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DIFFERENTIAL SANDWICH THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY A NEW LINEAR DERIVATIVE OPERATOR

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Abstract: In this paper, we consider a new operator $RI_{p,\alpha,\beta}^{m,\lambda} : A_p \to A_p, p \in N,$ defined by $RI_{p,\alpha,\beta}^{m,\lambda} f(z) = (1-\lambda)R_p^m f(z) + \lambda I_{p,\alpha,\beta}^m f(z), \lambda \geq 0,$ where A_p denote the class of analytic functions in the unit disc $U = \{z : z \in C, |z| < 1\}$, of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, R_p^m f(z), p \in N, m \in N_0 = N \cup \{0\}$ is the Ruscheweyh operator and $I_{p,\alpha,\beta}^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + p\beta}\right)^m a_k z^k$, $p \in N, m \in N_0 = N \cup \{0\},$ $\beta \geq 0,$ and α a real number with $\alpha + p\beta > 0.$ Few interesting results of differential subordination and superordination are obtained using the new operator $RI_{p,\alpha,\beta}^{m,\lambda}$. Further, we also consider the sandwich-type results for this operator.

Keywords: Multiplier transformation, differential subordination, differential superordination, subordinant, dominant.

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1. Introduction

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Denote by U the open unit disc of the complex plane, $U=\{z\in C:|z|<1\}$. Let $\mathrm{H}(U)$ be the space of analytic functions in U. For $p\in N, a\in C$ we define:

$$\mathrm{H}[a,p]=\{f\in\mathrm{H}(U): f(z)=a+a_p z^p+a_{p+1}z^{p+1}+...\}, z\in U,$$

$$A_p=\{f\in\mathrm{H}(U): f(z)=z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+...,\}, z\in U,$$

and we set $A_1=A,$ a well-known class of normalized analytic functions in U.

For $f,g\in\mathrm{H}(U),$ we say that the function f is subordinate to g, or the function g is superordinate to f , if there exists a Schwarz function $w,$ analytic in U, with $w(0)=0$ and $|w(z)|<1$, for all $z\in U,$ such that $f(z)=g(w(z)),$ for $z\in U.$ In such a case we write $f\prec g.$ Furthermore, if the function g is univalent in U ,then we have the following equivalence (See [15],[16] and [17]):

$$f(z)\prec g(z) \text{ if and only if } f(0)=g(0) \text{ and } f(U)\subset g(U).$$

Supposing that h and g are two analytic functions in U, let $\varphi(r,s,t;z):C^3\times U\to C.$ If h and $\varphi(h(z),zh'(z),z^2h''(z);z)$ are univalent functions in U and if h satisfies the second-order superordination

$$g(z)\prec\varphi(h(z),zh'(z),z^2h''(z);z), \tag{1.1}$$

then g is called to be a solution of the differential superordination (1.1). A function $q\in\mathrm{H}(U)$ is called a subordinant of (1.1), if $q(z)\prec h(z)$ for all the functions h satisfying (1.1). A univalent subordinant $\tilde{q}$ that satisfies $q(z)\prec\tilde{q}(z)$ for all of the subordinants q of (1.1), is said to be the best subordinant.

Recently, Miller and Mocanu [20] obtained sufficient conditions on the functions g,q and φ for which the following implication holds:

$$g(z)\prec\varphi(h(z),zh'(z),z^2h''(z);z)\Rightarrow g(z)\prec h(z).$$

Using the results of Miller and Mocanu [20], Bulboaca [7] considered certain classes of first order differential superordinations. Ali et. al. [2], have used the results of Bulboaca [7] to obtain sufficient conditions for normalised analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent normalised functions in U.

Very recently, Macovei ([13] and [14]) obtained differential subordinations and superordinations for analytic functions defined by the Ruscheweyh linear operator and the author [30] extended and improved these results for certain subclasses of analytic functions defined by the Ruscheweyh derivative and a new generalized multiplier transformation(see [28]).

We now state the following definitions with few remarks.

