A CLASS OF SIXTH ORDER HYBRID EXTENDED BLOCK BACKWARD DIFFERENTIATION FORMULAE FOR COMPUTATIONAL SOLUTIONS OF FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

C. CHIBUISI[1], B. O. OSU[2,*], S. A. IHEDIOHA[3], C. OLUNKWA[4], E. E. AKPANIBAH[5], P. U. UZOMA[6]

[1]Department of Insurance, University of Jos, Jos, Nigeria

[2]Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria

[3]Department of Mathematics, Plateau State University, Bokkos, Jos, Nigeria

[4]Department of Mathematics, Abia State University, Nigeria

[5]Department of Mathematics and Statistics, Federal University Otuoke, Bayelsa, Nigeria

[6]Department of Mathematics and Statistics, Federal Polytechnic, Nekede Owerri Nigeria

Abstract: In this paper, we established and carried-out the computational solution of some first order delay differential equations (DDEs) using hybrid extended backward differentiation formulae method in block forms without the application of interpolation techniques in determining the delay term. The discrete schemes were worked-out through the linear multistep collocation technique by matrix inversion approach from the continuous construction of each step number which clearly demonstrated the order and error constants, consistency, zero stability, convergence and region of absolute stability of this method after investigations. The results obtained after the implementation of this method validate that the lower step number integrated with hybrid extended future points performed better than the higher step numbers integrated with hybrid extended future points when compared with the exact solutions and other existing methods.

Keywords: first order delay differential equations; hybrid block method; off-grid point; extended future points;

---

*Corresponding author

E-mail address: osu.bright@mouau.edu.ng

backward differentiation formulae.

2010 AMS Subject Classification: 34K28; 65F30.

1. INTRODUCTION

Delay differential equations have displayed great importance in its applications and diversities in modeling real life situations. Delay differential equations (DDEs) are differential equations which considers time series of past values of dependent variables and derivatives, whereas the evolution of ODEs depends only on the current values of these quantities. Every real life phenomenon involves delay and has been studied by many researchers. For decades, notable scholars have carried-out intensive research activities in the area of computational solutions for delay differential equations which have shown the numerous advantages of DDEs over ODEs for mathematical modeling of real life problems. The applications of DDEs can be seen in many real life situations involving celestial and quantum mechanics, nuclear and theoretical physics, astrophysics, quantum chemistry, molecular dynamics, engineering, medicine and economic dynamics and control. In physical sciences, Tziperman*., Stone., Cane., Jarosh.,(1994), delay differential equations was used in the modeling of El Nino temperature oscillations in the Equatorial Pacific to determine the model single-species population growth. In electrical circuits, delays are introduced because it takes time for a signal to travel through a transmission line.

In this research work, we look forward to obtaining the computational solutions of the first order delay differential equations (DDEs) of the form:

$$x'(t) = f(t, x(t), x(t-\tau)), \quad \text{for } t > t_0, \tau > 0$$

$$x(t) = \alpha(t), \quad \text{for } t \le t_0 \tag{1}$$

where $\alpha(t)$ is the initial function, τ is called the delay, $(t-\tau)$ is called the delay argument and $x(t-\tau)$ is the solution of the delay argument using hybrid extended block backward differentiation formulae methods.

Renowned scholars such as Ballen, Zennaro, (1985), formulated the algorithms for solving non-stiff DDEs with time dependent delays using predictor-corrector version of the one-step

collocation method at Gaussion points. Evans, Raslan, (2005), constructed a numerical method for linear and non-linear higher order delay differential equations through Adomain decomposition method. Seong, Majid, (2015), applied the direct two-point fourth and fifth order multistep block method in the form of Adams- Moulton Method in predictor-corrected (PECE) mode to compute the numerical solutions at two points simultaneously for second order delay differential equations. Al-Mutib, (1977), Oberle, Pesh, (1981) used linear multistep approach in solving delay differential equations with the aid of interpolation techniques in estimating the delay argument. Ishak, Suleiman, Omar, (2008) applied interpolation method to investigate the delay argument while solving delay differential equations in a predictor-corrector mode. Bocharov, Marchuk, Romanyukha, (1996) adopted linear multistep methods for the numerical solution of initial value problem for stiff delay differential equations with several constant delays with Nordsieck's interpolation technique. Majid, Radzi, Ismail, (2012) solved delay differential equations by the five -point one-step block method using *Neville's* interpolation in approximating the delay term. Heng, Ibrahim, Suleiman, Ismail, (2013) applied implicit 2-point Block Backward Differential Formulae to solve a set of delay differential equations with the help of interpolation technique in approximating the delay term. Edeki and Akinlabi (2017), used the Zhou method to obtain approximate-analytical solutions of certain system of functional differential equations (SFDEs) induced by proportional delays. Edeki, Akinlabi, Adeosun, (2017), procured the results of some system of time-fractional differential equations (TFDDEs) using proportional delays. Sirisena, Yakubu, (2019), used Reformulated Block Backward Differentiation Formulae Methods to study the numerical solutions of first order delay differential equations without the application of any interpolation technique for the delay term computation. Jena, Chakraverty, Edeki, and Ofuyatan, (2020), examined the results of fractional order delay differential equations by applying Shifted Legendre polynomials with Collocation and Galerkin settings. Yakubu and Chibuisi (2020), investigated the numerical solution of stiff differential equations using a class of fifth order block hybrid Adams Moulton's Method to determine the accuracy and efficiency of each step number incorporated with off-grid points. Chibuisi, Osu, and Ogbogbo. (2020) implemented block simpson's methods in solving first order

delay differential equations without using any interpolation techniques in examining the delay term. Osu, Chibuisi, Okwuchukwu, Olunkwa, and Okore. (2020), implemented third derivative block backward differentiation formulae for numerical solutions of first order delay differential equations without interpolation techniques in investigating the delay argument. Chibuisi, Osu, Amaraihu and Okore. (2020), solved first order delay differential equations using multiple off-grid hybrid block simpson's methods without the application of any interpolation technique in estimating the delay term. Chibuisi, Osu, Edeki and Akinlabi. (2020), solved first DDEs using extended block backward differentiation formulae for efficiency of the numerical solution without the introduction of interpolation techniques in finding the delay argument.

The difficulty in applying these interpolation techniques by Majid *et al*., (2012) is that the numerical methods to be implemented in solving DDEs should be the same with the interpolating polynomials which is very difficult to carry out; otherwise, the accuracy of the method will not be preserved. It is important that in the evaluation of the delay argument, applying a reliable and coherent formula shall be highly recommended.

In order to prevail over the difficulty caused by using interpolation techniques in evaluating the delay argument; we applied idea of the sequence constructed by Sirisena *et al*. (2019) which we merge into the first order delay differential equations. Then we implemented hybrid extended block backward differentiation formulae methods to solve some first order delay differential equations containing the evaluated delay argument to improve the performance of the existing extended BBDF been studied by Chibuisi et al. (2020) in terms of efficiency, accuracy, consistency, convergence and region of absolute stability at constant step width a.

2. CONSTRUCTION STEPS

2.1 Construction of Multistep Collocation Approach

The k-step multistep collocation approach with e collocation points was derived In Sirisena (1997) as;

$$y(x) = \sum_{s=0}^{g-1} \alpha_s(x) y_{v+s} + a \sum_{s=0}^{e-1} \beta_s(x) f_{v+s}(x, y(x)) \tag{2}$$

where $\alpha_s(x)$ and $\beta_s(x)$ are continuous coefficients of the technique defined as;

$$\alpha_s(x) = \sum_{r=0}^{g+e-1} \alpha_{s,r+1} x^r \text{ for } s = \{0,1,...,g-1\} \tag{3}$$

$$a\beta_s(x) = \sum_{r=0}^{g+e-1} a\beta_{s,r+1} x^r \text{ for } s = \{0,1,...,e-1\} \tag{4}$$

where $x_0, \ldots, x_{e-1}$ are the e collocation points, $x_{v+s}, s = 0,1,2,...,g-1$ are the g arbitrarily chosen interpolation points and a is the constant step width.

To get $\alpha_s(x)$ and $\beta_s(x)$, Sirisena (1997) constructed a matrix equation of the form

$$AB = I \tag{5}$$

where I is the elementary matrix of dimension $(g+e) \times (g+e)$ while B and A are matrices defined as

$$B = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{g-1,1} & a\beta_{0,1} & \cdots & a\beta_{e-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{g-1,2} & a\beta_{0,2} & \cdots & a\beta_{e-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,g+e} & \alpha_{1,g+e} & \cdots & \alpha_{g-1,g+e} & a\beta_{0,g+e} & \cdots & a\beta_{e-1,g+e} \end{bmatrix} \tag{6}$$

$$A = \begin{bmatrix} 1 & x_v & x_v^2 & \cdots & x_v^{g+e-1} \\ 1 & x_{v+1} & x_{v+1}^2 & \cdots & x_{v+1}^{g+e-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{v+g-1} & x_{v+g-1}^2 & \cdots & x_{v+g-1}^{g+e-1} \\ 0 & 1 & 2x_0 & \cdots & (g+e-1)x_0^{g+e-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{e-1} & \cdots & (g+e-1)x_{e-1}^{g+e-2} \end{bmatrix} \tag{7}$$

From the matrix equation (5), the columns of $B = A^{-1}$ give the continuous coefficients of the continuous scheme of (2).

