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# STRONG DOUBLY EDGE GEODETIC PROBLEM IN GRAPHS

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**Abstract.** For a graph $G(V(G),E(G)$, the problem to find a set $S \subseteq V(G)$ where every edge in $E(G)$ is covered by least two fixed geodesics between the vertices in $S$ is called the strong doubly edge geodetic problem and the cardinality of the smallest such $S$ is the strong doubly edge geodetic number of $G$. In this paper the computational complexity for strong doubly edge geodetic problem is studied and also some bounds for general graphs are derived.

**Keywords:** strong edge geodetic number; strong geodetic set; strong doubly edge geodetic number; geodetic set.
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## 1. Introduction

Consider a graph $G(V(G),E(G))$, with order $|V(G)|$ and size $|E(G)|$. An $(x-y)$ geodesic is the length of the shortest path between the vertices $x$ and $y$. For a graph $G$, the length of the maximum geodesic is called the graph diameter, denoted as $diam(G)$. Harary et al introduced a graph theoretical parameter in [2] called the geodetic number of a graph and it was further studied in [3]. Let $I[u,v]$ be the set of all vertices lying on some $u-v$ geodesic of $G$, and for some non empty subset $S$ of $V(G)$, $I[S] = \cup_{u,v\in S} I[u,v]$. The set $S$ of vertices of $G$ is called

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a geodetic set of $G$, if $I[S] = V$. A geodetic set of minimum cardinality is called minimum geodetic set of $G$. The cardinality of the minimum geodetic set of $G$ is the geodetic number $g(G)$ of $G$. The geodetic set decision problem is NP-complete [12]. The set $S \subseteq V(G)$ is an edge geodetic cover of $G$ if every edge of $G$ is contained in the geodesic between some pair of vertices in $S$, and the cardinality of minimum edge geodetic cover is called the edge geodetic number of $G$ denoted as $g_1(G)$ [13].

Strong geodetic problem is a variation of geodetic problem and is defined in [10] as follows. For a graph $G(V(G),E(G))$, given a set $S \subseteq V(G)$, for each pair of vertices $(x,y) \subseteq S$, $x \neq y$, let $\tilde{g}(x,y)$ be a selected fixed shortest path between $x$ and $y$. Let $\tilde{I}(S) = \{\tilde{g}(x,y) : x,y \in S\}$ and $V(\tilde{I}(S)) = \cup_{\tilde{P} \in \tilde{I}(S)} V(\tilde{P})$. If $V(\tilde{I}(S)) = V$ for some $\tilde{I}(S)$, then $S$ is called a strong geodetic set . The cardinality of the minimum strong geodetic set is the strong geodetic number of $G$ and is denoted by $sg(G)$. The strong geodetic problem was later studied in [4][5][6][7][8][9][15].The edge version of the strong geodetic problem is defined in [11] i.e. a set $S \subseteq V(G)$ is called a strong edge geodetic set if for any pair $x,y \in S$ a shortest path $P_{xy}$ can be assigned such that $\cup_{\{x,y\} \in \binom{S}{2}} E(P_{xy}) = E(G)$. The cardinality of the smallest strong edge geodetic set of $G$ is called the strong edge geodetic number and is denoted as $sg_e(G)$.

In [16] another variant of geodetic problem named doubly geodetic problem is introduced and is defined as follows: For any graph $G(V,E)$ two geodesics $g_p(x,y)$ and $g_q(u,v)$ are distinct if $I[g_p(x,y)] \neq I[g_q(u,v)]$ where $\{u,v,x,y\} \in V(G)$. A set $S \subseteq V(G)$ is called the doubly geodetic set if each vertex in $V(G) \backslash S$ lies on at least two geodesics between the vertices in $S$ and the cardinality of the smallest such $S$ is the doubly geodetic number of $G$, denoted as $\ddot{d}g(G)$. Later in [17] the edge version of this problem is defined. A set $S \subseteq V(G)$is called a doubly edge geodetic set of if each edge $e \in E(G) \backslash E(S)$ lies on at least two distinct geodesics of vertices of $S$, where $E(S)$ is the edge set of the sub graph induced by the vertices of $S$. The doubly edge geodetic number $\ddot{d}g_e(G)$ is the minimum cardinality of a doubly edge geodetic set.

## 2. Motivation

Smarandache geometries and Smarandache multispaces are sigficant topics in Mathematics. In graph theory the study of Smarandache path $k-$ cover is initiated in [1]. Also, in [14] Smarandache edge geodetic set is defined. A set $T \subseteq V(G)$ is a Smarandache edge geodetic set if each edge in $E(G)$ lies on at least two geodesic between the vertices in $T$.