Definition 1.1 ([28]). For $f \in A_p, m \in N_0 = N \cup \{0\}, \beta \geq 0$ and α a real number with $\alpha + p\beta > 0$, a new generalized multiplier operator $I^m_{p,\alpha,\beta}$ is defined by the following infinite series:

$$I^m_{p,\alpha,\beta} f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + p\beta} \right)^m a_k z^k, z \in U. \tag{1.2}$$

It follows from (1.2) that

$$I^m_{p,\alpha,0} f(z) = f(z),$$

$$(\alpha + p\beta) I^{m+1}_{p,\alpha,\beta} f(z) = \alpha I^m_{p,\alpha,\beta} f(z) + \beta z (I^m_{p,\alpha,\beta} f(z))', \tag{1.3}$$

We note that

- $I^m_{1,\alpha,\beta} f(z) = I^m_{\alpha,\beta} f(z)$ (See [27]).

- $I_{p,\alpha,1}^{m} f(z) = I_p^m(\alpha) f(z), \alpha > -p$ (See [1], [25] and [26]).
- $I_{p,l+p-p\beta,\beta}^{m} f(z) = I_p^m(\beta, l) f(z), l > -p, \beta \geq 0$ (See Catas [8]).
- $I_{p,0,\beta}^{m} f(z) = D_p^m f(z)$ (See [5], [12] and [22]).

Remark 1.3 i) $I_p^m(\alpha) f(z)$ was considered in [1], [25] and [26] for $\alpha \geq 0$ and $I_p^m(\beta, l) f(z)$ was defined in [8] for $l \geq 0, \beta \geq 0$, ii) $I_p^m(l) f(z) = I_p^m(1, l) f(z), l > -p$, iii) $I_p^m(\beta, 0) f(z) = D_p^m(\beta) f(z)$, $\beta \geq 0$, was mentioned in Aouf et.al. [4], iv) $D_1^m(\beta), \beta \geq 0$, was introduced by Al-Oboudi [3],v) $D_1^m(1) f(z) = D^m f(z)$ was defined by Salagean [24] and was considered for $m \geq 0$ in [6] , vi) $I_1^m(\alpha) f(z), \alpha \geq 0,$ was investigated in [9] and [10] and vii) $I_1^m(1)$ was due to Uralegaddi and Somanatha[33].

Definition 1.3 (Goel and sohi [11]). For $m \in N_0$, $f \in A_p$, the operator R_p^m is defined by $R_p^m : A_p \to A_p$,

$$R_p^0 f(z) = f(z),$$

$$R_p^1 f(z) = zf'(z)/p,$$

$$\ldots$$

(1.4) $$(m+p) R_p^{m+1} f(z) = z(R_p^m f(z))' + m R_p^m f(z), z \in U.$$

Remark 1.4. The operator $R_1^m = R^m$ was introduced and studied by Ruscheweyh in [23].

Definition 1.5. Let $m \in N_0, \lambda \geq 0, \beta \geq 0$ and α a real number with $\alpha + p\beta > 0.$ Denote by $RI_{p,\alpha,\beta}^{m,\lambda}$ the operator given by $RI_{p,\alpha,\beta}^{m,\lambda} : A_p \to A_p$,

$$RI_{p,\alpha,\beta}^{m,\lambda} f(z) = (1-\lambda) R_p^m f(z) + \lambda I_{p,\alpha,\beta}^{m} f(z), z \in U.$$

Remark 1.6. Clearly $RI_{p,\alpha,\beta}^{m,0} = R_p^m$ and $RI_{p,\alpha,\beta}^{m,1} = I_{p,\alpha,\beta}^{m}$. The operator $RI_{1,\alpha,\beta}^{m,\lambda} = RI_{\alpha,\beta}^{m,\lambda}$ was introduced in [29] and examined in [30], [31] and [32].

In this paper, we investigate interesting results of differential subordination and superordination, using the new operator $RI_{p,\alpha,\beta}^{m,\lambda}$. Further, we also consider the sandwich-type results for this operator.

2. **Preliminaries**

In order to prove our results, we need the following definition and lemmas.

