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f_q-DERIVATIONS OF B-ALGEBRAS

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Abstract. Let $X = (X, *, 0)$ be a B-algebra and f a self-map on X. We study some properties of X for the self-map d_q^f is an outside and inside f_q-derivation of X, respectively, as follows:

$$(\forall x, y \in X)(d_q^f(x * y) = f(x) * d_q^f(y)),$$
$$(\forall x, y \in X)(d_q^f(x * y) = d_q^f(x) * f(y)).$$

In addition, we define and study some properties of (right-left) and (left-right) f_q-derivation of X, respectively, as follows:

$$(\forall x, y \in X)(d_q^f(x * y) = (f(x) * d_q^f(y)) \wedge (d_q^f(x) * f(y))),$$
$$(\forall x, y \in X)(d_q^f(x * y) = (d_q^f(x) * f(y)) \wedge (f(x) * d_q^f(y))).$$

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1. Introduction and Preliminaries

In 1966, Iséki [8] introduced the class of BCI-algebras as follows:

Definition 1.1. Let X be a non-empty set with a binary operation $*$ and a constant 0 in X. An algebra $X = (X, *, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(BCI1) $(\forall x \in X)(x * x = 0)$,

(BCI2) $(\forall x, y, z \in X)(((x * y) * (x * z)) * (z * y) = 0)$,

(BCI3) $(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y)$,

(BCI4) $(\forall x, y \in X)((x * (x * y)) * y = 0)$.

In any BCI-algebra X, the following property holds:

(BCI5) $(\forall x \in X)(x * 0 = x)$.

In 1983, Hu and Li [6] introduced a new class of algebras so-called a BCH-algebra. They proved that the class of BCI-algebras is a proper subclass of BCH-algebras and studied some properties of this algebra.

Definition 1.2. A *BCH-algebra* is an algebra $X = (X, *, 0)$ satisfying the following axioms:

(BCH1) $(\forall x \in X)(x * x = 0)$,

(BCH2) $(\forall x \in X)(x * 0 = x)$,

(BCH3) $(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y)$.

Next, Bandru and Rafi [5] introduced a new notion, called G-algebra. This notion played an important role in algebra and many applications as follows:

Definition 1.3. A *G-algebra* is an algebra $X = (X, *, 0)$ satisfying the following axioms:

(G1) $(\forall x \in X)(x * x = 0)$,

(G2) $(\forall x, y \in X)(x * (x * y) = y)$.

In 2002, Neggers and Kim [11] introduced a new algebraic structure, they took some properties from BCI and BCH-algebras, called B-algebra.

Definition 1.4. A *B-algebra* is an algebra $X = (X, *, 0)$ satisfying the following axioms:

(B1) $(\forall x \in X)(x * x = 0)$,

(B2) $(\forall x \in X)(x * 0 = x)$,

(B3) $(\forall x, y, z \in X)((x * y) * z = x * (z * (0 * y)))$.

Example 1.5. Let $X = \{0, 1, 2, 3\}$ with the Cayley table (Table 1) as follows:

*	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Table 1

Then $X = (X, *, 0)$ is a B-algebra.

Theorem 1.6. [11] *If* $X = (X, *, 0)$ *is a B-algebra, then:*

(B4) $(\forall x, y \in X)((x * y) * (0 * y) = x)$,

(B5) $(\forall x, y, z \in X)(x * (y * z) = (x * (0 * z)) * y)$,

(B6) $(\forall x, y \in X)(x * y = 0 \Rightarrow x = y)$,

(B7) $(\forall x \in X)(0 * (0 * x) = x)$,

(B8) $(\forall x, y, z \in X)(x * z = y * z \Rightarrow x = y)$ *(right cancelation law),*

(B9) $(\forall x, y, z \in X)(z * x = z * y \Rightarrow x = y)$ *(left cancelation law).*

Theorem 1.7. [11] *An algebra* $X = (X, *, 0)$ *is a B-algebra if and only if it satisfies the following axioms:*

(B10) $(\forall x \in X)(x * x = 0)$,

(B11) $(\forall x \in X)(0 * (0 * x) = x)$,

(B12) $(\forall x, y, z \in X)((x * z) * (y * z) = x * y)$,

(B13) $(\forall x, y \in X)(0 * (x * y) = y * x)$.

Definition 1.8. [10] A B-algebra $X = (X, *, 0)$ is said to be 0-*commutative* if it satisfies the following axioms:

$$(\forall x, y \in X)(x * (0 * y) = y * (0 * x)).$$

Example 1.9. In Example 1.5, we have $X = (X, *, 0)$ is a 0-commutative B-algebra.

