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GENERALIZED PARIKH MATRICES OF PICTURE ARRAY

K. JANAKI[1], R. ARULPRAKASAM[1,*], V.R. DARE[2]

[1]Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur-603203, Tamilnadu, India

[2]Department of Mathematics, Madras Christian College, Chennai-600059, Tamilnadu, India

Abstract. The theory of Parikh matrices and subword occurrences has led to extensive research in combinatorics on words. The extension of Parikh matrix of a word into picture array is an another interesting problem. Since the Parikh matrix does not say about the number of subword occurrences in the array. Therefore, in this paper we introduce the generalized Parikh matrix of a picture array which gives the number of subwords occurrences in the array. We also discuss some properties of generalized Parikh matrices.

Keywords: subword indicator; M-ambiguity; Parikh matrix; picture array.

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1. Introduction

In formal language theory [9], the ‘Parikh vector’ of a word over the alphabet **A** was introduced by [8] which tells the number of presence of the symbols in a word. Parikh vector is not ‘injective’ and so many words can have same Parikh vector. In [5] began as a natural extension of the Parikh vector to ‘Parikh matrix’, which provides several information about a word than a Parikh vector. Parikh vector tells the number of occurrence of the ‘symbols’ in

*Corresponding author

E-mail address: r.aruljeeva@gmail.com

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a word whereas Parikh matrix tells about the number of subwords in a word. The various aspects of Parikh matrix, was appeared in [1, 2, 3, 4, 6, 7, 10, 11, 12, 13, 15, 16] for studying properties related to subwords. Further Subramanian et. al extended the Parikh matrix concept of a 'word' to a 'picture array' [17]. However the Parikh matrix fails to tell few additional information about the number of subwords occurrences in an array. Motivated by the work of [14, 17], in this paper we introduce the notion of generalized Parikh matrix of an array to fulfill the number of subwords occurrences in an array and also results relating to subword indicators and M-ambiguity of picture arrays are derived. This paper comprises four sections. In section 2, to collect basic definitions of Parikh matrix of words which are used in subsequent section. In section 3, horizontal and vertical generalized Parikh matrix of picture array are introduced. Also discussed results relating to subword indicators and M-ambiguity of picture arrays are derived with conclusion in section 4.

2. Preliminaries

In this section we recollect certain notions. Consider an alphabet $\mathbf{A}$ and set of all words over $\mathbf{A}$ is $\mathbf{A}^*$. For any word $x \in \mathbf{A}^*$, the length of x is represented by $|x|$. A scattered subword is a subsequence of a given word x, which itself is a finite sequence of symbols over $\mathbf{A}$. For $1 \leq i \leq j \leq k$, subword '$a_i a_{i+1} \cdots a_j$' is represented by $a_{i,j}$.

Consider the ordered alphabet $\mathbf{A}_k = \{a_1 < a_2 < \cdots < a_k\}$. The Parikh matrix M, is the morphism on word over an ordered alphabet $\mathbf{A}_k$ is defined as $M_k : \mathbf{A}_k \to M_{k+1}$ such that $M_k = (m_{r,s})_{1 \leq r,s \leq k+1}$ where for every $1 \leq r \leq k+1$, $m_{r,r} = 1, m_{r,r+1} = 1$, and the remaining elements of the matrices are zero.

Consider $x = abbc$ over $\mathbf{A}_2$ then

$$\begin{aligned} M_3(abbc) &= M_3(a)M_3(b)M_3(b)M_3(c) \\ &= \begin{bmatrix} 1\,1\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,1\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,1\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,1 \\ 0\,0\,0\,1 \end{bmatrix} = \begin{bmatrix} 1\,1\,2\,1 \\ 0\,1\,2\,2 \\ 0\,0\,1\,1 \\ 0\,0\,0\,1 \end{bmatrix}. \end{aligned}$$

The Parikh matrix is not "injective". If the words x and y over $\mathbf{A}_k$ are said to be 'M-equivalent' then $M_k(x) = M_k(y)$. The triangle matrix $Z = X \oplus Y$ formed by the sum of all elements of two triangle matrices X and Y such that the elements in the leading diagonals of X and Y are excluded while adding.