Definition 2.1 ([20]). We denote by Q, the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\}$ and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

Lemma 2. 2([21]). Let the function q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that either

i) h is convex in U or

ii) Q is starlike in U.

In addition, assume that

iii) $\mathrm{Re}\left(\dfrac{zh'(z)}{Q(z)}\right) > 0, z \in U$.

If $p(z) \in \mathrm{H}(U)$, with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$, and q is the best dominant.

Lemma 2. 3([7]). let θ and φ be analytic in a domain D and let q be univalent in U, with $q(U) \subset D$. Set $Q(z) = zq'(z)\varphi(q(z))$ and suppose that

i) Q is starlike in U

and

$$\text{ii) } \mathrm{Re}\left(\frac{\theta'(q(z))}{\varphi(q(z))}\right) > 0, z \in U.$$

If $p(z) \in \mathrm{H}[q(0),1] \cap \mathrm{Q}$, $p(U) \subset D$ and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U and $\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z))$ then $q(z) \prec p(z), z \in U$ and q is the best subordinant.

3. Main Results

Theorem 3.1. Let $f \in A_p, m \in N_0 = N \cup \{0\}, \mu > 0, \lambda \geq 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0$. Let the function q be univalent in U and suppose that it satisfies the conditions

(3.1) $$\mathrm{Re}(q(z)) > 0, z \in U$$

and

(3.2) $$\mathrm{Re}\left[\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + 1\right] > 0, z \in U.$$

Let

(3.3) $$\Phi_p(m,\mu,\lambda,\alpha,\beta;z) = \left(\frac{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}{z^p}\right)^{\mu} + \mu(m+p)\frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)} - \mu(m+p)$$

$$+\mu\lambda\left[\left(\frac{\alpha+p\beta}{\beta} - (m+p)\right)\frac{I_{p,\alpha,\beta}^{m+1} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)} - \left(\frac{\alpha}{\beta} - m\right)\frac{I_{p,\alpha,\beta}^{m} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}\right].$$

If

(3.4) $$\Phi_p(m,\mu,\lambda,\alpha,\beta;z) \prec q(z) + \frac{zq'(z)}{q(z)}, z \in U,$$

then

$$\left(\frac{RI^{m,\lambda}_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu} \prec q(z), \text{and } q \text{ is the best dominant.}$$

Proof. Define the function $p(z)$ by

$$p(z)=\left(\frac{RI^{m,\lambda}_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}, z\in U. \tag{3.5}$$

Differentiating (3.5) logarithmically, with respect to z, and making use of (1.3) and (1.4), we get,

$$p(z)+\frac{zp'(z)}{p(z)}=\left(\frac{RI^{m,\lambda}_{p,\alpha,\beta}f(z)}{z^p}\right)^{\mu}+\mu(m+p)\frac{RI^{m+1,\lambda}_{p,\alpha,\beta}f(z)}{RI^{m,\lambda}_{p,\alpha,\beta}f(z)}-\mu(m+p)+ \tag{3.6}$$

$$+\mu\lambda\left[\left(\frac{\alpha+p\beta}{\beta}-(m+p)\right)\frac{I^{m+1}_{p,\alpha,\beta}f(z)}{RI^{m,\lambda}_{p,\alpha,\beta}f(z)}-\left(\frac{\alpha}{\beta}-m\right)\frac{I^{m}_{p,\alpha,\beta}f(z)}{RI^{m,\lambda}_{p,\alpha,\beta}f(z)}\right].$$

From (3.3), (3.4) and (3.6), we obtain $p(z)+\frac{zp'(z)}{p(z)}\prec q(z)+\frac{zq'(z)}{q(z)}, z\in U.$

By setting $\theta(w)=w$ and $\phi(w)=1/w$, it can easily be observed that $\theta(w)$ is analytic in the complex plane C and $\phi(w)$ is analytic in the complex plane $C\setminus\{0\}$ and that $\phi(w)\neq 0, w\in C\setminus\{0\}$. Also, by letting

$$Q(z)=zq'(z)\phi(q(z))=\frac{zq'(z)}{q(z)}$$

and

$$h(z)=\theta(q(z))+Q(z)=q(z)+\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike in U (on using (3.2)) and that $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)>0, z\in U$ (on using (3.1) and (3.2)).Hence the result now follows by an application of Lemma 2.2.