In [11] an urban road network problem is modelled into a graph where the vertices represent the bus stops or junctions and the edges represent the roads connecting them, subjected to the condition that a road is a geodesic and it is patrolled by a pair of road inspectors by placing one inspector at each end. Also, one pair of road inspectors is not assigned to more than one road segment. The strong edge geodetic problem is to identify the minimum number of road inspectors to patrol the urban road network i.e. each edge is patrolled by at least one pair of inspectors. Assume that the inspectors are working under the following condition. In order to guard and secure a road(geodesic), there should necessarily be a communication between the inspectors who are guarding that particular road(geodesic). Suppose there is a communication problem or other network related issues between a pair of inspectors then the edges in the fixed geodesic patrolled by that particular pair of inspectors are unsecured. To avoid this situation we can arrange at least two pair of inspectors for an edge. In this arrangement, even if one pair of inspector had lost the communication, then the other pair of inspectors can guard that particular edge. With this motivation the strong doubly edge geodetic set can be defined.

## 3. Strong Doubly Edge Geodetic Number

A set $S \subseteq V(G)$ is called a strong doubly edge geodetic set of if each edge $e \in E(G)$ lies on at least two distinct fixed geodesics between vertices of $S$. The strong doubly edge geodetic number $s\ddot{d}g_e(G)$ is the minimum cardinality of a strong doubly edge geodetic set.

For the graph given in Figure 1: the set $\{b, f, g\}$ forms a minimum strong edge geodetic set whereas the $\{a, b, c, f, g\}$ is a strong doubly edge geodetic set. Thus $sg_e(G) = 3$ and $s\ddot{d}g_e(G) = 5$.

A graph $G$ which has a strong doubly edge geodetic set is called a strong doubly edge geodetic graph. The complete graph $K_n$ and the graph $(K_n - e)$ are not strong doubly edge geodetic graphs.

A graph may have strong doubly geodetic set but not strong doubly edge geodetic set. For example, complete graph $K_n$ is a strong doubly geodetic graph but not a strong doubly edge geodetic graph.

Throughout this paper, we assume $G$ to be a strong doubly edge geodetic graph.

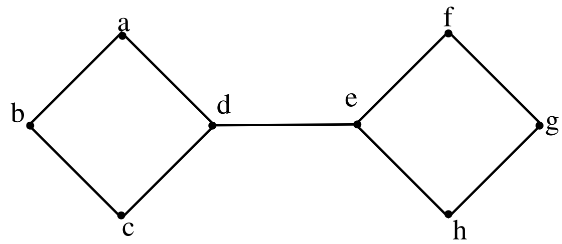


FIGURE 1. Illustration of Strong doubly edge geodetic number of a graph.

**Result 1.** Every simplicial vertices belongs to each strong doubly edge geodetic set.

**Result 2.** For a graph $G$, $3 \leq s\ddot{d}g_e(G) \leq n$.
The bounds are sharp, i.e.if $G = P_n$, then $s\ddot{d}g_e(P_n) = 3$ and for $G = C_5$, $s\ddot{d}g_e(C_5) = 5$.

## 4. COMPUTATIONAL COMPLEXITY

The proof for the NP-completeness of the strong doubly edge geodetic problem for general graphs can be reduced from the vertex cover problem which is already proved to be NP-complete.