Consider a word say 'subword indicator' $u = c_1c_2...c_t$ of length t over $\mathbf{A}$ in positions $q_1, q_2, \dots q_p$. The generalized Parikh matrix mapping M_u, is the morphism on word over an ordered alphabet $\mathbf{A}_k$ is defined as $M_u : \mathbf{A}_k \to M_{t+1}$ such that $M_u = (m_{r,s})_{1 \le r \le s \le t+1}$ where for every $1 \le r \le s \le t+1$, $m_{r,r} = 1$ and $m_{r,r+1} = 1$ if r is one of the numbers $q_1, q_2, \dots q_p$ and remaining elements of the matrix are zero.

Consider $x = abbac$ over $\mathbf{A}_2$ and the subword indicator $u = aabc$ over $\mathbf{A}$ then

$$
\begin{aligned}
M_{aabc}(abbc) &= M_u(a)M_u(b)M_u(b)M_u(a)M_u(c) \\
&= \begin{bmatrix} 1&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} 1&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{bmatrix} \\
&= \begin{bmatrix} 1&2&1&0&0 \\ 0&1&2&2&2 \\ 0&0&1&2&2 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{bmatrix}.
\end{aligned}
$$

An arrangment in 'm' rows and 'n' columns of symbols over an alphabet called 'picture array' and it is represented by '$\wp$'.

For example, $X \in \wp$ over $\mathbf{A}_2 = \{a < b\}$ is $X = \begin{matrix} a\,a\,b\,a\,a \\ b\,a\,a\,a\,a \\ a\,b\,a\,a\,b \\ b\,b\,a\,a\,a \end{matrix}$.

Consider an array $X \in \wp$ over $\mathbf{A}_k$ and a_i, b_j are the words of X in 'm' rows and 'n' columns respectively, where $i \in [1,m]$, $j \in [1,n]$ for all m, $n \ge 1$. The 'horizontal Parikh matrix' of X is

$M^r(X)$ is defined as $M^r(X) = M(a_1) \oplus M(a_2) \oplus \ldots \oplus M(a_m)$ and the 'vertical Parikh matrix' of X is $M^c(X)$ is defined as $M^c(X) = M\left(b_1^T\right) \oplus M\left(b_2^T\right) \oplus \ldots \oplus M\left(b_n^T\right)$. The row and column Parikh matrix gives the number of presence of horizontal subwords ab and vertical subwords $\begin{matrix} a \\ b \end{matrix}$ respectively and both the matrices gives the number of presence of a and the number of presence of b in X. The column concatenation of two arrays say X and Y is represented by '$X \circ Y$' where X and Y having same number of rows. Similarly, the row concatenation is represented by '$X \diamond Y$' where X and Y having same number of columns.

3. Generalized Parikh Matrices of Picture Array

In this section we introduce horizontal and vertical generalized Parikh matrix for a picture array and prove some of its properties. Further we define horizontal Parikh-friendly permutation and vertical Parikh-friendly permutation based on subword indicator. Finally deduce that every circular permutation is Parikh-friendly.

Definition 3.1. *Consider an array $X \in \wp$ over $\mathbf{A}_k$ and a_i, b_j are the words of X in 'm' rows and 'n' columns respectively, where $i \in [1,m]$, $j \in [1,n]$ for all m, $n \geq 1$. Consider a subword indicator $u = c_1c_2...c_t$ of length t over $\mathbf{A}$. The 'horizontal generalized Parikh matrix' of X is defined as $M_u^r(X) = M_u(a_1) \oplus M_u(a_2) \oplus \ldots \oplus M_u(a_m)$ and the 'vertical generalized Parikh matrix' of X is defined as $M_u^c(X) = M_u\left(b_1^T\right) \oplus M_u\left(b_2^T\right) \oplus \ldots \oplus M_u\left(b_n^T\right)$.*

Example 1. Let $u = abab$ over $\mathbf{A} = \{a,b\}$ and $X = \begin{matrix} a\,b\,b\,a\,b \\ b\,a\,a\,b\,a \\ a\,b\,a\,a\,b \\ a\,b\,a\,b\,b \\ b\,b\,b\,a\,a \end{matrix}$ in $\wp$ over $\mathbf{A}_2 = \{a < b\}$.