Theorem 3.2. Let the function q be convex in U and suppose that it satisfies the relations (3.2) and

(3.7) $\operatorname{Re}(q(z)q'(z)) > 0, z \in U.$

Let $f \in A_p, m \in N_0 = N \cup \{0\}, \mu > 0, \lambda \geq 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0,$ and

$\left(\frac{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}{z^p}\right)^{\mu} \in \mathrm{H}[q(0),1] \cap \mathrm{Q}$. If the function $\Phi_p(m,\mu,\lambda,\alpha,\beta;z)$, given by (3.3), is

univalent in U and $q(z) + \frac{zq'(z)}{q(z)} \prec \Phi_p(m,\mu,\lambda,\alpha,\beta;z)$, then

$$q(z) \prec \left(\frac{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}{z^p}\right)^{\mu}, \text{and } q \text{ is the best subordinant.}$$

Proof. It can be proved easily by using the same technique of Theorem 3.1 and by an application of Lemma 2.3.

Combining the results of Theorem 3.1 and Theorem 3.2, we state the following Sandwich theorem.

Theorem 3.3. Let $f \in A_p, m \in N_0 = N \cup \{0\}, \mu > 0, \lambda \geq 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0,$ and $\left(\frac{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}{z^p}\right)^{\mu} \in \mathrm{H}[q(0),1] \cap \mathrm{Q}$. Let $\Phi_p(m,\mu,\lambda,\alpha,\beta;z)$, given by (3.3), be univalent in U. Let q_1 be convex in U and q_2 be univalent in U. Suppose that the function q_1 satisfy relations (3.2) and (3.7) and the function q_2 satisfy relations (3.1) and (3.2). If

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \Phi_p(m,\mu,\lambda,\alpha,\beta;z) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}{z^p}\right)^{\mu} \prec q_2(z),$$

q_1 and q_2 are respectively the best subordinant and best dominant.

Theorem 3.4. Let $f \in A_p, m \in N_0 = N \cup \{0\}, \mu > 0, \lambda \geq 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0.$ Let the function q be univalent in U and suppose that it satisfies the conditions (3.1) and (3.2). Let

$$(3.9)\quad \Psi_p(m,\mu,\lambda,\alpha,\beta;z) = \left(\frac{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)}{z^p}\right)\left(\frac{z^p}{RI^{m,\lambda}_{p,\alpha,\beta} f(z)}\right)^{\mu} + (m+p+1)\left[\frac{RI^{m+2,\lambda}_{p,\alpha,\beta} f(z)}{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)} - 1\right]$$

$$+ \mu(m+p)\left[1 - \frac{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)}{RI^{m,\lambda}_{p,\alpha,\beta} f(z)}\right] + \lambda\left(\frac{\alpha + p\beta}{\beta} - (m+p+1)\right)\frac{I^{m+2}_{p,\alpha,\beta} f(z)}{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)}$$

$$- \lambda\mu\left(\frac{\alpha + p\beta}{\beta} - (m+p)\right)\frac{I^{m+1}_{p,\alpha,\beta} f(z)}{RI^{m,\lambda}_{p,\alpha,\beta} f(z)} - \lambda\left(\frac{\alpha}{\beta} - (m+1)\right)\frac{I^{m+1}_{p,\alpha,\beta} f(z)}{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)}$$

$$+ \lambda\mu\left(\frac{\alpha}{\beta} - m\right)\frac{I^{m}_{p,\alpha,\beta} f(z)}{RI^{m,\lambda}_{p,\alpha,\beta} f(z)}, z \in U.$$