**Theorem 1.** Strong doubly edge geodetic problem is NP-complete for general graphs.

*Proof.* The graph $\bar{G}(\bar{V},\bar{E})$ is constructed from a given graph $G(V,E)$ as follows : The vertex set $\bar{V} = \{a,b\} \cup V \cup V^{'} \cup V^{''}$ where $V^{'}$ induces a clique in $\bar{G}$ and $V^{''}$ is an independent set of order $2|V|$. The edge set of $\bar{G}$ is $\bar{E} = E \cup E^{'} \cup E^{''} \cup E^{'''}$ where $E^{'}$ is the edge set of $\bar{G}(V^{'})$ which is a complete graph with $|V|$ vertices, $E^{''} = \{vv^{'}\} \cup \{v^{'}v_1^{''}\} \cup \{v^{'}v_2^{''}\}: v \in V, v' \in V', v_1^{''}, v_2^{''} \in V''$ and $E^{'''} = \{vb\} \cup \{ab\}: v \in V$. Refer figure 2. Let $X = V'' \cup \{a\}$. Since the vertices of $V'' \cup \{a\}$ forms a set of simplicial vertices and they are the elements of any strong doubly edge geodetic set in $\bar{G}$. Let $T$ be a vertex cover set of $G$. We will prove that $T \cup X$ forms a strong doubly edge geodetic set set for $\bar{G}$. For $v \in T$ and $u \in N(v)$, the geodesics $v \to u \to u' \to u_1^{''}$, $v \to u \to u' \to u_2^{''}$, $v \to v' \to v_1^{''}$, $v \to v' \to v_2^{''}$, $a \to b \to v \to v' \to v_1^{''}, a \to b \to v \to v' \to v_2^{''}, v_1^{''} \to v' \to u' \to u_1^{''}$ and $v_1^{''} \to v' \to u' \to u_2^{''}$ are the fixed geodesics between the vertices of $T \cup X$ and they will cover all the edges in $\bar{E}(\bar{G})$ at least twice. Thus $T \cup X$ forms a strong doubly edge geodetic set for $\bar{G}$. Conversely, assume that $A$ is a strong doubly edge geodetic set of $\bar{G}$. If $b \in A$ then $A^* = A \backslash \{b\}$

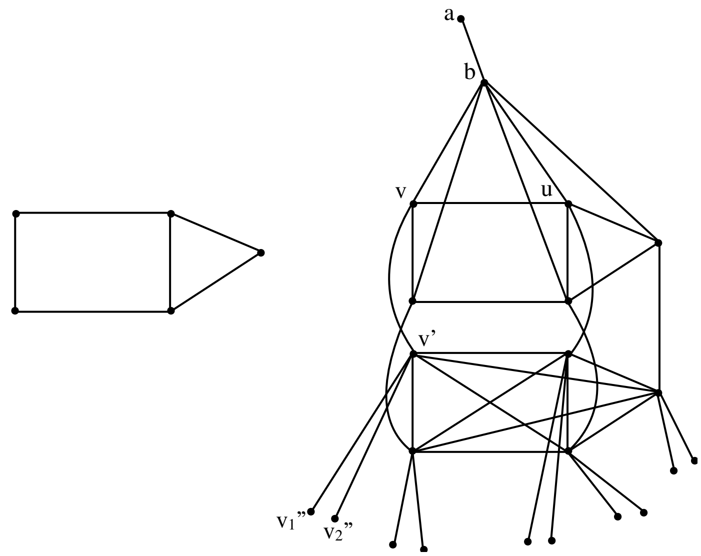


FIGURE 2. Illustration of $G$ and $\bar{G}$.

otherwise $A^* = A$. Clearly, $A^*$ is a strong doubly edge geodetic set of $\bar{G}$ and $|A^*| \leq |A|$. The geodesics between the vertices in $V'' \cup \{a\}$ does not cover any edges in $V$ i.e. $A^* \cap V \neq \phi$. Also, if $v' \in A$ then replace $v'$ by its corresponding vertex $v$. Let $A^{**} = \{v \backslash v' \in A\} \cup \{A \cap V\} \cup X$. It is straightforward that $|A^{**}| \leq |A^*|$ and $A^{**}$ is a strong doubly edge geodetic set of $\bar{G}$. It is easy to see that $A^{**} \backslash X$ is a vertex cover set for $G$.

□

## 5. MAIN RESULTS

**Theorem 2.** If $G(V,E)$ is a graph with diameter $d$ then $s\ddot{d}g_e(G) \geq \frac{d+\sqrt{d^2+16d|E|}}{2d}$.

*Proof.* Let $S \subseteq V(G)$ be a minimum strong doubly geodetic set, where $|S| = s\ddot{d}g_e(G)$. This implies that every $e \in E(G)$ lies on at least 2 geodesics between the vertices in $S$ and each geodesic covers $d$ edges. Thus $2|E| \leq d\binom{|S|}{2}$ which in turn implies that $s\ddot{d}g_e(G) \geq \frac{d+\sqrt{d^2+16d|E|}}{2d}$.

□

This bound is sharp for the graphs in Figure 3.