2.1 Construction of HEBBDF Method with Integrated Three Off-grids Extended Future Points and One Extended Future Point for $k = 2$

FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

Here, we integrated three off-grids extended future points at $x = x_{v+\frac{9}{4}}, x = x_{v+\frac{5}{2}}, x = x_{v+\frac{11}{4}}$ and one

extended future point at $x = x_{v+3}$ as collocation points, thus the interpolation points, $g = 2$ and

the collocation points $e = 5$ are considered, therefore, (2) becomes:

$$y(x) = \alpha_0(x)y_v + \alpha_1(x)y_{v+1} + a[\beta_2(x)f_{v+2} + \beta_{\frac{9}{4}}(x)f_{v+\frac{9}{4}} + \beta_{\frac{5}{2}}(x)f_{v+\frac{5}{2}} + \beta_{\frac{11}{4}}(x)f_{v+\frac{11}{4}} + \beta_3(x)f_{v+3}] \quad (8)$$

The matrix A in (5) becomes

$$A = \begin{pmatrix} 1 & x_v & x_v^2 & x_v^3 & x_v^4 & x_v^5 & x_v^6 \\ 1 & x_v+a & (x_v+a)^2 & (x_v+a)^3 & (x_v+a)^4 & (x_v+a)^5 & (x_v+a)^6 \\ 0 & 1 & 2x_v+4a & 3(x_v+2a)^2 & 4(x_v+2a)^3 & 5(x_v+2a)^4 & 6(x_v+2a)^5 \\ 0 & 1 & 2x_v+\frac{9}{2}a & 3\left(x_v+\frac{9}{4}a\right)^2 & 4\left(x_v+\frac{9}{4}a\right)^3 & 5\left(x_v+\frac{9}{4}a\right)^4 & 6\left(x_v+\frac{9}{4}a\right)^5 \\ 0 & 1 & 2x_v+5a & 3\left(x_v+\frac{5}{2}a\right)^2 & 4\left(x_v+\frac{5}{2}a\right)^3 & 5\left(x_v+\frac{5}{2}a\right)^4 & 6\left(x_v+\frac{5}{2}a\right)^5 \\ 0 & 1 & 2x_v+\frac{11}{2}a & 3\left(x_v+\frac{11}{4}a\right)^2 & 4\left(x_v+\frac{11}{4}a\right)^3 & 5\left(x_v+\frac{11}{4}a\right)^4 & 6\left(x_v+\frac{11}{4}a\right)^5 \\ 0 & 1 & 2x_v+3a & 3(x_v+3a)^2 & 4(x_v+3a)^3 & 5(x_v+3a)^4 & 6(x_v+3a)^5 \end{pmatrix}$$

The inverse of the matrix $B = A^{-1}$ is examined using Maple 18 from which the continuous

scheme is obtained using (2), evaluating and simplifying it at

$x = x_{v+2}, x = x_{v+\frac{9}{4}}, x = x_{v+\frac{5}{2}}, x = x_{v+\frac{11}{4}}, x = x_{v+3}$ and its derivative at $x = x_{v+1}$, the following discrete

schemes are obtained;

$$y_{v+1} = \frac{248}{45}af_{v+1} - \frac{1517}{10}af_{v+2} - \frac{4333}{90}af_{v+3} - \frac{21194}{45}af_{v+\frac{5}{2}} + \frac{19088}{45}af_{v+\frac{9}{4}} + \frac{3632}{15}af_{v+\frac{11}{4}} + y_v$$

$$y_{v+2} = -\frac{73}{1736}y_v + \frac{1809}{1736}y_{v+1} + \frac{1859461}{156240}af_{v+2} - \frac{288754}{9765}af_{v+\frac{9}{4}} + \frac{404263}{13020}af_{v+\frac{5}{2}} - \frac{21502}{1395}af_{v+\frac{11}{4}} + \frac{66763}{22320}af_{v+3}$$

$$y_{v+\frac{9}{4}} = -\frac{10675}{253952}y_v + \frac{264627}{253952}y_{v+1} + \frac{6090795}{507904}af_{v+2} - \frac{465975}{15872}af_{v+\frac{9}{4}} + \frac{3932955}{126976}af_{v+\frac{5}{2}} - \frac{244215}{15872}af_{v+\frac{11}{4}} + \frac{1516995}{507904}af_{v+3}$$

$$y_{v+\frac{5}{2}} = -\frac{4671}{111104}y_v + \frac{115775}{111104}y_{v+1} + \frac{2662935}{222208}af_{v+2} - \frac{202995}{6944}af_{v+\frac{9}{4}} + \frac{1729095}{55552}af_{v+\frac{5}{2}} - \frac{15285}{992}af_{v+\frac{11}{4}} + \frac{94905}{31744}af_{v+3}$$

$$y_{v+\frac{11}{4}} = -\frac{10675}{253952}y_v + \frac{264627}{253952}y_{v+1} + \frac{274022287}{22855680}af_{v+2} - \frac{20901419}{714240}af_{v+\frac{9}{4}} + \frac{59597461}{1904640}af_{v+\frac{5}{2}} - \frac{10922219}{714240}af_{v+\frac{11}{4}}$$
$$+\frac{68201287}{22855680}af_{v+3}$$

$$y_{v+3} = -\frac{73}{1736}y_v + \frac{1809}{1736}y_{v+1} + \frac{207957}{17360}af_{v+2} - \frac{31698}{1085}af_{v+\frac{9}{4}} + \frac{135333}{4340}af_{v+\frac{5}{2}} - \frac{2334}{155}af_{v+\frac{11}{4}} + \frac{7611}{2480}af_{v+3} \quad (9)$$

2.2 Construction of HEBBDF Method with Integrated Two Off-grids Extended Future Points and One Extended Future Point for $k = 3$

In this case, we integrated two off-grids extended future points at $x = x_{v+\frac{7}{2}}, x = x_{v+\frac{15}{4}}$ and one extended future point at $x = x_{v+4}$ as collocation points, thus the interpolation points, $g = 3$ and the collocation points $e = 4$ are considered, therefore, (2) becomes:

$$y(x) = \alpha_0(x)y_v + \alpha_1(x)y_{v+1} + \alpha_2(x)y_{v+2} + a[\beta_3(x)f_{v+3} + \beta_{\frac{7}{2}}(x)f_{v+\frac{7}{2}} + \beta_{\frac{15}{4}}(x)f_{v+\frac{15}{4}} + \beta_4(x)f_{v+4}] \quad (10)$$

The matrix A in (5) becomes

$$A = \begin{pmatrix} 1 & x_v & x_v^2 & x_v^3 & x_v^4 & x_v^5 & x_v^6 \\ 1 & x_v+a & (x_v+a)^2 & (x_v+a)^3 & (x_v+a)^4 & (x_v+a)^5 & (x_v+a)^6 \\ 1 & x_v+2a & (x_v+2a)^2 & (x_v+2a)^3 & (x_v+2a)^4 & (x_v+2a)^5 & (x_v+2a)^6 \\ 0 & 1 & 2x_v+6a & 3(x_v+3a)^2 & 4(x_v+3a)^3 & 5(x_v+3a)^4 & 6(x_v+3a)^5 \\ 0 & 1 & 2x_v+7a & 3\left(x_v+\frac{7}{2}a\right)^2 & 4\left(x_v+\frac{7}{2}a\right)^3 & 5\left(x_v+\frac{7}{2}a\right)^4 & 6\left(x_v+\frac{7}{2}a\right)^5 \\ 0 & 1 & 2x_v+\frac{15}{2}a & 3\left(x_v+\frac{15}{4}a\right)^2 & 4\left(x_v+\frac{15}{4}a\right)^3 & 5\left(x_v+\frac{15}{4}a\right)^4 & 6\left(x_v+\frac{15}{4}a\right)^5 \\ 0 & 1 & 2x_v+8a & 3(x_v+4a)^2 & 4(x_v+4a)^3 & 5(x_v+4a)^4 & 6(x_v+4a)^5 \end{pmatrix}$$

The inverse of the matrix $B = A^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (2), evaluating and simplifying it

at $x = x_{v+3}, x = x_{v+\frac{7}{2}}, x = x_{v+\frac{15}{4}}, x = x_{v+4}$ and its derivative at $x = x_{v+1}$ and $x = x_{v+2}$, the following

discrete schemes are obtained;

$$y_{v+1} = -\frac{9373517}{11408760}af_{v+1} - \frac{696013}{207432}af_{v+3} + \frac{292489}{103716}af_{v+4} + \frac{428528}{43215}af_{v+\frac{7}{2}} - \frac{2804480}{285219}af_{v+\frac{15}{4}} - \frac{3151}{23048}y_v$$

$$+\frac{26199}{23048}y_{v+2}$$

$$y_{v+2} = \frac{9373517}{4371885}af_{v+2} - \frac{4075204}{624555}af_{v+3} + \frac{927397}{208185}af_{v+4} + \frac{10557376}{624555}af_{v+\frac{7}{2}} - \frac{70227968}{4371885}af_{v+\frac{15}{4}} - \frac{1193}{13879}y_v$$

$$+\frac{15072}{13879}y_{v+1}$$

$$y_{v+3} = \frac{144377}{9373517}y_v - \frac{1564731}{9373517}y_{v+1} + \frac{10793871}{9373517}y_{v+2} + \frac{23388822}{9373517}af_{v+3} - \frac{45109440}{9373517}af_{v+\frac{7}{2}} + \frac{40722432}{9373517}af_{v+\frac{15}{4}}$$

$$-\frac{10904274}{9373517}af_{v+4}$$

$$y_{v+\frac{7}{2}} = \frac{9134775}{599905088}y_v - \frac{24820803}{149976272}y_{v+1} + \frac{690053525}{599905088}y_{v+2} + \frac{402702825}{149976272}af_{v+3} - \frac{40561395}{9373517}af_{v+\frac{7}{2}} + \frac{38661000}{9373517}af_{v+\frac{15}{4}}$$

$$-\frac{335171025}{299952544}af_{v+4}$$

$$y_{v+\frac{15}{4}} = \frac{146309933}{9598481408}y_v - \frac{24839325}{149976272}y_{v+1} + \frac{11041888275}{9598481408}y_{v+2} + \frac{12864892425}{4799240704}af_{v+3} - \frac{5038612425}{1199810176}af_{v+\frac{7}{2}}$$

$$+\frac{319538835}{74988136}af_{v+\frac{15}{4}} - \frac{676350675}{599905088}af_{v+4}$$

$$y_{v+4} = \frac{142641}{9373517}y_v - \frac{1550656}{9373517}y_{v+1} + \frac{10781532}{9373517}y_{v+2} + \frac{25178016}{9373517}af_{v+3} - \frac{39817728}{9373517}af_{v+\frac{7}{2}}$$

$$+\frac{41828352}{9373517}af_{v+\frac{15}{4}} - \frac{9706980}{9373517}af_{v+4} \qquad (11)$$