If

$$(3.10)\qquad \Psi_p(m,\mu,\lambda,\alpha,\beta;z) \prec q(z) + \frac{zq'(z)}{q(z)}$$

then

$$\left(\frac{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)}{z^p}\right)\left(\frac{z^p}{RI^{m,\lambda}_{p,\alpha,\beta} f(z)}\right)^{\mu} \prec q(z), z \in U, \text{ and } q \text{ is the best dominant.}$$

Proof. Let $p(z) = \left(\frac{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)}{z^p}\right)\left(\frac{z^p}{RI^{m,\lambda}_{p,\alpha,\beta} f(z)}\right)^{\mu}$. Then the function $p(z)$ is analytic in U and $p(0) = 1.$ Differentiating this function logarithmically, with respect to z, and making use of (1.3) and (1.4), we obtain

$$(3.11)\quad p(z) + \frac{zp'(z)}{p(z)} = \left(\frac{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)}{z^p}\right)\left(\frac{z^p}{RI^{m,\lambda}_{p,\alpha,\beta} f(z)}\right)^{\mu} + (m+p+1)\left[\frac{RI^{m+2,\lambda}_{p,\alpha,\beta} f(z)}{RI^{m+1,\lambda}_{p,\alpha,\beta} f(z)} - 1\right]$$

$$+\mu(m+p)\left[1-\frac{RI_{p,\alpha,\beta}^{m+1,\lambda}f(z)}{RI_{p,\alpha,\beta}^{m,\lambda}f(z)}\right]+\lambda\left(\frac{\alpha+p\beta}{\beta}-(m+p+1)\right)\frac{I_{p,\alpha,\beta}^{m+2}f(z)}{RI_{p,\alpha,\beta}^{m+1,\lambda}f(z)}$$

$$-\lambda\mu\left(\frac{\alpha+p\beta}{\alpha}-(m+p)\right)\frac{I_{p,\alpha,\beta}^{m+1}f(z)}{RI_{p,\alpha,\beta}^{m,\lambda}f(z)}-\lambda\left(\frac{\alpha}{\beta}-(m+1)\right)\frac{I_{p,\alpha,\beta}^{m+1}f(z)}{RI_{p,\alpha,\beta}^{m+1,\lambda}f(z)}$$

$$+\lambda\mu\left(\frac{\alpha}{\beta}-m\right)\frac{I_{p,\alpha,\beta}^{m}f(z)}{RI_{p,\alpha,\beta}^{m,\lambda}f(z)},z\in U.$$

From (3.10) and (3.11), we have $p(z)+\frac{zp'(z)}{p(z)}\prec q(z)+\frac{zq'(z)}{q(z)},z\in U.$

We apply now Lemma 2.2, with the functions $\theta(w)=w$ and $\phi(w)=1/w$ to obtain the conclusion of our theorem.

Theorem 3.5. Let $f\in A_p, m\in N_0=N\cup\{0\},\mu>0,\lambda\geq 0,\beta>0,\alpha$ a real number such that $\alpha+p\beta>0$, and $\left(\frac{RI_{p,\alpha,\beta}^{m+1,\lambda}f(z)}{z^p}\right)\left(\frac{z^p}{RI_{p,\alpha,\beta}^{m,\lambda}f(z)}\right)^{\mu}\in \mathrm{H}[q(0),1]\cap$ Q. Let $\Psi_p(m,\mu,\lambda,\alpha,\beta;z)$, defined by (3.9), be univalent in U. Let the function q be convex in U and suppose that it satisfies the relations (3.2) and (3.7). If

$$q(z)+\frac{zq'(z)}{q(z)}\prec\Psi_p(m,\mu,\lambda,\alpha,\beta;z),z\in U,\tag{3.12}$$

then

$$q(z)\prec\left(\frac{RI_{p,\alpha,\beta}^{m+1,\lambda}f(z)}{z^p}\right)\left(\frac{z^p}{RI_{p,\alpha,\beta}^{m,\lambda}f(z)}\right)^{\mu},z\in U,$$

and q is the best subordinant.

Proof. Theorem 3.5 follows by using the same technique of proof of Theorem 3.4 and by an application of Lemma 2.3.