Then the generalized Parikh matrices $M_u(a_i)$, $1 \leq i \leq 5$ of the horizontal words $a_1 = abbab$, $a_2 = baaba$, $a_3 = abaab$, $a_4 = ababb$ and $a_5 = bbbaa$ are

$$M_u(a_1) = \begin{bmatrix} 1\,2\,4\,2\,2 \\ 0\,1\,3\,2\,2 \\ 0\,0\,1\,2\,4 \\ 0\,0\,0\,1\,3 \\ 0\,0\,0\,0\,1 \end{bmatrix}$$

$$M_u(a_2) = \begin{bmatrix} 1\,3\,2\,2\,0 \\ 0\,1\,2\,4\,2 \\ 0\,0\,1\,3\,2 \\ 0\,0\,0\,1\,2 \\ 0\,0\,0\,0\,1 \end{bmatrix}$$

$$M_u(a_3) = \begin{bmatrix} 1\,3\,4\,2\,2 \\ 0\,1\,2\,2\,2 \\ 0\,0\,1\,3\,4 \\ 0\,0\,0\,1\,2 \\ 0\,0\,0\,0\,1 \end{bmatrix}$$

$$M_u(a_4) = \begin{bmatrix} 1\,2\,5\,1\,2 \\ 0\,1\,3\,1\,2 \\ 0\,0\,1\,2\,5 \\ 0\,0\,0\,1\,3 \\ 0\,0\,0\,0\,1 \end{bmatrix}$$

$$M_u(a_5) = \begin{bmatrix} 1\,2\,0\,0\,0 \\ 0\,1\,3\,6\,0 \\ 0\,0\,1\,2\,0 \\ 0\,0\,0\,1\,3 \\ 0\,0\,0\,0\,1 \end{bmatrix}.$$

The horizontal generalized Parikh matrix $M_u^r(X)$ of X is

$$M_u^r(X) = \begin{bmatrix} 1 & 12 & 15 & 7 & 6 \\ 0 & 1 & 13 & 15 & 8 \\ 0 & 0 & 1 & 12 & 15 \\ 0 & 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly the vertical generalized Parikh matrix $M_u^c(X)$ of X is

$$M_u^c(X) = \begin{bmatrix} 1 & 12 & 15 & 8 & 3 \\ 0 & 1 & 13 & 13 & 11 \\ 0 & 0 & 1 & 12 & 15 \\ 0 & 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Theorem 1. *For a word u over* $\mathbf{A}$ *then for the generalized Parikh matrices of the column and row concatenation of* $X, Y \in \wp$ *over* $\mathbf{A}_k$ *are of the form* $M_u^r(X \circ Y)$ *and* $M_u^c(X \diamond Y)$ *such that*
(i) $M_u^r(X \circ Y) = M_u^r(X) + M_u^r(Y)$
(ii) $M_u^c(X \diamond Y) = M_u^c(X) + M_u^c(Y)$.
Proof *Let* a_i, b_i *are the words in the* i^{th} *rows of* X, Y *respectively. From def 3.1,*

$$M_u^r(X) = M_u(a_1) \oplus M_u(a_2) \oplus \ldots \oplus M_u(a_m)$$

$$M_u^r(Y) = M_u(b_1) \oplus M_u(b_2) \oplus \ldots \oplus M_u(b_m).$$

Then

$$\begin{aligned} M_u^r(X \circ Y) &= M_u(a_1) \oplus M_u(a_2) \oplus \ldots \oplus M_u(a_m) \oplus M_u(b_1) \oplus M_u(b_2) \oplus \ldots \oplus M_u(b_m) \\ &= M_u^r(X) + M_u^r(Y). \end{aligned}$$

The proof is similar for $M_u^c(X \diamond Y) = M_u^c(X) + M_u^c(Y)$.

Example 2. *Consider the word $u = aba$ over $\mathbf{A} = \{a < b\}$ and the arrays $X, Y \in \wp$ over $\mathbf{A}_2$ where* $X = \begin{matrix} a\ b\ a \\ a\ a\ b \\ b\ a\ a \end{matrix}$, $Y = \begin{matrix} b\ b\ a \\ a\ b\ b \\ a\ a\ b \end{matrix}$. *The column and row concatenation X and Y are* $X \circ Y = \begin{matrix} a\ b\ a \\ a\ a\ b \\ b\ a\ a \\ b\ b\ a \\ a\ b\ b \\ a\ a\ b \end{matrix}$, $X \diamond Y = \begin{matrix} a\ b\ a\ b\ b\ a \\ a\ a\ b\ a\ b\ a \\ b\ a\ a\ a\ a\ b \end{matrix}$ *respectively. Thus we get,*

$$M_u^r(X) = \begin{bmatrix} 1\,6\,3\,1 \\ 0\,1\,3\,3 \\ 0\,0\,1\,6 \\ 0\,0\,0\,1 \end{bmatrix}$$

$$M_u^r(Y) = \begin{bmatrix} 1\,4\,4\,0 \\ 0\,1\,5\,2 \\ 0\,0\,1\,4 \\ 0\,0\,0\,1 \end{bmatrix}$$

$$M_u^r(X \circ Y) = \begin{bmatrix} 1 & 10 & 7 & 1 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$M_u^r(X) + M_u^r(Y) = \begin{bmatrix} 1 & 10 & 7 & 1 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