**Result 3.** For path, $P_n$ where $n \geq 3$, $s\ddot{d}g_e(P_n) = 3$.

**Result 4.** For cycle $C_n$ where $n \geq 5$, $s\ddot{d}g_e(C_n) = 5$

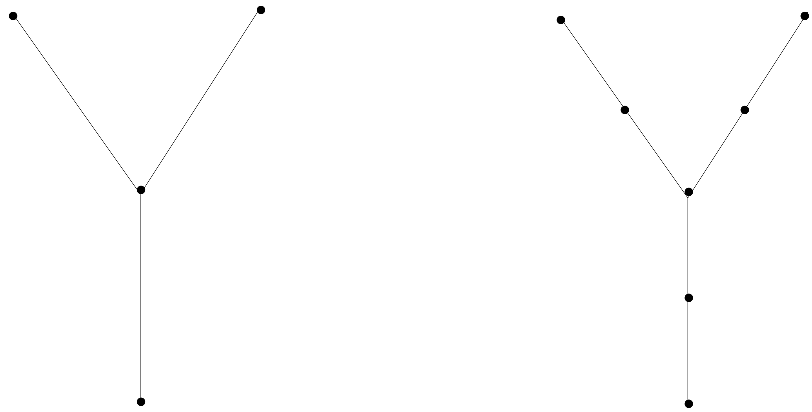
(a) (b)

FIGURE 3. Ilustration of graphs with $s\ddot{d}g_e(G) = \frac{d+\sqrt{d^2+16d|E|}}{2d}$

**Result 5.** For star graph, $K_{1,m}$ where $m \geq 3$, $s\ddot{d}g_e(K_{1,m}) = m$.

**Result 6.** For wheel graph, $W_n$ where $n \geq 7$, $s\ddot{d}g_e(W_n) = n-1$.

**Result 7.** For fan graph, $F_{1,m}$ where $m \geq 7$, $s\ddot{d}g_e(F_{1,m}) = m$

**Result 8.** For a tree $T$ on $l \geq 3$ leaves, $s\ddot{d}g_e(T) = l$.

Two edges $e, f \in E(G)$ are geodesic if they belong to some shortest path of $G$ and are otherwise called non-geodesic edges [11].

**Theorem 3.** Consider a graph $G$ with a set of pairwise non-geodesic edges $A$. If $G-A$ consists convex components then $s\ddot{d}g_e(G) \geq \left\lceil a\sqrt[a]{\frac{4|A|}{a(a-1)}}\right\rceil$ where $a \geq 2$ is the number of components in $G-A$.

*Proof.* Let the components in $G-A$ be $G_1, G_2, \ldots, G_a$ and $S$ be the strong doubly edge geodetic set of $G$. Also let $S_i$ be the strong doubly edge geodetic set of $G_i$ and $S_i \cap S_j = \phi$ where $i, j \in [1, 2 \ldots, a]$. As each component of $G-A$ is assumed to be convex, no geodesics between the vertices in $S_i$ will contain an edge in $A$. Since $S$ is assumed to be the strong doubly edge geodetic set of $G$, there exists two geodesic between the vertices of $S_i$ and $S_j$ where each geodesics contains at most one edge from $A$. This implies that $\sum\limits_{i \neq j} |S_i||S_j| \geq 2|A|$. Also for any positive real numbers, their arithmetic mean is greater than or equal to their geometric mean, i.e. $|S| =$

$\sum_{k=1}^{a}|S_k| \geq a\sqrt[a]{\prod_{k=1}^{a}|S_k|}$. But for all $|S_k| \geq 1$, $\prod_{k=1}^{a}|S_k| \geq |S_i||S_j|$. This inequality holds for all $\binom{a}{2}$ pairs i.e. $\binom{a}{2}\prod_{k=1}^{a}|S_k| \geq \sum_{i\neq j}|S_i||S_j|$. Thus $|S| \geq a\sqrt[a]{\frac{\sum_{i\neq j}|S_i||S_j|}{\binom{a}{2}}}$. From this we get,$|S| \geq a\sqrt[a]{\frac{2|A|}{\binom{a}{2}}}$. On solving this inequality and considering only the integral part, $|S| \geq \left\lceil a\sqrt[a]{\frac{4|A|}{a(a-1)}}\right\rceil$.

□

If $a = 2$ in the above theorem then the set $A$ is a convex edge cut and hence the following theorem.