2.3 Construction of HEBBDF Method with Integrated One Off-grid Extended Future Point and One Extended Future Point for $k = 4$

With the same procedure, we integrated one off-grid extended future points at $x = x_{v+\frac{9}{2}}$ and one

extended future point at $x = x_{v+5}$ as collocation points, thus the interpolation points, $g = 4$ and

the collocation points $e = 3$ are considered, therefore, (2) becomes:

$$y(x) = \alpha_0(x)y_v + \alpha_1(x)y_{v+1} + \alpha_2(x)y_{v+2} + \alpha_3(x)y_{v+3} + a[\beta_4(x)f_{v+4} + \beta_{\frac{9}{2}}(x)f_{v+\frac{9}{2}} + \beta_5(x)f_{v+5}] \qquad (12)$$

The matrix A in (5) becomes

$$A = \begin{pmatrix} 1 & x_v & x_v^2 & x_v^3 & x_v^4 & x_v^5 & x_v^6 \\ 1 & x_v+a & (x_v+a)^2 & (x_v+a)^3 & (x_v+a)^4 & (x_v+a)^5 & (x_v+a)^6 \\ 1 & x_v+2a & (x_v+2a)^2 & (x_v+2a)^3 & (x_v+2a)^4 & (x_v+2a)^5 & (x_v+2a)^6 \\ 1 & x_v+3a & (x_v+3a)^2 & (x_v+3a)^3 & (x_v+3a)^4 & (x_v+3a)^5 & (x_v+3a)^6 \\ 0 & 1 & 2x_v+8a & 3(x_v+4a)^2 & 4(x_v+4a)^3 & 5(x_v+4a)^4 & 6(x_v+4a)^5 \\ 0 & 1 & 2x_v+9a & 3\left(x_v+\frac{9}{2}a\right)^2 & 4\left(x_v+\frac{9}{2}a\right)^3 & 5\left(x_v+\frac{9}{2}a\right)^4 & 6\left(x_v+\frac{9}{2}a\right)^5 \\ 0 & 1 & 2x_v+10a & 3(x_v+5a)^2 & 4(x_v+5a)^3 & 5(x_v+5a)^4 & 6(x_v+5a)^5 \end{pmatrix}$$

The inverse of the matrix $B = A^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (2), evaluating and simplifying it at $x = x_{v+4}, x = x_{v+\frac{9}{2}}, x = x_{v+5}$ and its derivative at $x = x_{v+1}, x = x_{v+2}$ and $x = x_{v+3}$, the following discrete schemes are obtained;

$$y_{v+1} = -\frac{17939}{22302}af_{v+1} + \frac{8284}{11151}af_{v+4} + \frac{1325}{7434}af_{v+5} - \frac{128}{189}af_{v+\frac{9}{2}} - \frac{4594}{33453}y_v + \frac{2294}{1239}y_{v+2} - \frac{3413}{4779}y_{v+3}$$

$$y_{v+2} = -\frac{71756}{32355}af_{v+2} - \frac{2638}{2157}af_{v+4} - \frac{1706}{6471}af_{v+5} + \frac{33856}{32355}af_{v+\frac{9}{2}} + \frac{1565}{19413}y_v - \frac{683}{719}y_{v+1} + \frac{36289}{19413}y_{v+3}$$

$$y_{v+3} = \frac{215268}{173263}af_{v+3} - \frac{1253502}{1212841}af_{v+4} - \frac{219294}{1212841}af_{v+5} + \frac{939840}{1212841}af_{v+\frac{9}{2}} + \frac{34291}{1212841}y_v - \frac{43929}{173263}y_{v+1} + \frac{1486053}{1212841}y_{v+2}$$

$$y_{v+4} = -\frac{1657}{125573}y_v + \frac{1982}{17939}y_{v+1} - \frac{55890}{125573}y_{v+2} + \frac{24178}{17939}y_{v+3} + \frac{152304}{125573}af_{v+4} - \frac{75648}{125573}af_{v+\frac{9}{2}} + \frac{15804}{125573}af_{v+5}$$

$$y_{v+\frac{9}{2}} = -\frac{230125}{18369536}y_v + \frac{1937925}{18369536}y_{v+1} - \frac{7890939}{18369536}y_{v+2} + \frac{24552675}{18369536}y_{v+3} + \frac{13480425}{9184768}af_{v+4} - \frac{94815}{287024}af_{v+\frac{9}{2}}$$
$$+ \frac{978075}{9184768}af_{v+5}$$

$$y_{v+5} = -\frac{1702}{125573}y_v + \frac{2025}{17939}y_{v+1} - \frac{56650}{125573}y_{v+2} + \frac{24250}{17939}y_{v+3} + \frac{171900}{125573}af_{v+4} + \frac{9600}{125573}af_{v+\frac{9}{2}}$$
$$+ \frac{36240}{125573}af_{v+5} \tag{13}$$

3. CONVERGENCE ANALYSIS

Here, the examinations of order, error constant, consistency, zero stability and region of the absolute stability of (9), (11) and (13) are presented.

3.1 Order and Error Constant

The order and error constants for (9) are obtained as follows:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_{\frac{9}{4}} + \alpha_{\frac{5}{2}} + \alpha_{\frac{11}{4}} + \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \alpha_1 + 2\alpha_2 + \frac{9}{4}\alpha_{\frac{9}{4}} + \frac{5}{2}\alpha_{\frac{5}{2}} + \frac{11}{4}\alpha_{\frac{11}{4}} + 3\alpha_3 - \beta_0 - \beta_1 - \beta_2 - \beta_{\frac{9}{4}} - \beta_{\frac{5}{2}} - \beta_{\frac{11}{4}} - \beta_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{81}{32}\alpha_{\frac{9}{4}} + \frac{25}{8}\alpha_{\frac{5}{2}} + \frac{121}{32}\alpha_{\frac{11}{4}} + \frac{9}{2}\alpha_3 - \beta_1 - 2\beta_2 - \frac{9}{4}\beta_{\frac{9}{4}} - \frac{5}{2}\beta_{\frac{5}{2}} - \frac{11}{4}\beta_{\frac{11}{4}} - 3\beta_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{243}{128}\alpha_{\frac{9}{4}} + \frac{125}{48}\alpha_{\frac{5}{2}} + \frac{1331}{384}\alpha_{\frac{11}{4}} + \frac{9}{2}\alpha_3 - \frac{1}{2}\beta_1 - 2\beta_2 - \frac{81}{32}\beta_{\frac{9}{4}} - \frac{25}{8}\beta_{\frac{5}{2}} - \frac{121}{32}\beta_{\frac{11}{4}} - \frac{9}{2}\beta_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{2187}{2048}\alpha_{\frac{9}{4}} + \frac{625}{384}\alpha_{\frac{5}{2}} + \frac{14641}{6144}\alpha_{\frac{11}{4}} + \frac{27}{8}\alpha_3 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{243}{128}\beta_{\frac{9}{4}} - \frac{125}{48}\beta_{\frac{5}{2}}$$

$$-\frac{1331}{384}\beta_{\frac{11}{4}} - \frac{9}{2}\beta_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \frac{1}{120}\alpha_1 + \frac{4}{15}\alpha_2 + \frac{19683}{40960}\alpha_{\frac{9}{4}} + \frac{625}{768}\alpha_{\frac{5}{2}} + \frac{161051}{122880}\alpha_{\frac{11}{4}} + \frac{81}{40}\alpha_3 - \frac{1}{24}\beta_1 - \frac{2}{3}\beta_2 - \frac{2187}{2048}\beta_{\frac{9}{4}} - \frac{625}{384}\beta_{\frac{5}{2}}$$

$$-\frac{14641}{6144}\beta_{\frac{11}{4}} - \frac{27}{8}\beta_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \frac{1}{720}\alpha_1 + \frac{4}{45}\alpha_2 + \frac{59049}{327680}\alpha_{\frac{9}{4}} + \frac{3125}{9216}\alpha_{\frac{5}{2}} + \frac{1771561}{2949120}\alpha_{\frac{11}{4}} + \frac{81}{80}\alpha_3 - \frac{1}{120}\beta_1 - \frac{4}{15}\beta_2 - \frac{19683}{40960}\beta_{\frac{9}{4}} - \frac{625}{768}\beta_{\frac{5}{2}}$$

$$-\frac{161051}{122880}\beta_{\frac{11}{4}} - \frac{81}{40}\beta_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_7 = \frac{1}{5040}\alpha_1 + \frac{8}{315}\alpha_2 + \frac{531441}{9175040}\alpha_{\frac{9}{4}} + \frac{15625}{129024}\alpha_{\frac{5}{2}} + \frac{19487171}{82575360}\alpha_{\frac{11}{4}} + \frac{243}{560}\alpha_3 - \frac{1}{720}\beta_1 - \frac{4}{45}\beta_2 - \frac{59049}{327680}\beta_{\frac{9}{4}}$$

$$-\frac{3125}{9216}\beta_{\frac{5}{2}} - \frac{1771561}{2949120}\beta_{\frac{11}{4}} - \frac{81}{80}\beta_3 = \begin{pmatrix} \frac{67031}{1935360} \\ -\frac{6280327}{3359784960} \\ -\frac{6800355}{3640655872} \\ -\frac{2976175}{1592786944} \\ -\frac{26228807}{14042529792} \\ -\frac{232669}{124436480} \end{pmatrix}$$