From (1) *and* (2) *we have*

$$M_u^r(X \circ Y) = M_u^r(X) + M_u^r(Y).$$

Similarly,

(3)
$$M_u^c(X \diamond Y) = \begin{bmatrix} 1 & 10 & 5 & 1 \\ 0 & 1 & 8 & 7 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(4)
$$M_u^c(X) + M_u^c(Y) = \begin{bmatrix} 1 & 10 & 5 & 1 \\ 0 & 1 & 8 & 7 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $M_u^c(X) = \begin{bmatrix} 1 & 6 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *and* $M_u^c(Y) = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

From (3) *and (4) we have*

$$M_u^c(X \diamond Y) \quad = \quad M_u^c(X) + M_u^c(Y).$$

Remark 1. *The above theorem 1 does not hold for changing into horizontal generalized Parikh matrix of row concatenation instead of horizontal generalized Parikh matrix of column concatenation and vise-versa.*

Example 3. *Consider the word* $u = aba$ *over* $\mathbf{A} = \{a < b\}$ *and consider the arrays* $X = \begin{matrix} a & b & a \\ a & a & b \\ b & a & a \end{matrix}$,

$Y = \begin{matrix} b & b & a \\ a & b & b \\ a & a & b \end{matrix}$ *. where* $X, Y \in \wp$ *over* $\mathbf{A}_2$*. The column and row concatenation of* X, Y *are* $X \circ Y =$

$$\begin{matrix} a\ b\ a \\ a\ a\ b \\ b\ a\ a \\ b\ b\ a \\ a\ b\ b \\ a\ a\ b \end{matrix}, X \diamond Y = \begin{matrix} a\ b\ a\ b\ b\ a \\ a\ a\ b\ a\ b\ a \\ b\ a\ a\ a\ a\ b \end{matrix}$$ *respectively. Thus we get,*

$$M_u^r(X) = \begin{bmatrix} 1 & 6 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$M_u^r(Y) = \begin{bmatrix} 1 & 4 & 4 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$M_u^r(X \diamond Y) = \begin{bmatrix} 1 & 10 & 18 & 6 \\ 0 & 1 & 8 & 9 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$$M_u^r(X) + M_u^r(Y) = \begin{bmatrix} 1 & 10 & 7 & 1 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

From (5) *and* (6) *we have*

$$M_u^r(X \diamond Y) \neq M_u^r(X) + M_u^r(Y). \tag{9}$$

Similarly,

(10)
$$M_u^c(X \circ Y) = \begin{bmatrix} 1 & 10 & 15 & 14 \\ 0 & 1 & 8 & 11 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(11)
$$M_u^c(X) + M_u^c(Y) = \begin{bmatrix} 1 & 10 & 5 & 1 \\ 0 & 1 & 8 & 7 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From (7) *and* (8)

(12)
$$M_u^c(X \circ Y) \neq M_u^c(X) + M_u^c(Y).$$

Theorem 2. *Suppose a word u of length s over* $\mathbf{A}_s$ *then the horizontal and vertical generalized Parikh matrix of* $X \in \wp$ *are of the form* $M_u^r(X)$ *and* $M_u^c(X)$ *if there exist* $X' \in \wp$ *over* $\mathbf{A}_s$ *such that*

(i) $M_u^r(X) = M_s^r(X')$

(ii) $M_u^c(X) = M_s^c(X')$.

Proof *Consider the word u of length s over* $\mathbf{A}_s$. *The occurrences of the letters in u by* $1, \ldots, s$. *A morphism h of* $\mathbf{A}$ *into* $\mathbf{A}_s$ *is defined by* $h(a) = q_p q_{p-1} \ldots q_1$ *where* $a \in \mathbf{A}$ *appear in u in positions* $q_1, q_2, \ldots q_p$, $1 \leq p \leq s$, $q_i < q_{i+1}$, $1 \leq i \leq p-1$. *Choose* $X' = h(X)$, *where* $X \in \wp$ *over* $\mathbf{A}$. *By example 1 ,* $h(a) = 31$ *and* $h(b) = 42$.