Combining the results of Theorem 3.4 and Theorem 3.5, we get the following sandwich theorem.

Theorem 3.6. Let $f \in A_p, m \in N_0 = N \cup \{0\}, \mu > 0, \lambda \geq 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0,$ and $\left(\frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{z^p}\right)\left(\frac{z^p}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}\right)^{\mu} \in \mathrm{H}[q(0),1] \cap$ Q. Let $\Psi_p(m,\mu,\lambda,\alpha,\beta;z)$, given by (3.9), be univalent in U. Let q_1 be convex in U and q_2 be univalent in U . Suppose the function q_1 satisfies relations (3.2) and (3.7) and the function q_2 satisfies relations (3.1) and (3.2). If

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \Psi_p(m,\mu,\lambda,\alpha,\beta;z) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)}, z \in U,$$

then

$$q_1(z) \prec \left(\frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{z^p}\right)\left(\frac{z^p}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}\right)^{\mu} \prec q_2(z),$$

where q_1 and q_2 are respectively the best subordinant and best dominant.

$\lambda = 1$ in Theorem 3.6 yields

Corollary 3.7. Let $f \in A_p, m \in N_0 = N \cup \{0\}, \mu > 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0,$ and $\left(\frac{I_{p,\alpha,\beta}^{m+1} f(z)}{z^p}\right)\left(\frac{z^p}{I_{p,\alpha,\beta}^{m} f(z)}\right)^{\mu} \in \mathrm{H}[q(0),1] \cap$ Q. Let

$$\tau_p(m,\mu,\lambda,\alpha,\beta;z) = \left(\frac{I_{p,\alpha,\beta}^{m+1} f(z)}{z^p}\right)\left(\frac{z^p}{I_{p,\alpha,\beta}^{m} f(z)}\right)^{\mu} + \left(\frac{\alpha + p\beta}{\beta}\right)\left[\left(\frac{I_{p,\alpha,\beta}^{m+2}}{I_{p,\alpha,\beta}^{m+1}} - 1\right) + \mu\left(1 - \frac{I_{p,\alpha,\beta}^{m+1}}{I_{p,\alpha,\beta}^{m}}\right)\right]$$

be univalent in U. Let q_1 be convex in U and q_2 be univalent in U . Suppose the function q_1 satisfies relations (3.2) and (3.7) and the function q_2 satisfies relations (3.1) and (3.2). If

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \tau_p(m,\mu,\alpha,\beta;z) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)}, z \in U,$$

then

$$q_1(z) \prec \left(\frac{I_{p,\alpha,\beta}^{m+1} f(z)}{z^p}\right)\left(\frac{z^p}{I_{p,\alpha,\beta}^{m} f(z)}\right)^{\mu} \prec q_2(z),$$

where q_1 and q_2 are respectively the best subordinant and best dominant.

Theorem 3.8. Let $f \in A_p, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0.$ Let the function q be univalent in U and suppose that it satisfies the conditions (3.1) and (3.2). If

(3.13) $$\Theta_p(m,\lambda,\alpha,\beta;z) = (m+p+1)\frac{RI_{p,\alpha,\beta}^{m+2,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)} - (m+p-1)\frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)} - 1$$

$$+\lambda\left(\frac{\alpha+p\beta}{\beta} - (m+p+1)\right)\frac{I_{p,\alpha,\beta}^{m+2} f(z)}{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)} - \lambda\left(\frac{\alpha+p\beta}{\beta} - (m+p)\right)\frac{I_{p,\alpha,\beta}^{m+1} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}$$

$$-\lambda\left(\frac{\alpha}{\beta} - (m+1)\right)\frac{I_{p,\alpha,\beta}^{m+1} f(z)}{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)} + \lambda\left(\frac{\alpha}{\beta} - m\right)\frac{I_{p,\alpha,\beta}^{m} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}.$$

If

(3.14) $$\Theta_p(m,\lambda,\alpha,\beta;z) \prec q(z) + \frac{zq'(z)}{q(z)}, z \in U,$$

then $\frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)} \prec q(z), z \in U,$ and q is the best dominant.