Therefore, (9) has order $d = 6$ and error constants,

$$\left(\frac{67031}{1935360}, -\frac{6280327}{3359784960}, -\frac{6800355}{3640655872}, -\frac{2976175}{1592786944}, -\frac{26228807}{14042529792}, -\frac{232669}{124436480}\right)^{T}$$

With the same approach, (11) can be presented as:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_{\frac{7}{2}} + \alpha_{\frac{15}{4}} + \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \frac{7}{2}\alpha_{\frac{7}{2}} + \frac{15}{4}\alpha_{\frac{15}{4}} + 4\alpha_4 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_{\frac{7}{2}} - \beta_{\frac{15}{4}} - \beta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{9}{2}\alpha_3 + \frac{49}{8}\alpha_{\frac{7}{2}} + \frac{225}{32}\alpha_{\frac{15}{4}} + 8\alpha_4 - \beta_1 - 2\beta_2 - 3\beta_3 - \frac{7}{2}\beta_{\frac{7}{2}} - \frac{15}{4}\beta_{\frac{15}{4}} - 4\beta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3 + \frac{343}{48}\alpha_{\frac{7}{2}} + \frac{1125}{128}\alpha_{\frac{15}{4}} + \frac{32}{3}\alpha_4 - \frac{1}{2}\beta_1 - 2\beta_2 - \frac{9}{2}\beta_3 - \frac{49}{8}\beta_{\frac{7}{2}} - \frac{225}{32}\beta_{\frac{15}{4}} - 8\beta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{27}{8}\alpha_3 + \frac{2401}{384}\alpha_{\frac{7}{2}} + \frac{16875}{2048}\alpha_{\frac{15}{4}} + \frac{32}{3}\alpha_4 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{9}{2}\beta_3 - \frac{343}{48}\beta_{\frac{7}{2}}$$

$$-\frac{1125}{128}\beta_{\frac{15}{4}} - \frac{32}{3}\beta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \frac{1}{120}\alpha_1 + \frac{4}{15}\alpha_2 + \frac{81}{40}\alpha_3 + \frac{16807}{3840}\alpha_{\frac{7}{2}} + \frac{50625}{8192}\alpha_{\frac{15}{4}} + \frac{128}{15}\alpha_4 - \frac{1}{24}\beta_1 - \frac{2}{3}\beta_2 - \frac{27}{8}\beta_3 - \frac{2401}{384}\beta_{\frac{7}{2}}$$

$$-\frac{16875}{2048}\beta_{\frac{15}{4}} - \frac{32}{3}\beta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \frac{1}{720}\alpha_1 + \frac{4}{45}\alpha_2 + \frac{81}{80}\alpha_3 + \frac{117649}{46080}\alpha_{\frac{7}{2}} + \frac{253125}{65536}\alpha_{\frac{15}{4}} + \frac{256}{45}\alpha_4 - \frac{1}{120}\beta_1 - \frac{4}{15}\beta_2 - \frac{81}{40}\beta_3 - \frac{16807}{3840}\beta_{\frac{7}{2}}$$

$$-\frac{50625}{8192}\beta_{\frac{15}{4}} - \frac{128}{15}\beta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_7 = \frac{1}{5040}\alpha_1 + \frac{8}{315}\alpha_2 + \frac{243}{560}\alpha_3 + \frac{117649}{92160}\alpha_{\frac{7}{2}} + \frac{3796875}{1835008}\alpha_{\frac{15}{4}} + \frac{1024}{315}\alpha_4 - \frac{1}{720}\beta_1 - \frac{4}{45}\beta_2 - \frac{81}{80}\beta_3$$

$$-\frac{117649}{46080}\beta_{\frac{7}{2}} - \frac{253125}{65536}\beta_{\frac{15}{4}} - \frac{256}{45}\beta_4 = \begin{pmatrix} -\dfrac{1835179}{139394304} \\[6pt] -\dfrac{6496457}{419700960} \\[6pt] \dfrac{17421171}{5249169520} \\[6pt] \dfrac{250360775}{76787851264} \\[6pt] \dfrac{16053110535}{4914422480896} \\[6pt] \dfrac{610393}{187470340} \end{pmatrix}$$

Therefore, (11) has order $d = 6$ and error constants,

$$\left(-\frac{1835179}{139394304}, -\frac{6496457}{419700960}, \frac{17421171}{5249169520}, \frac{250360775}{76787851264}, \frac{16053110535}{4914422480896}, \frac{610393}{187470340} \right)^T$$

Applying the same approach, (13) can be obtained as:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_{\frac{9}{2}} + \alpha_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \frac{9}{2}\alpha_{\frac{9}{2}} + 5\alpha_5 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 - \beta_{\frac{9}{2}} - \beta_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_2 = \frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{9}{2}\alpha_3 + 8\alpha_4 + \frac{81}{8}\alpha_{\frac{9}{2}} + \frac{25}{2}\alpha_5 - \beta_1 - 2\beta_2 - 3\beta_3 - 4\beta_4 - \frac{9}{2}\beta_{\frac{9}{2}} - 5\beta_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_3 = \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3 + \frac{32}{3}\alpha_4 + \frac{243}{16}\alpha_{\frac{9}{2}} + \frac{125}{6}\alpha_5 - \frac{1}{2}\beta_1 - 2\beta_2 - \frac{9}{2}\beta_3 - 8\beta_4 - \frac{81}{8}\beta_{\frac{9}{2}} - \frac{25}{2}\beta_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_4 = \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2 + \frac{27}{8}\alpha_3 + \frac{32}{3}\alpha_4 + \frac{2187}{128}\alpha_{\frac{9}{2}} + \frac{625}{24}\alpha_5 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_2 - \frac{9}{2}\beta_3 - \frac{32}{3}\beta_4$$

$$- \frac{243}{16}\beta_{\frac{9}{2}} - \frac{125}{6}\beta_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_5 = \frac{1}{120}\alpha_1 + \frac{4}{15}\alpha_2 + \frac{81}{40}\alpha_3 + \frac{128}{15}\alpha_4 + \frac{19683}{1280}\alpha_{\frac{9}{2}} + \frac{625}{24}\alpha_5 - \frac{1}{24}\beta_1 - \frac{2}{3}\beta_2 - \frac{27}{8}\beta_3 - \frac{32}{3}\beta_4$$

$$-\frac{2187}{128}\beta_{\frac{9}{2}} - \frac{625}{24}\beta_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_6 = \frac{1}{720}\alpha_1 + \frac{4}{45}\alpha_2 + \frac{81}{80}\alpha_3 + \frac{256}{45}\alpha_4 + \frac{59049}{5120}\alpha_{\frac{9}{2}} + \frac{3125}{144}\alpha_5 - \frac{1}{120}\beta_1 - \frac{4}{15}\beta_2 - \frac{81}{40}\beta_3 - \frac{128}{15}\beta_4$$

$$-\frac{19683}{1280}\beta_{\frac{9}{2}} - \frac{625}{24}\beta_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_7 = \frac{1}{5040}\alpha_1 + \frac{8}{315}\alpha_2 + \frac{243}{560}\alpha_3 + \frac{1024}{315}\alpha_4 + \frac{531441}{71680}\alpha_{\frac{9}{2}} + \frac{15625}{1008}\alpha_5 - \frac{1}{720}\beta_1 - \frac{4}{45}\beta_2 - \frac{81}{80}\beta_3$$

$$-\frac{256}{45}\beta_4 - \frac{59049}{5120}\beta_{\frac{9}{2}} - \frac{3125}{144}\beta_5 = \begin{pmatrix} -\dfrac{180601}{9366840} \\ -\dfrac{48523}{2174256} \\ \dfrac{7688973}{679190960} \\ -\dfrac{216271}{35160440} \\ -\dfrac{1684125}{293912576} \\ -\dfrac{45505}{7032088} \end{pmatrix}$$

Therefore, (13) has order $d = 6$ and error constants,

$$\left(-\frac{180601}{9366840}, -\frac{48523}{2174256}, \frac{7688973}{679190960}, -\frac{216271}{35160440}, -\frac{1684125}{293912576}, -\frac{45505}{7032088} \right)^T$$

3.2 Consistency

Since the schemes in (9), (11) and (13) satisfy the condition for consistency of order $d \geq 1$, then

they are consistent.

