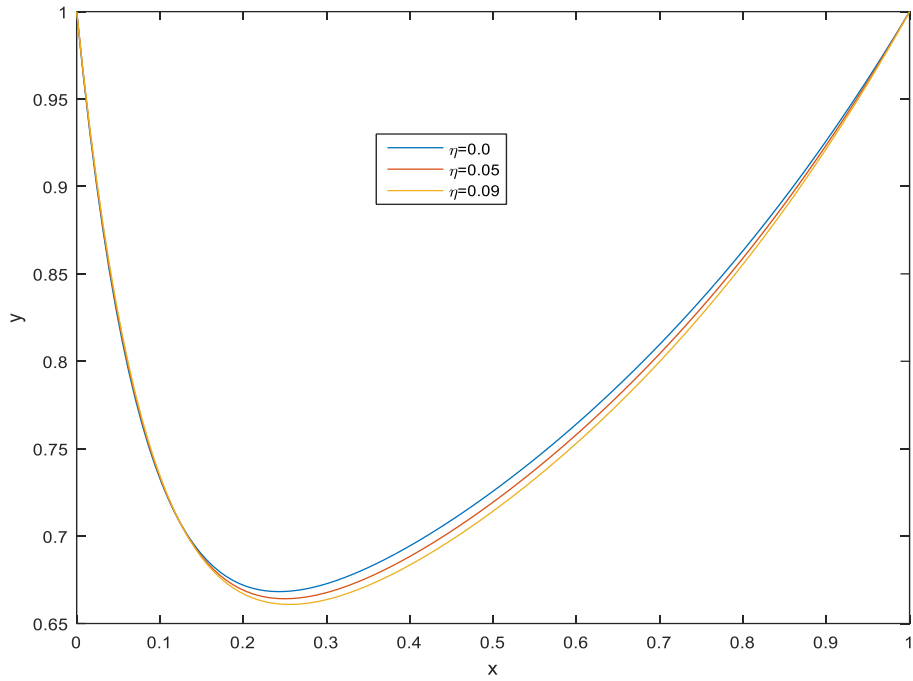


Fig. 6. Numerical solution of Problem 4 with $\varepsilon = 0.1$ and $\delta = 0.05$.

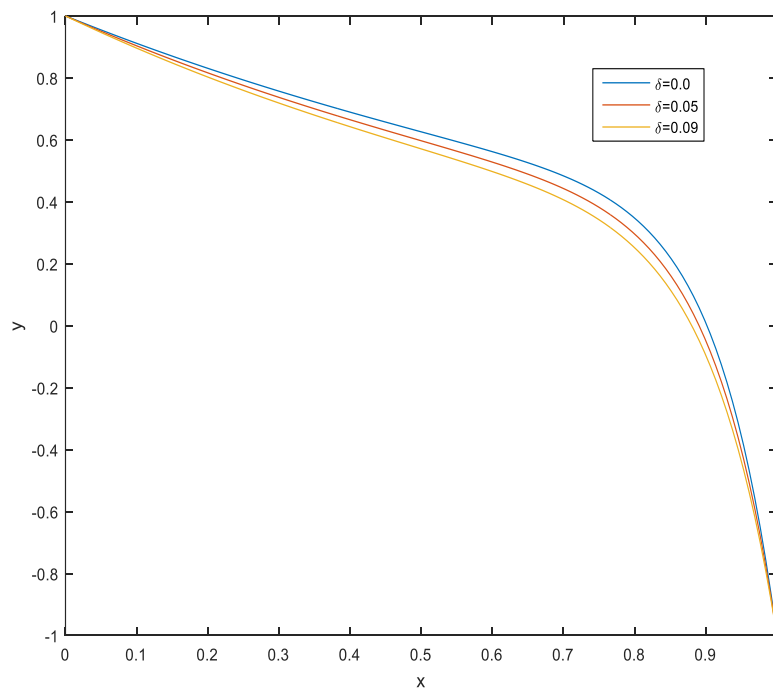


Fig. 7. Numerical solution of Problem 5 with $\varepsilon = 0.1$ and $\eta = 0$.