For picture array

$$X = \begin{matrix} a & b & b & a & b \\ b & a & a & b & a \\ a & b & a & a & b \\ a & b & a & b & b \\ b & b & b & a & a \end{matrix},$$

we obtain

$$X^{'} = \begin{matrix} 31\,42\,42\,31\,42 \\ 42\,31\,31\,42\,31 \\ 31\,42\,31\,31\,42 \\ 31\,42\,31\,42\,42 \\ 42\,42\,42\,31\,31 \end{matrix}$$

For ordered alphabet $\mathbf{A}_4 = \{1,2,3,4\}$, *we have*

$$M_s^r\left(X'\right) = \begin{bmatrix} 1 & 12 & 15 & 7 & 6 \\ 0 & 1 & 13 & 15 & 8 \\ 0 & 0 & 1 & 12 & 15 \\ 0 & 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_s^c\left(X'\right) = \begin{bmatrix} 1 & 12 & 15 & 8 & 3 \\ 0 & 1 & 13 & 13 & 11 \\ 0 & 0 & 1 & 12 & 15 \\ 0 & 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We prove by induction on the length of each horizontal words and length of each vertical words respectively of X. *Now consider* h *be the morphism and* X' *be the picture array. Consider the theorem holds for each horizontal words and vertical words of* X *be length* n. *Consider* $h(w) \in \mathbf{A}_s^+$ *and* $w \in \mathbf{A}$. *Now we have* $M_u^r(X \circ w) = M_u^r(X) + M_u(w)$. *By induction hypothesis,* $M_u^r(X) = M_s^r(h(X))$. *Therefore,*

$$\begin{aligned} M_u^r(X \circ w) &= M_s^r(h(X)) + M_u(w) \\ &= M_s^r(h(X)) + M_s(h(w)) \\ &= M_s^r(h(X \circ w)). \end{aligned}$$

Similarly,

$$\begin{aligned} M_u^c(X \diamond w) &= M_u^c(X) + M_u(w) \\ &= M_s^c(h(X)) + M_s(h(w)) \\ &= M_s^c(h(X \diamond w)). \end{aligned}$$

Assume that we have $A = \begin{matrix} a\,b\,b\,a\,b \\ b\,a\,a\,b\,a \\ a\,b\,a\,a\,b \end{matrix}$ *and* $A \subseteq X$ *and* $w = ababb$. *We have*

$$M_u^r(A) = \begin{bmatrix} 1 & 8 & 10 & 6 & 4 \\ 0 & 1 & 7 & 8 & 6 \\ 0 & 0 & 1 & 8 & 10 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$M_u^r(w) = \begin{bmatrix} 1 & 2 & 5 & 1 & 2 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then

$$M_u^r(A \circ w) = \begin{bmatrix} 1 & 10 & 15 & 7 & 6 \\ 0 & 1 & 10 & 10 & 8 \\ 0 & 0 & 1 & 10 & 15 \\ 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, we have $M_u^r(A \circ w) = M_u^r(h(A \circ w))$.

Lemma 3. *The equation* $M_{ab}^r(X) = M_{ba}^r(X)$ *and* $M_{ab}^c(X) = M_{ba}^c(X)$, $X \in \wp$ *over* $\mathbf{A}_2 = \{a < b\}$ *holds for ambiguous arrays* X.

Proof *Consider the entries in the horizontal generalized Parikh matrix and vertical generalized Parikh matrix, we deduce that a necessary and sufficient condition for the equations* $M^r_{ab}(X) = M^r_{ba}(X)$ *and* $M^c_{ab}(X) = M^c_{ba}(X)$ *are that both of the equations,* $|a_i|_a = |a_i|_b$, $|a_i|_{ab} = |a_i|_{ba}$ *where* $i \in [1,m]$ *and* $|b^T_j|_a = |b^T_j|_b$ *and* $|b^T_j|_{ab} = |b^T_j|_{ba}$ *where* $j \in [1,n]$.