3.3 Stability Analysis

The zero stability for (9) is estimated as follows:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-\dfrac{1809}{1736} & 1 & 0 & 0 & 0 & 0 \\
-\dfrac{264627}{253952} & 0 & 1 & 0 & 0 & 0 \\
-\dfrac{115775}{111104} & 0 & 0 & 1 & 0 & 0 \\
-\dfrac{264627}{253952} & 0 & 0 & 0 & 1 & 0 \\
-\dfrac{1809}{1736} & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_{v+1} \\ y_{v+2} \\ y_{v+\frac{9}{4}} \\ y_{v+\frac{5}{2}} \\ y_{v+\frac{11}{4}} \\ y_{v+3}
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & \dfrac{73}{1736} \\
0 & 0 & 0 & 0 & 0 & \dfrac{10675}{253952} \\
0 & 0 & 0 & 0 & 0 & \dfrac{4671}{111104} \\
0 & 0 & 0 & 0 & 0 & \dfrac{10675}{253952} \\
0 & 0 & 0 & 0 & 0 & \dfrac{73}{1736}
\end{pmatrix}
\begin{pmatrix}
y_{v-\frac{11}{4}} \\ y_{v-\frac{5}{2}} \\ y_{v-\frac{9}{4}} \\ y_{v-2} \\ y_{v-1} \\ y_v
\end{pmatrix}
$$

$$
+a
\begin{pmatrix}
\dfrac{248}{45} & -\dfrac{1517}{10} & \dfrac{19088}{45} & -\dfrac{21194}{15} & \dfrac{3632}{15} & -\dfrac{4333}{90} \\
0 & \dfrac{1859461}{156240} & -\dfrac{288754}{9765} & \dfrac{404263}{13020} & -\dfrac{21502}{1395} & \dfrac{66763}{22320} \\
0 & \dfrac{6090795}{507904} & -\dfrac{465975}{15872} & \dfrac{3932955}{126976} & -\dfrac{244215}{15872} & \dfrac{1516995}{507904} \\
0 & \dfrac{2662935}{222208} & -\dfrac{202995}{6944} & \dfrac{1729095}{55552} & -\dfrac{15285}{992} & \dfrac{94905}{31744} \\
0 & \dfrac{274022287}{22855680} & -\dfrac{20901419}{714240} & \dfrac{59597461}{1904640} & -\dfrac{10922219}{714240} & \dfrac{68201287}{22855680} \\
0 & \dfrac{207957}{17360} & -\dfrac{31698}{1085} & \dfrac{135333}{4340} & -\dfrac{2334}{155} & \dfrac{7611}{2480}
\end{pmatrix}
\begin{pmatrix}
f_{v+1} \\ f_{v+2} \\ f_{v+\frac{9}{4}} \\ f_{v+\frac{5}{2}} \\ f_{v+\frac{11}{4}} \\ f_{v+3}
\end{pmatrix}
$$

$$
+a
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
f_{v-\frac{11}{4}} \\ f_{v-\frac{5}{2}} \\ f_{v-\frac{9}{4}} \\ f_{v-2} \\ f_{v-1} \\ f_v
\end{pmatrix}
$$

where $J_2^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{1809}{1736} & 1 & 0 & 0 & 0 & 0 \\ -\dfrac{264627}{253952} & 0 & 1 & 0 & 0 & 0 \\ -\dfrac{115775}{111104} & 0 & 0 & 1 & 0 & 0 \\ -\dfrac{264627}{253952} & 0 & 0 & 0 & 1 & 0 \\ -\dfrac{1809}{1736} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, J_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{73}{1736} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{4671}{111104} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{73}{1736} \end{pmatrix}$

and $E_2^{(1)} = \begin{pmatrix} \dfrac{248}{45} & -\dfrac{1517}{10} & \dfrac{19088}{45} & -\dfrac{21194}{15} & \dfrac{3632}{15} & -\dfrac{4333}{90} \\ 0 & \dfrac{1859461}{156240} & -\dfrac{288754}{9765} & \dfrac{404263}{13020} & -\dfrac{21502}{1395} & \dfrac{66763}{22320} \\ 0 & \dfrac{6090795}{507904} & -\dfrac{465975}{15872} & \dfrac{3932955}{126976} & -\dfrac{244215}{15872} & \dfrac{1516995}{507904} \\ 0 & \dfrac{2662935}{222208} & -\dfrac{202995}{6944} & \dfrac{1729095}{55552} & -\dfrac{15285}{992} & \dfrac{94905}{31744} \\ 0 & \dfrac{274022287}{22855680} & -\dfrac{20901419}{714240} & \dfrac{59597461}{1904640} & -\dfrac{10922219}{714240} & \dfrac{68201287}{22855680} \\ 0 & \dfrac{207957}{17360} & -\dfrac{31698}{1085} & \dfrac{135333}{4340} & -\dfrac{2334}{155} & \dfrac{7611}{2480} \end{pmatrix}$

$$\Phi(\phi) = \det\left(\phi J_2^{(1)} - J_1^{(1)}\right)$$
$$= \left|\phi J_2^{(1)} - J_1^{(1)}\right| = 0. \tag{14}$$

Now we have,

$$\Phi(\phi) = \left| \phi \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{1809}{1736} & 1 & 0 & 0 & 0 & 0 \\ -\dfrac{264627}{253952} & 0 & 1 & 0 & 0 & 0 \\ -\dfrac{115775}{111104} & 0 & 0 & 1 & 0 & 0 \\ -\dfrac{264627}{253952} & 0 & 0 & 0 & 1 & 0 \\ -\dfrac{1809}{1736} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{73}{1736} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{4671}{111104} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{73}{1736} \end{pmatrix} \right|$$

$$
= \left\| \begin{pmatrix} \phi & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{1809}{1736}\phi & \phi & 0 & 0 & 0 & 0 \\ -\dfrac{264627}{253952}\phi & 0 & \phi & 0 & 0 & 0 \\ -\dfrac{115775}{111104}\phi & 0 & 0 & \phi & 0 & 0 \\ -\dfrac{264627}{253952}\phi & 0 & 0 & 0 & \phi & 0 \\ -\dfrac{1809}{1736}\phi & 0 & 0 & 0 & 0 & \phi \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{73}{1736} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{4671}{111104} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{73}{1736} \end{pmatrix} \right\|
$$

$$
\Rightarrow \Phi(\phi) = \begin{pmatrix} \phi & 0 & 0 & 0 & 0 & 1 \\ -\dfrac{1809}{1736}\phi & \phi & 0 & 0 & 0 & -\dfrac{73}{1736} \\ -\dfrac{264627}{253952}\phi & 0 & \phi & 0 & 0 & -\dfrac{10675}{253952} \\ -\dfrac{115775}{111104}\phi & 0 & 0 & \phi & 0 & -\dfrac{4671}{111104} \\ -\dfrac{264627}{253952}\phi & 0 & 0 & 0 & \phi & -\dfrac{10675}{253952} \\ -\dfrac{1809}{1736}\phi & 0 & 0 & 0 & 0 & \phi-\dfrac{73}{1736} \end{pmatrix}.
$$

Using Maple (18) software, we obtain:

$$\Phi(\phi) = \phi^5\left(\phi+1\right)$$

$$\Rightarrow \phi^5\left(\phi+1\right) = 0$$

$$\Rightarrow \phi_1 = -1, \phi_2 = 0, \phi_3 = 0, \phi_4 = 0, \phi_5 = 0, \phi_6 = 0 . \text{Since } |\phi_i| < 1, \ i = 1,2,3,4,5,6 \text{, (9) is zero stable.}$$

CHIBUISI, OSU, IHEDIOHA, OLUNKWA, AKPANIBAH, UZOMA

Implementing the same approach, then (11) is presented as:

$$
\begin{pmatrix}
1 & -\dfrac{26199}{23048} & 0 & 0 & 0 & 0 \\[2mm]
-\dfrac{15072}{13879} & 1 & 0 & 0 & 0 & 0 \\[2mm]
\dfrac{1564731}{9373517} & -\dfrac{10793871}{9373517} & 1 & 0 & 0 & 0 \\[2mm]
\dfrac{24820803}{149976272} & -\dfrac{690053525}{599905088} & 0 & 1 & 0 & 0 \\[2mm]
\dfrac{24839325}{149976272} & -\dfrac{11041888275}{9598481408} & 0 & 0 & 1 & 0 \\[2mm]
\dfrac{1550656}{9373517} & -\dfrac{10781532}{9373517} & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_{v+1} \\ y_{v+2} \\ y_{v+3} \\ y_{v+\frac{7}{2}} \\ y_{v+\frac{15}{4}} \\ y_{v+4}
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \dfrac{3151}{23048} \\[2mm]
0 & 0 & 0 & 0 & 0 & \dfrac{1193}{13879} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{144377}{9373517} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{9134775}{599905088} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{146309933}{9598481408} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{142641}{9373517}
\end{pmatrix}
\begin{pmatrix}
y_{v-\frac{15}{4}} \\ y_{v-\frac{7}{2}} \\ y_{v-3} \\ y_{v-2} \\ y_{v-1} \\ y_{v}
\end{pmatrix}
$$

$$
+a
\begin{pmatrix}
-\dfrac{9373517}{11408760} & 0 & -\dfrac{696013}{207432} & \dfrac{428528}{43215} & -\dfrac{2804480}{285219} & \dfrac{292489}{103716} \\[2mm]
0 & \dfrac{9373517}{4371885} & -\dfrac{4075204}{624555} & \dfrac{10557376}{624555} & -\dfrac{70227968}{4371885} & \dfrac{927397}{208185} \\[2mm]
0 & 0 & \dfrac{23388822}{9373517} & -\dfrac{45109440}{9373517} & \dfrac{40722432}{9373517} & -\dfrac{10904274}{9373517} \\[2mm]
0 & 0 & \dfrac{402702825}{149976272} & -\dfrac{40561395}{9373517} & \dfrac{38661000}{9373517} & -\dfrac{335171025}{299952544} \\[2mm]
0 & 0 & \dfrac{12864892425}{4799240704} & -\dfrac{5038612425}{1199810176} & \dfrac{319538835}{74988136} & -\dfrac{676350675}{599905088} \\[2mm]
0 & 0 & \dfrac{25178016}{9373517} & -\dfrac{39817728}{9373517} & \dfrac{41828352}{9373517} & -\dfrac{9706980}{9373517}
\end{pmatrix}
\begin{pmatrix}
f_{v+1} \\ f_{v+2} \\ f_{v+3} \\ f_{v+\frac{7}{2}} \\ f_{v+\frac{15}{4}} \\ f_{v+4}
\end{pmatrix}
$$