GAUSSIAN QUADRATURE METHOD

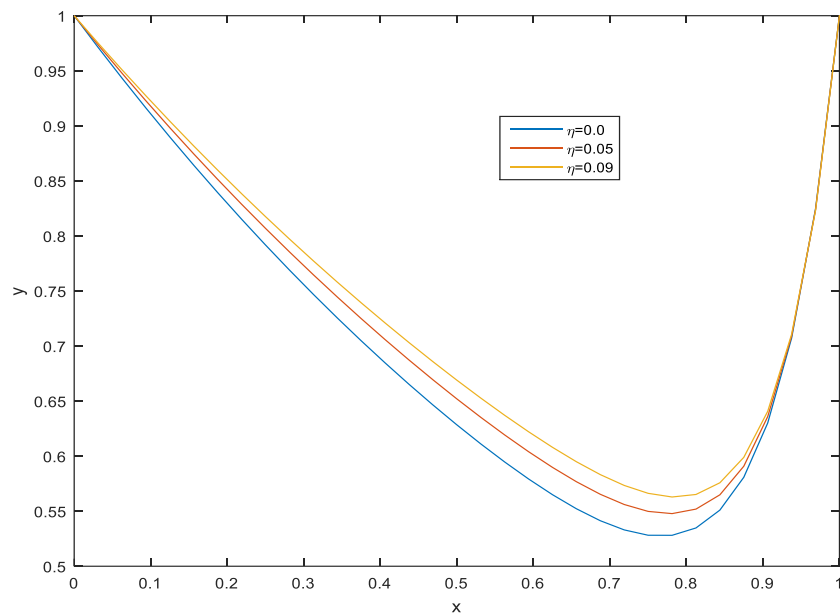


Fig. 8. Numerical solution of Problem 6 with $\varepsilon = 0.1$ and $\delta = 0$.

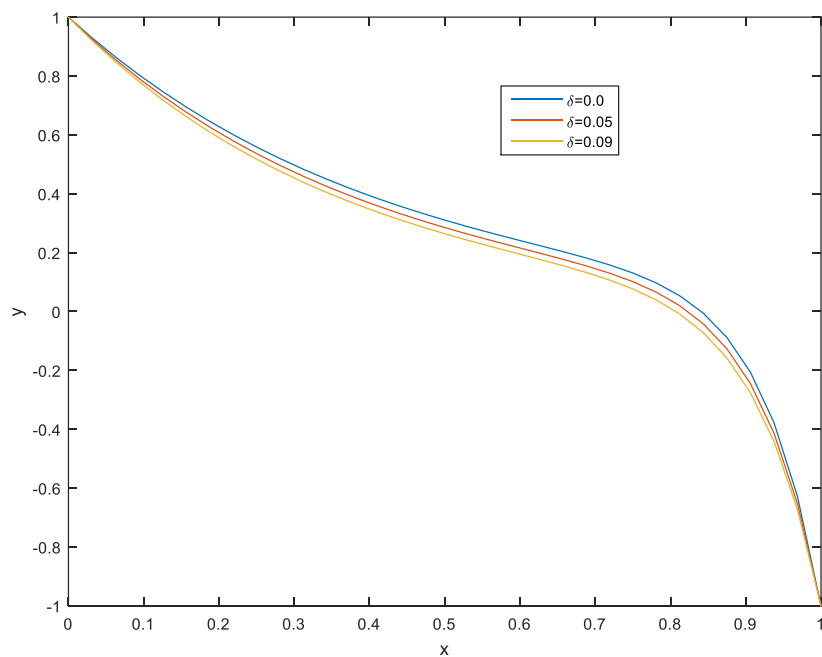


Fig. 9. Numerical solution of Problem 7 with $\varepsilon = 0.1$ and $\eta = 0.05$.

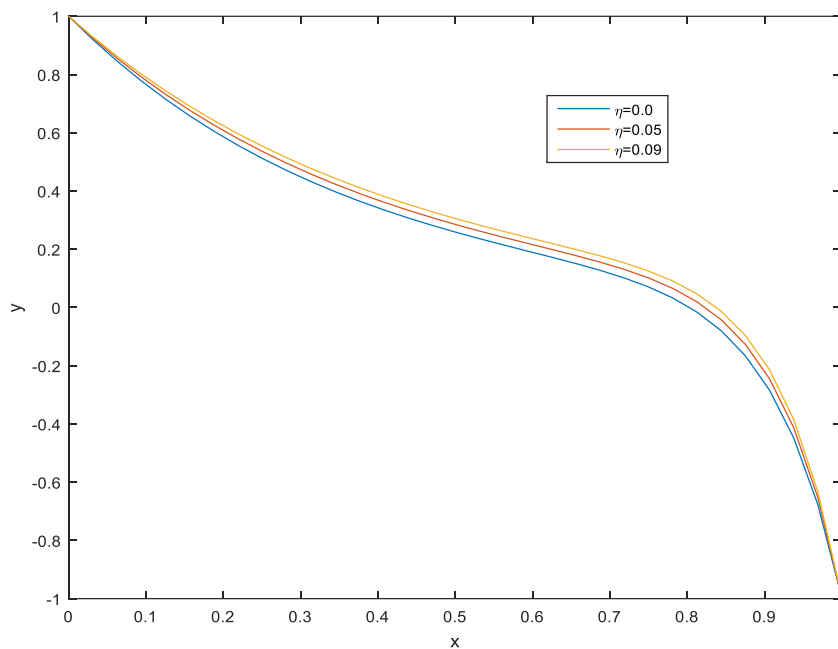


Fig. 10. Numerical solution of Problem 7 with $\varepsilon = 0.1$ and $\delta = 0.05$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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