**Corollary 1.** For a graph $G$ with convex edge cut $A$, $s\ddot{d}g_e(G) \geq \lceil 2\sqrt{2|A|}\rceil$.

This bound is sharp for Glued binary trees without randomization $GT_1(r)$ and $GT_2(r)$. The glued binary tree $GT_2(r)$ is obtained by adding cross edges to each leaves of $GT_1(r)$ (Refer Figure 4 and Figure 5). For $GT_1(r)$, $A = 2^r$ and it can be easily verified that $s\ddot{d}g_e(GT_1(r)) = \lceil 2\sqrt{2^{r+1}}\rceil$. Similarly for $GT_2(r)$, $A = 2^{r+1}$ and it can be easily verified that $s\ddot{d}g_e(GT_2(r)) = \lceil 4\sqrt{2^r}\rceil$.

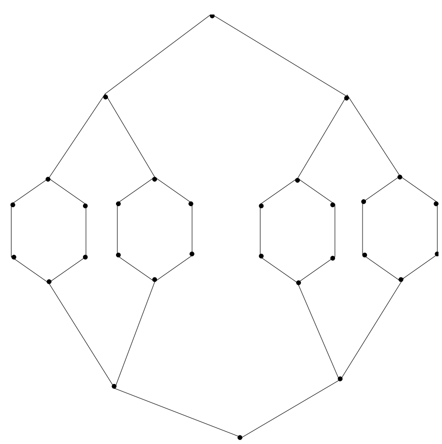

FIGURE 4. Illustration of $GT_1(3)$

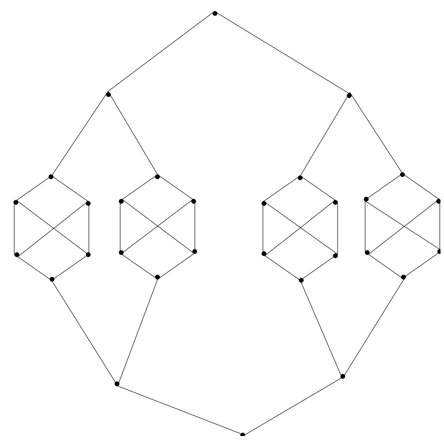

FIGURE 5. Illustration of $GT_2(3)$

**Theorem 4.** For three positive integers $r,d,a$ where $4 < r < d \leq 2r$ and $a \geq 5$, there exists a connected graph $G$ with $radius(G) = G$, $diameter(G) = d$ and $s\ddot{d}g_e(G) = a$.

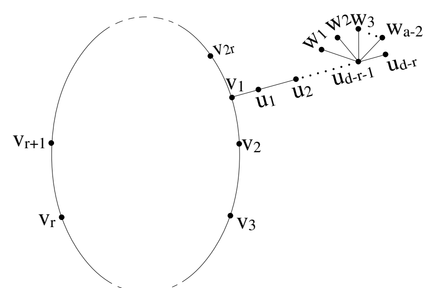


FIGURE 6. Graph $G$ with $radius(G) = r$, $diameter(G) = d$ and $s\ddot{d}g_e(G) = a$.

*Proof.* For $G = K_a$, $s\ddot{d}g_e(G) = a$ with $r = d = 1$. For $G = K_{1,a}$, $s\ddot{d}g_e(G) = a$ with $r = 1$ and $d = 2$. For $r \geq 2$, the graph $G$ is constructed by identifying a vertex $u_0$ of the path $P : u_0, u_1, u_2, \ldots, u_{d-r}$ with the vertex $v_1$ of the cycle $C : v_1, v_2, v_3, \ldots, v_{2r}, v_1$ and adding new vertices $w_1, w_2, \ldots, w_{a-2}$ where each $w_i, 1 \leq i \leq a-2$ to the vertex $v_{d-r-1}$ (Refer Figure: 6). Clearly, the set $\{w_1, w_2, \ldots, w_{a-2}, u_{r+1}, v_{d-r}\}$ forms a strong doubly edge geodetic set for $G$. Thus the graph $G$ is obtained where $s\ddot{d}g_e(G) = a$, $radius(G) = G$ and $diameter(G) = d$.

□

**Theorem 5.** For two positive integers $a,b$ where $5 \leq a < b$, there exists a connected graph $G$ with $\ddot{d}g_e(G) = a$ and $s\ddot{d}g_e(G) = b$.