$$
+a
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
f_{v-\frac{11}{4}} \\ f_{v-\frac{7}{2}} \\ f_{v-3} \\ f_{v-2} \\ f_{v-1} \\ f_{v}
\end{pmatrix}
$$

where

$$
J_2^{(2)} = \begin{pmatrix}
1 & -\dfrac{26199}{23048} & 0 & 0 & 0 & 0 \\[2mm]
-\dfrac{15072}{13879} & 1 & 0 & 0 & 0 & 0 \\[2mm]
\dfrac{1564731}{9373517} & -\dfrac{10793871}{9373517} & 1 & 0 & 0 & 0 \\[2mm]
\dfrac{24820803}{149976272} & -\dfrac{690053525}{599905088} & 0 & 1 & 0 & 0 \\[2mm]
\dfrac{24839325}{149976272} & -\dfrac{11041888275}{9598481408} & 0 & 0 & 1 & 0 \\[2mm]
\dfrac{1550656}{9373517} & -\dfrac{10781532}{9373517} & 0 & 0 & 0 & 1
\end{pmatrix},
J_1^{(2)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \dfrac{3151}{23048} \\[2mm]
0 & 0 & 0 & 0 & 0 & \dfrac{1193}{13879} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{144377}{9373517} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{9134775}{599905088} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{146309933}{9598481408} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{142641}{9373517}
\end{pmatrix}
$$

and

$$
E_2^{(2)} = \begin{pmatrix}
-\dfrac{9373517}{11408760} & 0 & -\dfrac{696013}{207432} & \dfrac{428528}{43215} & -\dfrac{2804480}{285219} & \dfrac{292489}{103716} \\[2mm]
0 & \dfrac{9373517}{4371885} & -\dfrac{4075204}{624555} & \dfrac{10557376}{624555} & -\dfrac{70227968}{4371885} & \dfrac{927397}{208185} \\[2mm]
0 & 0 & \dfrac{23388822}{9373517} & -\dfrac{45109440}{9373517} & \dfrac{40722432}{9373517} & -\dfrac{10904274}{9373517} \\[2mm]
0 & 0 & \dfrac{402702825}{149976272} & -\dfrac{40561395}{9373517} & \dfrac{38661000}{9373517} & -\dfrac{335171025}{299952544} \\[2mm]
0 & 0 & \dfrac{12864892425}{4799240704} & -\dfrac{5038612425}{1199810176} & \dfrac{319538835}{74988136} & -\dfrac{676350675}{599905088} \\[2mm]
0 & 0 & \dfrac{25178016}{9373517} & -\dfrac{39817728}{9373517} & \dfrac{41828352}{9373517} & -\dfrac{9706980}{9373517}
\end{pmatrix}
$$

$$
\Phi(\phi) = \det\left(\phi J_2^{(2)} - J_1^{(2)}\right)
$$
$$
= \left| \phi J_2^{(2)} - J_1^{(2)} \right| = 0.
$$
(15)

Now we have,

$$\Phi(\phi) = \left\| \phi \begin{pmatrix} 1 & -\dfrac{26199}{23048} & 0 & 0 & 0 & 0 \\[2mm] -\dfrac{15072}{13879} & 1 & 0 & 0 & 0 & 0 \\[2mm] \dfrac{1564731}{9373517} & -\dfrac{10793871}{9373517} & 1 & 0 & 0 & 0 \\[2mm] \dfrac{24820803}{149976272} & -\dfrac{690053525}{599905088} & 0 & 1 & 0 & 0 \\[2mm] \dfrac{24839325}{149976272} & -\dfrac{11041888275}{9598481408} & 0 & 0 & 1 & 0 \\[2mm] \dfrac{1550656}{9373517} & -\dfrac{10781532}{9373517} & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dfrac{3151}{23048} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1193}{13879} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{144377}{9373517} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{9134775}{599905088} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{146309933}{9598481408} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{142641}{9373517} \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} \phi & -\dfrac{26199}{23048}\phi & 0 & 0 & 0 & 0 \\[2mm] -\dfrac{15072}{13879}\phi & \phi & 0 & 0 & 0 & 0 \\[2mm] \dfrac{1564731}{9373517}\phi & -\dfrac{10793871}{9373517}\phi & \phi & 0 & 0 & 0 \\[2mm] \dfrac{24820803}{149976272}\phi & -\dfrac{690053525}{599905088}\phi & 0 & \phi & 0 & 0 \\[2mm] \dfrac{24839325}{149976272}\phi & -\dfrac{11041888275}{9598481408}\phi & 0 & 0 & \phi & 0 \\[2mm] \dfrac{1550656}{9373517}\phi & -\dfrac{10781532}{9373517}\phi & 0 & 0 & 0 & \phi \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dfrac{3151}{23048} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1193}{13879} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{144377}{9373517} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{9134775}{599905088} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{146309933}{9598481408} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{142641}{9373517} \end{pmatrix} \right\|$$

$$\Rightarrow \Phi(\phi) = \begin{pmatrix} \phi & -\dfrac{26199}{23048}\phi & 0 & 0 & 0 & -\dfrac{3151}{23048} \\[2mm] -\dfrac{15072}{13879}\phi & \phi & 0 & 0 & 0 & -\dfrac{1193}{13879} \\[2mm] \dfrac{1564731}{9373517}\phi & -\dfrac{10793871}{9373517}\phi & \phi & 0 & 0 & \dfrac{144377}{9373517} \\[2mm] \dfrac{24820803}{149976272}\phi & -\dfrac{690053525}{599905088}\phi & 0 & \phi & 0 & \dfrac{9134775}{599905088} \\[2mm] \dfrac{24839325}{149976272}\phi & -\dfrac{11041888275}{9598481408}\phi & 0 & 0 & \phi & \dfrac{146309933}{9598481408} \\[2mm] \dfrac{1550656}{9373517}\phi & -\dfrac{10781532}{9373517}\phi & 0 & 0 & 0 & \phi+\dfrac{142641}{9373517} \end{pmatrix}$$

Using Maple (18) software, we obtain:

$$\Phi(\phi) = -\frac{9373517}{39985399}\phi^5(\phi+1)$$

$$\Rightarrow -\frac{9373517}{39985399}\phi^5(\phi+1) = 0$$

$$\Rightarrow \phi_1 = -1, \phi_2 = 0, \phi_3 = 0, \phi_4 = 0, \phi_5 = 0, \phi_6 = 0. \text{Since } |\phi_i| < 1,\ i = 1,2,3,4,5,6,\ (11) \text{ is zero stable.}$$

With the same procedure (13) can be presented as follows:

$$
\begin{pmatrix}
1 & -\dfrac{2294}{1239} & \dfrac{3413}{4779} & 0 & 0 & 0 \\[2mm]
\dfrac{683}{719} & 1 & -\dfrac{36289}{19413} & 0 & 0 & 0 \\[2mm]
\dfrac{43929}{173263} & -\dfrac{1486053}{1212841} & 1 & 0 & 0 & 0 \\[2mm]
-\dfrac{1982}{17939} & \dfrac{55890}{125573} & -\dfrac{24178}{17939} & 1 & 0 & 0 \\[2mm]
-\dfrac{1937925}{18369536} & \dfrac{7890939}{18369536} & -\dfrac{24552675}{18369536} & 0 & 1 & 0 \\[2mm]
-\dfrac{2025}{17939} & \dfrac{56650}{125573} & -\dfrac{24250}{17939} & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y_{v+1} \\ y_{v+2} \\ y_{v+3} \\ y_{v+4} \\ y_{v+\frac{9}{2}} \\ y_{v+5}
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \dfrac{4594}{33453} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{1565}{19413} \\[2mm]
0 & 0 & 0 & 0 & 0 & -\dfrac{34291}{1212841} \\[2mm]
0 & 0 & 0 & 0 & 0 & \dfrac{1657}{125573} \\[2mm]
0 & 0 & 0 & 0 & 0 & \dfrac{230125}{18369536} \\[2mm]
0 & 0 & 0 & 0 & 0 & \dfrac{1702}{125573}
\end{pmatrix}
\begin{pmatrix}
y_{v-\frac{9}{2}} \\ y_{v-4} \\ y_{v-3} \\ y_{v-2} \\ y_{v-1} \\ y_v
\end{pmatrix}
$$

$$
+a
\begin{pmatrix}
-\dfrac{17939}{22302} & 0 & 0 & \dfrac{8284}{11151} & -\dfrac{128}{189} & \dfrac{1325}{7434} \\[2mm]
0 & -\dfrac{71756}{32355} & 0 & -\dfrac{2638}{2157} & \dfrac{33856}{32355} & -\dfrac{1706}{6471} \\[2mm]
0 & 0 & \dfrac{215268}{173263} & -\dfrac{1253502}{1212841} & -\dfrac{939840}{1212841} & -\dfrac{219294}{1212841} \\[2mm]
0 & 0 & 0 & \dfrac{152304}{125573} & -\dfrac{75648}{125573} & \dfrac{15804}{125573} \\[2mm]
0 & 0 & 0 & \dfrac{13480425}{9184768} & -\dfrac{94815}{287024} & \dfrac{978075}{9184768} \\[2mm]
0 & 0 & 0 & \dfrac{171900}{125573} & \dfrac{9600}{125573} & \dfrac{36240}{125573}
\end{pmatrix}
\begin{pmatrix}
f_{v+1} \\ f_{v+2} \\ f_{v+3} \\ f_{v+4} \\ f_{v+\frac{9}{2}} \\ f_{v+5}
\end{pmatrix}
$$