Theorem 1.10. [10] *If $X = (X, *, 0)$ is a 0-commutative B-algebra, then:*

(B14) $(\forall x, y \in X)((0 * x) * (0 * y) = y * x)$,

(B15) $(\forall x, y, z \in X)((z * y) * (z * x) = x * y)$,

(B16) $(\forall x, y, z \in X)((x * y) * z = (x * z) * y)$,

(B17) $(\forall x, y \in X)((x * (x * y)) * y = 0)$,

(B18) $(\forall x, y, z, t \in X)((x * z) * (y * t) = (t * z) * (y * x))$,

(B19) $(\forall x, y, z \in X)((x * y) * z = x * (y * z))$,

(B20) $(\forall x, y \in X)(x * (x * y) = y)$.

For a B-algebra $X = (X, *, 0)$, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition 1.11. [2, 9] A self-map d on a B-algebra $X = (X, *, 0)$ is called

(1) a *(left-right)-derivation* ((l, r)-derivation, in short) of X if

$$(\forall x, y \in X)(d(x * y) = (d(x) * y) \wedge (x * d(y))),$$

(2) a *(right-left)-derivation* ((r, l)-derivation, in short) of X if

$$(\forall x, y \in X)(d(x * y) = (x * d(y)) \wedge (d(x) * y)),$$

(3) a *derivation* of X if it is both an (l, r) and an (r, l)-derivation of X.

Definition 1.12. [2, 7, 9, 13] A self-map d on a B-algebra $X = (X, *, 0)$ is said to be *regular* if $d(0) = 0$; otherwise, d is said to be *irregular*.

Example 1.13. [2, 9] In Example 1.5, we define a self-map d on X by:

$$d(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2 & \text{otherwise.} \end{cases}$$

Then d is regular.

Example 1.14. [2, 9] In Example 1.5, we define a self-map d on X by:

$$d(x) = \begin{cases} 3 & \text{if } x = 0 \\ 2 & \text{if } x = 1 \\ 1 & \text{if } x = 2 \\ 0 & \text{if } x = 3. \end{cases}$$

Then d is a derivation of X, and we see that d is irregular.

Definition 1.15. A self-map f on a B-algebra $X = (X, *, 0)$ is called an *endomorphism* if

$$(\forall x, y \in X)(f(x * y) = f(x) * f(y)).$$

Definition 1.16. [3] Let f be an endomorphism of a B-algebra $X = (X, *, 0)$. A self-map d on X is called

(1) a *(left-right)-f-derivation* ((l,r)-f-derivation, in short) of X if

$$(\forall x, y \in X)(d(x * y) = (d(x) * f(y)) \wedge (f(x) * d(y))),$$

(2) a *(right-left)-f-derivation* ((r,l)-f-derivation, in short) of X if

$$(\forall x, y \in X)(d(x * y) = (f(x) * d(y)) \wedge (d(x) * f(y))),$$

(3) an *f-derivation* of X if it is both an (l,r) and an (r,l)-f-derivation of X.

Note that if f is the identity map on a B-algebra $X = (X, *, 0)$, then every f-derivation of X is a derivation.

Let f be an endomorphism of a B-algebra $X = (X, *, 0)$ and $q \in X$. The self-map d_q^f on X is defined by

$$(\forall x \in X)(d_q^f(x) = f(x) * q).$$

We note that $d_0^f = f$; indeed, $d_0^f(x) = f(x) * 0 = f(x)$ for all $x \in X$.

Definition 1.17. [1] Let f be an endomorphism of a B-algebra $X = (X, *, 0)$. A self-map d_q^f on X is called

(1) an *outside f_q-derivation* of X if

$$(\forall x, y \in X)(d_q^f(x * y) = f(x) * d_q^f(y)),$$

(2) an *inside f_q-derivation* of X if

$$(\forall x,y\in X)(d_q^f(x*y)=d_q^f(x)*f(y)),$$

(3) an *f_q-derivation* of X if it is both an outside and inside f_q-derivation of X.

Next, we introduce a new concept of a (left-right) and a (right-left) f_q-derivation by the concept of [1, 4] as follows:

Definition 1.18. Let f be an endomorphism of a B-algebra $X=(X,*,0)$. A self-map d_q^f on X is called

(1) an *(left-right) f_q-derivation* of X if

$$(\forall x,y\in X)(d_q^f(x*y)=(d_q^f(x)*f(y))\wedge(f(x)*d_q^f(y))),$$

(2) an *(right-left) f_q-derivation* of X if

$$(\forall x,y\in X)(d_q^f(x*y)=(f(x)*d_q^f(y))\wedge(d_q^f(x)*f(y))).$$

Moreover, we present some examples to illustrate and support our results.