Lemma 4. *The equation* $M^r_{ab}(X) = M^r_{ba}(X)$ *and* $M^c_{ab}(X) = M^c_{ba}(X)$, $X \in \wp$ *over* $\mathbf{A}_2 = \{a < b\}$ *implies that* $|a_i|_a = |a_i|_b$, $i \in [1,m]$ *and* $|b^T_j|_a = |b^T_j|_b$, $j \in [1,n]$ *are even.*

Proof *By condradiction, if* $|a_i|_a = |a_i|_b$, $i \in [1,m]$ *and* $|b^T_j|_a = |b^T_j|_b$, $j \in [1,n]$ *are odd then the equation* $(|w|_a).(|w|_a) = (|w|_{ab}) + (|w|_{ba})$ *be false. Here w be each horizontal words and each vertical words.*

Definition 3.2. *A permutation* $\mathbf{A} \in \mathbf{A}_s$ *where* $\mathbf{A}_s = \{a_1 < a_2 < \cdots a_s\}$ *is said to be horizontal (vertical) Parikh-friendly with respect to* $\mathbf{A}_s$ $\exists$ *a picture array* $X \in \wp$ *such* $\ni$ $M^r_u(X) = M^r_{\mathbf{A}(u)}(X)$ $\left(M^c_u(X) = M^c_{\mathbf{A}(u)}(X)\right)$. *Then the corresponding X is called horizontal (vertical) Parikh-friendly witness for* $\mathbf{A}$.

Example 4. *Consider the circular permutation* $\mathbf{A} = (abc)$ *and the picture array*

$$X = \begin{matrix} a\ b\ c \\ b\ c\ a \\ c\ b\ a \\ a\ c\ b \\ b\ a\ c \\ c\ a\ b \end{matrix}$$

over $\mathbf{A}_3 = \{a < b < c\}$. *We have*

$$M^r_{abc}(X) = M^r_{\mathbf{A}(abc)}(X) = \begin{bmatrix} 1\ 6\ 3\ 1 \\ 0\ 1\ 6\ 3 \\ 0\ 0\ 1\ 6 \\ 0\ 0\ 0\ 1 \end{bmatrix}$$

and clearly the circular permutation $\mathbf{A}$ *is horizontal Parikh-friendly. In the similar manner we obtained the vertical Parikh-friendly where*

$$M^c_{abc}(X) = M^c_{\mathbf{A}(abc)}(X) = \begin{bmatrix} 1\ 6\ 7\ 7 \\ 0\ 1\ 6\ 7 \\ 0\ 0\ 1\ 7 \\ 0\ 0\ 0\ 1 \end{bmatrix}.$$

Theorem 5. *The circular permutation $\mathbf{A} = (a_1 a_2 \cdots a_s)$ is horizontal Parikh-friendly as well as vertical Parikh-friendly if for any $s \geq 2$.*

Proof Consider the following picture array

$$X = \begin{matrix} a_1 & a_2 & \cdots & \cdots & a_s & a_s & \cdots & \cdots & a_1 \\ \vdots & & \ddots & & & & & & \vdots \\ \vdots & & & \ddots & & & & & \vdots \\ a_s & & & & \ddots & & & & \vdots \\ \vdots & & & & & \ddots & & & \vdots \\ \vdots & & & & & & & \ddots & \vdots \\ a_1 & a_2 & \cdots & \cdots & a_s & a_s & \cdots & \cdots & a_1 \end{matrix}$$

This yields that $M^r_u(X) = M^r_{\mathbf{A}(u)}(X)$ and $M^c_u(X) = M^c_{\mathbf{A}(u)}(X)$.

For instance, we consider the circular permutation $\mathbf{A} = (abcd)$ and the picture array

$$X = \begin{matrix} a & b & c & d & d & c & b & a \\ b & c & d & a & a & d & c & b \\ c & d & a & b & b & a & d & c \\ d & a & b & c & c & b & a & d \\ d & a & b & c & c & b & a & d \\ c & d & a & b & b & a & d & c \\ b & c & d & a & a & d & c & b \\ a & b & c & d & d & c & b & a \end{matrix}$$

Thus we get $M^r_{abcd}(X) = M^r_{bcda}(X)$ and $M^c_{abcd}(X) = M^c_{bcda}(X)$.

Corollary 6. *For any ordered alphabet, every horizontal Parikh-friendly is equal to vertical Parikh-friendly iff $X = X^T$ where $X \in \wp$.*

4. Conclusion

In this work, deals horizontal and vertical generalized Parikh matrix for picture array with illustrations. Through the concepts regarding M-ambiguity and subword indicators of picture array we defined horizontal and vertical Parikh-friendly permutation. Finally deduce that the circular permutation is Parikh-friendly. Many problem remains open for arbitrary circular permutation which characterize horizontal and vertical Parikh-friendly.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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