$$+a\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{v-\frac{9}{2}} \\ f_{v-4} \\ f_{v-3} \\ f_{v-2} \\ f_{v-1} \\ f_v \end{pmatrix}$$

where

$$J_2^{(3)} = \begin{pmatrix} 1 & -\dfrac{2294}{1239} & \dfrac{3413}{4779} & 0 & 0 & 0 \\[4pt] \dfrac{683}{719} & 1 & -\dfrac{36289}{19413} & 0 & 0 & 0 \\[4pt] \dfrac{43929}{173263} & -\dfrac{1486053}{1212841} & 1 & 0 & 0 & 0 \\[4pt] -\dfrac{1982}{17939} & \dfrac{55890}{125573} & -\dfrac{24178}{17939} & 1 & 0 & 0 \\[4pt] -\dfrac{1937925}{18369536} & \dfrac{7890939}{18369536} & -\dfrac{24552675}{18369536} & 0 & 1 & 0 \\[4pt] -\dfrac{2025}{17939} & \dfrac{56650}{125573} & -\dfrac{24250}{17939} & 0 & 0 & 1 \end{pmatrix}, \quad J_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dfrac{4594}{33453} \\[4pt] 0 & 0 & 0 & 0 & 0 & -\dfrac{1565}{19413} \\[4pt] 0 & 0 & 0 & 0 & 0 & -\dfrac{34291}{1212841} \\[4pt] 0 & 0 & 0 & 0 & 0 & \dfrac{1657}{125573} \\[4pt] 0 & 0 & 0 & 0 & 0 & \dfrac{230125}{18369536} \\[4pt] 0 & 0 & 0 & 0 & 0 & \dfrac{1702}{125573} \end{pmatrix}$$

and $E_2^{(3)} = \begin{pmatrix} -\dfrac{17939}{22302} & 0 & 0 & \dfrac{8284}{11151} & -\dfrac{128}{189} & \dfrac{1325}{7434} \\[4pt] 0 & -\dfrac{71756}{32355} & 0 & -\dfrac{2638}{2157} & \dfrac{33856}{32355} & -\dfrac{1706}{6471} \\[4pt] 0 & 0 & \dfrac{215268}{173263} & -\dfrac{1253502}{1212841} & -\dfrac{939840}{1212841} & -\dfrac{219294}{1212841} \\[4pt] 0 & 0 & 0 & \dfrac{152304}{125573} & -\dfrac{75648}{125573} & \dfrac{15804}{125573} \\[4pt] 0 & 0 & 0 & \dfrac{13480425}{9184768} & -\dfrac{94815}{287024} & \dfrac{978075}{9184768} \\[4pt] 0 & 0 & 0 & \dfrac{171900}{125573} & \dfrac{9600}{125573} & \dfrac{36240}{125573} \end{pmatrix}$

$$\Phi(\phi) = \det\left(\phi J_2^{(3)} - J_1^{(3)}\right)$$

$$= \left|\phi J_2^{(3)} - J_1^{(3)}\right| = 0. \tag{16}$$

Now we have,

$$\Phi(\phi) = \left| \phi \begin{pmatrix} 1 & -\dfrac{2294}{1239} & \dfrac{3413}{4779} & 0 & 0 & 0 \\[2mm] \dfrac{683}{719} & 1 & -\dfrac{36289}{19413} & 0 & 0 & 0 \\[2mm] \dfrac{43929}{173263} & -\dfrac{1486053}{1212841} & 1 & 0 & 0 & 0 \\[2mm] -\dfrac{1982}{17939} & \dfrac{55890}{125573} & -\dfrac{24178}{17939} & 1 & 0 & 0 \\[2mm] -\dfrac{1937925}{18369536} & \dfrac{7890939}{18369536} & -\dfrac{24552675}{18369536} & 0 & 1 & 0 \\[2mm] -\dfrac{2025}{17939} & \dfrac{56650}{125573} & -\dfrac{24250}{17939} & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dfrac{4594}{33453} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{1565}{19413} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{34291}{1212841} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1657}{125573} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{230125}{18369536} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1702}{125573} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} \phi & -\dfrac{2294}{1239}\phi & \dfrac{3413}{4779}\phi & 0 & 0 & 0 \\[2mm] \dfrac{683}{719}\phi & \phi & -\dfrac{36289}{19413}\phi & 0 & 0 & 0 \\[2mm] \dfrac{43929}{173263}\phi & -\dfrac{1486053}{1212841}\phi & \phi & 0 & 0 & 0 \\[2mm] -\dfrac{1982}{17939}\phi & \dfrac{55890}{125573}\phi & -\dfrac{24178}{17939}\phi & \phi & 0 & 0 \\[2mm] -\dfrac{1937925}{18369536}\phi & \dfrac{7890939}{18369536}\phi & -\dfrac{24552675}{18369536}\phi & 0 & \phi & 0 \\[2mm] -\dfrac{2025}{17939}\phi & \dfrac{56650}{125573}\phi & -\dfrac{24250}{17939}\phi & 0 & 0 & \phi \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dfrac{4594}{33453} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{1565}{19413} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{34291}{1212841} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1657}{125573} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{230125}{18369536} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1702}{125573} \end{pmatrix} \right|$$

CHIBUISI, OSU, IHEDIOHA, OLUNKWA, AKPANIBAH, UZOMA

$$\Rightarrow \Phi(\phi) = \begin{pmatrix} \phi & -\dfrac{2294}{1239}\phi & \dfrac{3413}{4779}\phi & 0 & 0 & -\dfrac{4594}{33453} \\[2mm] \dfrac{683}{719}\phi & \phi & -\dfrac{36289}{19413}\phi & 0 & 0 & \dfrac{1565}{19413} \\[2mm] \dfrac{43929}{173263}\phi & -\dfrac{1486053}{1212841}\phi & \phi & 0 & 0 & \dfrac{34291}{1212841} \\[2mm] -\dfrac{1982}{17939}\phi & \dfrac{55890}{125573}\phi & -\dfrac{24178}{17939}\phi & \phi & 0 & -\dfrac{1657}{125573} \\[2mm] -\dfrac{1937925}{18369536}\phi & \dfrac{7890939}{18369536}\phi & -\dfrac{24552675}{18369536}\phi & 0 & \phi & -\dfrac{230125}{18369536} \\[2mm] -\dfrac{2025}{17939}\phi & \dfrac{56650}{125573}\phi & -\dfrac{24250}{17939}\phi & 0 & 0 & \phi-\dfrac{1702}{125573} \end{pmatrix}$$

Using Maple (18) software, we obtain:

$$\Phi(\phi) = \frac{51489235360}{154349784183}\phi^5(\phi+1)$$

$$\Rightarrow \frac{51489235360}{154349784183}\phi^5(\phi+1) = 0$$

$$\Rightarrow \phi_1 = -1, \phi_2 = 0, \phi_3 = 0, \phi_4 = 0, \phi_5 = 0, \phi_6 = 0 . \text{Since } |\phi_i| < 1, \ i = 1,2,3,4,5,6 \text{, (13) is zero stable.}$$

3.4 Convergence

Since (9), (11) and (13) are both consistent and zero stable, therefore they are convergent.

3.5 Region of Absolute Stability

The regions of absolute stability of the numerical methods for DDEs are considered. We considered finding the U - and W -stability by applying (9), (11) and (13) to the DDEs of this form:

$$\begin{aligned} x'(t) &= \rho x(\mathrm{t}) + \delta x(t-\tau), \, t \geq t_0 \\ x(t) &= \alpha(t), \qquad\qquad t \leq t_0 \end{aligned} \tag{17}$$

where $\alpha(t)$ is the initial function, ρ, δ are complex coefficients, $\tau = va, v \in \mathbb{Z}^+, a$ is the step size

and $v = \dfrac{\tau}{a}, v$ is a positive integer. Let $L_1 = a\rho$ and $L_2 = a\delta$, then from (9), we have;

$$Y_{V+6} = \begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+\frac{9}{4}} \\ y_{v+\frac{5}{2}} \\ y_{v+\frac{11}{4}} \\ y_{v+3} \end{pmatrix}, Y_V = \begin{pmatrix} y_{v-\frac{11}{4}} \\ y_{v-\frac{5}{2}} \\ y_{v-\frac{9}{4}} \\ y_{v-2} \\ y_{v-1} \\ y_v \end{pmatrix}, F_{V+6} = \begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+\frac{9}{4}} \\ f_{v+\frac{5}{2}} \\ f_{v+\frac{11}{4}} \\ f_{v+3} \end{pmatrix} \text{ and } F_V = \begin{pmatrix} f_{v-\frac{11}{4}} \\ f_{v-\frac{5}{2}} \\ f_{v-\frac{9}{4}} \\ f_{v-2} \\ f_{v-1} \\ f_v \end{pmatrix}$$