Proof. Define the function $p(z)$ by $p(z) = \frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}$.Then

$p(z) + \frac{zp'(z)}{p(z)} = \Theta_p(m,\lambda,\alpha,\beta;z)$ which, in light of hypothesis (3.14) of Theorem 3.8, yields the following subordination

$$p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)}, z \in U.$$

The assertion of Theorem 3.8 now follows by an application of Lemma 2.3 with $\theta(w) = w$ and $\phi(w) = 1/w$.

Theorem 3.9. Let $f \in A_p, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0$, and $\left(\frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}\right) \in \mathrm{H}[q(0),1] \cap \mathrm{Q}$. Let $\Theta_p(m,\lambda,\alpha,\beta;z)$, as given in (3.13), be univalent in U. Let the function q be convex in U and suppose that it satisfies the relations (3.2) and (3.7). If

$$q(z) + \frac{zq'(z)}{q(z)} \prec \Theta_p(m,\lambda,\alpha,\beta;z)$$

then $q(z) \prec \frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}, z \in U$, and q is the best subordinant.

Proof. Theorem 3.9 follows by using the same technique of proof of Theorem 3.8 and by an application of Lemma 2.3.

Combining the results of Theorem 3.8 and Theorem 3.9, we have the following sandwich result.

Theorem 3.10. Let $f \in A_p, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \beta > 0, \alpha$ a real number such that $\alpha + p\beta > 0$, and $\left(\frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)}\right) \in \mathrm{H}[q(0),1] \cap \mathrm{Q}$. Let $\Theta_p(m,\lambda,\alpha,\beta;z)$, as defined in (3.13), be univalent in U. Let q_1 be convex in U and q_2 be univalent in U. Suppose the function q_1 satisfies relations (3.2) and (3.7) and the function q_2 satisfies relations (3.1) and (3.2). If

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \Theta_p(m,\lambda,\alpha,\beta;z) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)}, z \in U,$$

then $q_1(z) \prec \frac{RI_{p,\alpha,\beta}^{m+1,\lambda} f(z)}{RI_{p,\alpha,\beta}^{m,\lambda} f(z)} \prec q_2(z), z \in U$, q_1 and q_2 are respectively the best subordinant and best dominant.

Corollary 3.11. Let $f \in A_p, m \in N_0 = N \cup \{0\}$ and $\left(\frac{R^{m+1}_{p,\alpha,\beta} f(z)}{R^{m}_{p,\alpha,\beta} f(z)}\right) \in \mathrm{H}[q(0),1] \cap \mathrm{Q}$. Let

$\kappa_p(m;z) = (m+p+1)\frac{R_p^{m+2} f(z)}{R_p^{m+1} f(z)} - (m+p-1)\frac{R_p^{m+1} f)z)}{R_p^{m} f)z)} - 1$, be univalent in U. Let q_1 be convex in U and q_2 be univalent in U. Suppose the function q_1 satisfies relations (3.2) and (3.7) and the function q_2 satisfies relations (3.1) and (3.2). If

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \kappa_p(m;z) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)}, z \in U,$$

then $q_1(z) \prec \frac{R_p^{m+1} f(z)}{R_p^{m} f(z)} \prec q_2(z), z \in U$, q_1 and q_2 are respectively the best subordinant and best dominant.

Remark 3.12. For $p = 1$ in Theorem 3.1 to Theorem 3.10, we obtain Theorem 3.1 to Theorem 3.9 of the author [30], respectively. For $\lambda = 0$ in Theorem 3.1 to Theorem 3.6, we obtain results of Macovei [15] (Corrected versions). For $\lambda = 1$ and $\beta = 1$ in Theorem 3.1 to Theorem 3.3 and also in Theorem 3.8 to Theorem 3.10, we get corresponding results proved by Macovei in [16], for the operator $I^m_{p,\alpha,1} f(z) = I^m_{p,\alpha} f(z)$, considered for $\alpha \geq 0$. But our results hold true for $\alpha > -p$.

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