Example 1.19. Let $X=\{0,1,2\}$ with the Cayley table (Table 2) as follows:

*	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Table 2

Then $X=(X,*,0)$ is a B-algebra. Define an endomorphism $f:X\to X$ by

$$x\mapsto\begin{cases}0 & \text{if } x=0,\\ 2 & \text{if } x=1,\\ 1 & \text{if } x=2.\end{cases}$$

Then d_0^f is a (left-right) and a (right-left) f_q-derivation of X but d_2^f is not a (left-right) f_q-derivation or a (right-left) f_q-derivation of X. Indeed, $d_2^f(1*2)=2\neq 0=(d_2^f(1)*f(2))\wedge(f(1)*d_2^f(2))$ and $d_2^f(1*2)=2\neq 1=(f(1)*d_2^f(2))\wedge(d_2^f(1)*f(2))=1$.

2. MAIN RESULTS

In this section, our main results are divided into two parts as follows: 1. Outside and inside f_q-derivations, and 2. (Left-right) and (right-left) f_q-derivations.

From now on, we shall let X be a B-algebra $X = (X, *, 0)$.

2.1. Outside and inside f_q-derivations.

Theorem 2.1. *d_0^f is an f_q-derivation of X.*

Proof. Let $x, y \in X$. Then

$$d_0^f(x*y) = (f(x*y))*0 = (f(x)*f(y))*0 = f(x)*f(y). \tag{2.1}$$

$$f(x)*d_0^f(y) = f(x)*(f(y)*(0)) = f(x)*f(y). \tag{2.2}$$

$$d_0^f(x)*f(y) = (f(x)*0)*f(y) = f(x)*f(y). \tag{2.3}$$

By (2.1), (2.2) and (2.3), we get $f(x)*d_0^f(y) = d_0^f(x*y) = d_0^f(x)*f(y)$. Hence, d_0^f is an f_q-derivation of X. □

Theorem 2.2. *If X is an associative B-algebra, then d_q^f is an outside f_q-derivation of X for all $q \in X$.*

Proof. Let $q, x, y \in X$. Then

$$\begin{aligned} d_q^f(x*y) &= f(x*y)*q \\ &= (f(x)*f(y))*q \\ &= f(x)*(f(y)*q) \quad \text{(associative law)} \\ &= f(x)*d_q^f(y). \end{aligned}$$

Hence, d_q^f is an outside f_q-derivation of X. □

Proposition 2.3. *If X is a medial B-algebra, then d_q^f is an inside f_q-derivation of X for all $q \in X$.*

Proof. Let $q,x,y \in X$. Then

$$\begin{aligned} d_q^f(x*y) &= f(x*y)*q \\ &= (f(x)*f(y))*q \\ &= (f(x)*q)*f(y) && \text{(medial law)} \\ &= d_q^f(x)*f(y). \end{aligned}$$

Hence, d_q^f is an inside f_q-derivation of X. □

Corollary 2.4. *If X is an associative medial B-algebra, then d_q^f is an f_q-derivation of X for all $q \in X$.*

Proof. It is straightforward by Propositions 2.2 and 2.3. □

Theorem 2.5. *If d_q^f is an outside (resp., inside) f_q-derivation of X, then $d_q^f(0) = f(x)*d_q^f(x)$ (resp., $d_q^f(0) = d_q^f(x)*f(x)$) for all $x \in X$.*

Proof. We obtain the results from (B1). □

Theorem 2.6. *Let X be a medial B-algebra. If d_q^f is an outside f_q-derivation of X, then d_q^f is an f_q-derivation of X.*

Proof. It is straightforward by Proposition 2.3. □

Theorem 2.7. *Let X be an associative B-algebra. If d_q^f is an inside f_q-derivation of X, then d_q^f is an f_q-derivation of X.*

Proof. It is straightforward by Proposition 2.4. □

Theorem 2.8. *If d_q^f is a regular inside (outside) f_q-derivation of X, then $d_q^f = f$.*

Proof. Let $x \in X$. By (B1), we have $0 = d_q^f(0) = d_q^f(x*x) = d_q^f(x)*f(x)$. By (B6), we have $d_q^f(x) = f(x)$ for all $x \in X$, that is, $d_q^f = f$. □

2.2. (Left-right) and (right-left) f_q-derivations.