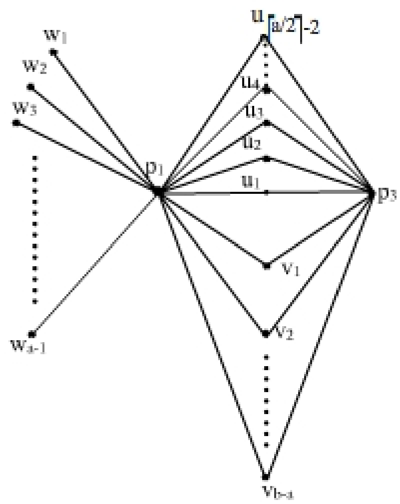


FIGURE 7. Graph $G$ with $\ddot{d}g_e(G) = a$ and $s\ddot{d}g_e(G) = b$

*Proof.* Let $G$ be a graph obtained from the path $P_3:\ p_1, u_1, p_3$ and adding new vertices $u_2, u_3, \ldots, u_{\lfloor \frac{a-1}{2} \rfloor}, v_1, v_2, \ldots, v_{b-a}$ in such a way that each $u_i, 2 \leq i \leq \lfloor \frac{a-1}{2} \rfloor$ and $v_j, 1 \leq j \leq b-a$ are joined to the vertices $p_1$ and $p_3$ respectively. Also, join the vertices $w_1, w_2, \ldots w_{a-1}$ to the vertex $p_1$ (Refer Figure: 7). It is straightforward to see that $S = \{w_1, w_2, \ldots, w_{a-1}, p_3\}$ forms a doubly edge geodetic set for $G$. But $S$ does-not form a strong doubly edge geodetic set for $G$. It can be easily seen that $T = S \cup \{v_1, v_2, \ldots, v_{b-a}\}$ forms a strong doubly edge geodetic set for $G$. Thus $\ddot{d}g_e(G) = a$ and $s\ddot{d}g_e(G) = b - a + a - 1 + 1 = b$.

□

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

## REFERENCES


[1] S. Arumugam, I.S. Hamid, Simple path covers in graphs, Int. J. Math. Comb. 3 (2008), 94.

[2] G. Chartrand, F. Harary, P. Zhang, On the geodetic number of a graph, Networks, 39 (2002), 1-6.

[3] G. Chartrand, E.M. Palmer, P. Zhang, The geodetic number of a graph: a survey, in: Proceedings of the Thirty-third Southeastern International Conference on Combinatorics, Graph Theory and Computing, Boca Raton, FL, 2002.

[4] H. Ge, Z. Wang, J. Zou, Strong geodetic number in some networks, J. Mat. Res. 11 (2019), 20-29. .

[5] V. Gledel, V. Iršič, Strong Geodetic Number of Complete Bipartite Graphs, Crown Graphs and Hypercubes, Bull. Malays. Math. Sci. Soc. 43 (2020), 2757–2767.

[6] V. Iršič, Strong geodetic number of complete bipartite graphs and of graphs with specified diameter, Graphs Comb. 34 (2018), 443-456.

[7] V. Iršič, M. Konvalinka, Strong geodetic problem on complete multipartite graphs, ArXiv:1806.00302 [Math]. (2018).

[8] V. Iršič, S. Klavžar, Strong geodetic problem on Cartesian products of graphs, RAIRO-Oper. Res. 52 (2018), 205-216.

[9] S. Klavžar, P. Manuel, Strong geodetic problem in grid-like architectures, Bull. Malays. Math. Sci. Soc. 41 (2018), 1671-1680.

[10] A. Arokiaraj, S. Klavžar, P. Manuel, E. Thomas, A. Xavier, Strong geodetic problems in networks, Discuss. Math. Graph Theory. 40 (2020), 307.

[11] P. Manuel, S. Klavžar, A. Xavier, A. Arokiaraj, E. Thomas, Strong edge geodetic problem in networks, Open Math. 15 (2017), 1225–1235.

[12] I.M. Pelayo, Geodesic Convexity in Graphs, Springer New York, New York, NY, 2013.

[13] A.P. Santhakumaran, J. John, Edge geodetic number of a graph, J. Discrete Math. Sci. Cryptogr. 10 (2007), 415–432.

[14] A.P. Santhakumaran, S.V. Ullas Chandran, On the edge geodetic and k-edge geodetic number of a graph, Int. J.Math. Comb. 3 (2008), 85-93

[15] Z. Wang, Y. Mao, H. Ge, C. Magnant, Strong geodetic number of graphs and connectivity, Bull. Malays. Math. Sci. Soc. 43 (2020), 2443–2453.

[16] D.A. Xavier, A. Arokiaraj, E. Thomas, Doubly geodetic number of a graph, IOSR J. Math. 12 (2016), 13-17.

[17] D.A. Xavier, E. Thomas, On the doubly edge geodetic number of a graph, Int. J. Computer Sci. Eng. 7 (2019), 208-212.