Theorem 2.9. If d_q^f is an (l,r)-f_q-derivation of X, then

$$(\forall x \in X)(d_q^f(0) = d_q^f(x) * f(x)).$$

Moreover, if X is 0-commutative, then

$$(\forall x \in X)(d_q^f(0) = d_q^f(x) * f(x) = 0 * q).$$

Proof. Let $x \in X$. Then

$$
\begin{aligned}
&((\mathrm{B1})) && d_q^f(0) = d_q^f(x * x)\\
& && = (d_q^f(x) * f(x)) \wedge (f(x) * d_q^f(x))\\
& && = (f(x) * d_q^f(x)) * ((f(x) * d_q^f(x)) * (d_q^f(x) * f(x)))\\
&((\mathrm{B5})) && = ((f(x) * d_q^f(x)) * (0 * (d_q^f(x) * f(x)))) * (f(x) * d_q^f(x))\\
&((\mathrm{B13})) && = ((f(x) * d_q^f(x)) * (f(x) * d_q^f(x))) * (f(x) * d_q^f(x))\\
&((\mathrm{B1})) && = 0 * (f(x) * d_q^f(x))\\
&((\mathrm{B13})) && = d_q^f(x) * f(x)\\
& && = (f(x) * q) * f(x)\\
&((\mathrm{B16})) && = (f(x) * f(x)) * q\\
&((\mathrm{B1})) && = 0 * q.
\end{aligned}
$$

Hence, $d_q^f(0) = d_q^f(x) * f(x) = 0 * q$ for all $x \in X$. □

Theorem 2.10. *If d_q^f is an (r,l)-f_q-derivation of X, then*

$$(\forall x \in X)(d_q^f(0) = f(x) * d_q^f(x)).$$

Moreover, if X is 0-commutative, then

$$(\forall x \in X)(d_q^f(0) = f(x) * d_q^f(x) = q).$$

Proof. Let $x \in X$. Then

$$
\begin{aligned}
d_q^f(0) &= d_q^f(x*x) && \text{((B1))}\\
&= (f(x)*d_q^f(x)) \wedge (d_q^f(x)*f(x))\\
&= (d_q^f(x)*f(x))*((d_q^f(x)*f(x))*(f(x)*d_q^f(x)))\\
&= ((d_q^f(x)*f(x))*(0*(f(x)*d_q^f(x))))*(d_q^f(x)*f(x)) && \text{((B5))}\\
&= ((d_q^f(x)*f(x))*(d_q^f(x)*f(x)))*(d_q^f(x)*f(x)) && \text{((B13))}\\
&= 0*(d_q^f(x)*f(x)) && \text{((B1))}\\
&= f(x)*d_q^f(x) && \text{((B13))}\\
&= f(x)*(f(x)*q)\\
&= q. && \text{((B20))}
\end{aligned}
$$

Hence, $d_q^f(0) = f(x)*d_q^f(x) = q$ for all $x \in X$. □

Theorem 2.11. *If d_q^f is an (l,r)-f_q-derivation ((r,l)-f_q-derivation) of X, then:*

(1) d_q^f is injective if and only if f is injective,

(2) if d_q^f is regular, then $d_q^f = f$,

(3) if there is an element $x_0 \in X$ such that $d_q^f(x_0) = f(x_0)$, then $d_q^f = f$.

Proof. (1) Suppose that d_q^f is injective and let $x,y \in X$ be such that $f(x) = f(y)$. By Theorem 2.9, we have $d_q^f(y)*f(y) = d_q^f(0) = d_q^f(x)*f(x) = d_q^f(x)*f(y)$. By (B8), we have $d_q^f(x) = d_q^f(y)$. Hence, $x = y$ because d_q^f is injective, so f is injective.

Conversely, suppose that f is injective and let $x,y \in X$ be such that $d_q^f(x) = d_q^f(y)$. By Theorem 2.9, we have $d_q^f(y)*f(y) = d_q^f(0) = d_q^f(x)*f(x) = d_q^f(y)*f(x)$. By (B9), we have $f(x) = f(y)$. Hence, $x = y$ because f is injective, so d_q^f is injective.

(2) Suppose that d_q^f is regular and let $x \in X$. By Theorem 2.9, we have $0 = d_q^f(0) = d_q^f(x)*f(x)$. By (B6), we have $d_q^f(x) = f(x)$ for all $x \in X$, that is, $d_q^f = f$.

(3) Suppose that there is an element $x_0 \in X$ such that $d_q^f(x_0) = f(x_0)$. By (B1) and Theorem 2.9, we have $d_q^f(0) = d_q^f(x_0)*f(x_0) = 0$. Thus d_q^f is regular. It follows from (2) that $d_q^f = f$.

Similarly, if d_q^f is an (r,l)-f_q-derivation of X, the proof follows by Theorem 2.10. □

3. Conclusion and Discussion

In this paper, we have introduced the concept of a (left-right) and a (right-left) f_q-derivation of B-algebras and some properties are provided. Moreover, we also get the results related to 0-commutative B-algebras and regular (l,r)-f_q-derivation ((r,l)-f_q-derivation) of B-algebras.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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