Since $J_2^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1809}{1736} & 1 & 0 & 0 & 0 & 0 \\ -\frac{264627}{253952} & 0 & 1 & 0 & 0 & 0 \\ -\frac{115775}{111104} & 0 & 0 & 1 & 0 & 0 \\ -\frac{264627}{253952} & 0 & 0 & 0 & 1 & 0 \\ -\frac{1809}{1736} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, J_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{73}{1736} \\ 0 & 0 & 0 & 0 & 0 & \frac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \frac{4671}{111104} \\ 0 & 0 & 0 & 0 & 0 & \frac{10675}{253952} \\ 0 & 0 & 0 & 0 & 0 & \frac{73}{1736} \end{pmatrix}$

and $E_2^{(1)} = \begin{pmatrix} \frac{248}{45} & -\frac{1517}{10} & \frac{19088}{45} & -\frac{21194}{15} & \frac{3632}{15} & -\frac{4333}{90} \\ 0 & \frac{1859461}{156240} & -\frac{288754}{9765} & \frac{404263}{13020} & -\frac{21502}{1395} & \frac{66763}{22320} \\ 0 & \frac{6090795}{507904} & -\frac{465975}{15872} & \frac{3932955}{126976} & -\frac{244215}{15872} & \frac{1516995}{507904} \\ 0 & \frac{2662935}{222208} & -\frac{202995}{6944} & \frac{1729095}{55552} & -\frac{15285}{992} & \frac{94905}{31744} \\ 0 & \frac{274022287}{22855680} & -\frac{20901419}{714240} & \frac{59597461}{1904640} & -\frac{10922219}{714240} & \frac{68201287}{22855680} \\ 0 & \frac{207957}{17360} & -\frac{31698}{1085} & \frac{135333}{4340} & -\frac{2334}{155} & \frac{7611}{2480} \end{pmatrix}$

we have,

$$J_2^{(1)} Y_{V+2} = J_1^{(1)} Y_{V+1} - a \sum_{i=1}^{2} E_i^{(1)} F_{V+i} \tag{18}$$

With the same technique for (11), we have

$$Y_{V+6} = \begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \\ y_{v+\frac{7}{2}} \\ y_{v+\frac{15}{4}} \\ y_{v+4} \end{pmatrix}, Y_V = \begin{pmatrix} y_{v-\frac{15}{4}} \\ y_{v-\frac{7}{2}} \\ y_{v-3} \\ y_{v-2} \\ y_{v-1} \\ y_v \end{pmatrix}, F_{V+6} = \begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \\ f_{v+\frac{7}{2}} \\ f_{v+\frac{15}{4}} \\ f_{v+4} \end{pmatrix} \text{ and } F_V = \begin{pmatrix} f_{v-\frac{15}{4}} \\ f_{v-\frac{7}{2}} \\ f_{v-3} \\ f_{v-2} \\ f_{v-1} \\ f_v \end{pmatrix}$$

Since

$$J_2^{(2)} = \begin{pmatrix} 1 & -\dfrac{26199}{23048} & 0 & 0 & 0 & 0 \\[2mm] -\dfrac{15072}{13879} & 1 & 0 & 0 & 0 & 0 \\[2mm] \dfrac{1564731}{9373517} & -\dfrac{10793871}{9373517} & 1 & 0 & 0 & 0 \\[2mm] \dfrac{24820803}{149976272} & -\dfrac{690053525}{599905088} & 0 & 1 & 0 & 0 \\[2mm] \dfrac{24839325}{149976272} & -\dfrac{11041888275}{9598481408} & 0 & 0 & 1 & 0 \\[2mm] \dfrac{1550656}{9373517} & -\dfrac{10781532}{9373517} & 0 & 0 & 0 & 1 \end{pmatrix}, J_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dfrac{3151}{23048} \\[2mm] 0 & 0 & 0 & 0 & 0 & \dfrac{1193}{13879} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{144377}{9373517} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{9134775}{599905088} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{146309933}{9598481408} \\[2mm] 0 & 0 & 0 & 0 & 0 & -\dfrac{142641}{9373517} \end{pmatrix}$$

and

$$E_2^{(2)} = \begin{pmatrix} -\dfrac{9373517}{11408760} & 0 & -\dfrac{696013}{207432} & \dfrac{428528}{43215} & -\dfrac{2804480}{285219} & \dfrac{292489}{103716} \\[2mm] 0 & \dfrac{9373517}{4371885} & -\dfrac{4075204}{624555} & \dfrac{10557376}{624555} & -\dfrac{70227968}{4371885} & \dfrac{927397}{208185} \\[2mm] 0 & 0 & \dfrac{23388822}{9373517} & -\dfrac{45109440}{9373517} & \dfrac{40722432}{9373517} & \dfrac{10904274}{9373517} \\[2mm] 0 & 0 & \dfrac{402702825}{149976272} & -\dfrac{40561395}{9373517} & \dfrac{38661000}{9373517} & -\dfrac{335171025}{299952544} \\[2mm] 0 & 0 & \dfrac{12864892425}{4799240704} & -\dfrac{5038612425}{1199810176} & \dfrac{319538835}{74988136} & -\dfrac{676350675}{599905088} \\[2mm] 0 & 0 & \dfrac{25178016}{9373517} & -\dfrac{39817728}{9373517} & \dfrac{41828352}{9373517} & -\dfrac{9706980}{9373517} \end{pmatrix}$$

we have,

$$J_2^{(2)} Y_{V+2} = J_1^{(2)} Y_{V+1} - a \sum_{i=1}^{2} E_i^{(2)} F_{V+i} \tag{19}$$

Applying the same approach for (13), we have

$$Y_{V+6} = \begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \\ y_{v+4} \\ y_{v+\frac{9}{2}} \\ y_{v+5} \end{pmatrix}, Y_V = \begin{pmatrix} y_{v-\frac{9}{2}} \\ y_{v-4} \\ y_{v-3} \\ y_{v-2} \\ y_{v-1} \\ y_v \end{pmatrix}, F_{V+6} = \begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \\ f_{v+4} \\ f_{v+\frac{9}{2}} \\ f_{v+5} \end{pmatrix} \text{ and } F_V = \begin{pmatrix} f_{v-\frac{9}{2}} \\ f_{v-4} \\ f_{v-3} \\ f_{v-2} \\ f_{v-1} \\ f_v \end{pmatrix}$$

Since

$$J_2^{(3)} = \begin{pmatrix} 1 & -\dfrac{2294}{1239} & \dfrac{3413}{4779} & 0 & 0 & 0 \\ \dfrac{683}{719} & 1 & -\dfrac{36289}{19413} & 0 & 0 & 0 \\ \dfrac{43929}{173263} & -\dfrac{1486053}{1212841} & 1 & 0 & 0 & 0 \\ -\dfrac{1982}{17939} & \dfrac{55890}{125573} & -\dfrac{24178}{17939} & 1 & 0 & 0 \\ -\dfrac{1937925}{18369536} & \dfrac{7890939}{18369536} & -\dfrac{24552675}{18369536} & 0 & 1 & 0 \\ -\dfrac{2025}{17939} & \dfrac{56650}{125573} & -\dfrac{24250}{17939} & 0 & 0 & 1 \end{pmatrix}, J_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dfrac{4594}{33453} \\ 0 & 0 & 0 & 0 & 0 & -\dfrac{1565}{19413} \\ 0 & 0 & 0 & 0 & 0 & -\dfrac{34291}{1212841} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1657}{125573} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{230125}{18369536} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1702}{125573} \end{pmatrix}$$

and $E_2^{(3)} = \begin{pmatrix} -\dfrac{17939}{22302} & 0 & 0 & \dfrac{8284}{11151} & -\dfrac{128}{189} & \dfrac{1325}{7434} \\ 0 & -\dfrac{71756}{32355} & 0 & -\dfrac{2638}{2157} & \dfrac{33856}{32355} & -\dfrac{1706}{6471} \\ 0 & 0 & \dfrac{215268}{173263} & -\dfrac{1253502}{1212841} & -\dfrac{939840}{1212841} & -\dfrac{219294}{1212841} \\ 0 & 0 & 0 & \dfrac{152304}{125573} & -\dfrac{75648}{125573} & \dfrac{15804}{125573} \\ 0 & 0 & 0 & \dfrac{13480425}{9184768} & -\dfrac{94815}{287024} & \dfrac{978075}{9184768} \\ 0 & 0 & 0 & \dfrac{171900}{125573} & \dfrac{9600}{125573} & \dfrac{36240}{125573} \end{pmatrix}$

we have,

$$J_2^{(3)} Y_{V+2} = J_1^{(3)} Y_{V+1} - a \sum_{i=1}^{2} E_i^{(3)} F_{V+i} \qquad (20)$$

The polynomials of V - and W -stability are constructed by introducing (18), (19) and (20) to (17) and (9), (11) and (13) to (17) as stated below START

$$\varpi^{(1)}(\varepsilon) = \det\left[\left(J_2^{(1)} - L_1 E_2^{(1)}\right) \varepsilon^{2+r} - \left(J_1^{(1)} - L_1 E_1^{(1)}\right) \varepsilon^{1+r} - L_2 \sum_{i=1}^{2} E_i^{(1)} \varepsilon^i \right] \qquad (21)$$

$$\varpi^{(2)}(\varepsilon) = \det\left[\left(J_2^{(2)} - L_1 E_2^{(2)}\right) \varepsilon^{2+r} - \left(J_1^{(2)} - L_1 E_1^{(2)}\right) \varepsilon^{1+r} - L_2 \sum_{i=1}^{2} E_i^{(2)} \varepsilon^i \right] \qquad (22)$$

$$\varpi^{(3)}(\varepsilon) = \det\left[\left(J_2^{(3)} - L_1 E_2^{(3)}\right) \varepsilon^{2+r} - \left(J_1^{(3)} - L_1 E_1^{(3)}\right) \varepsilon^{1+r} - L_2 \sum_{i=1}^{2} E_i^{(3)} \varepsilon^i \right] \qquad (23)$$

and

$$\Omega^{(1)}(\varepsilon) = \det\left[J_2^{(1)} \varepsilon^{2+r} - J_1^{(1)} \varepsilon^{1+r} - L_2 \sum_{i=1}^{2} E_i^{(1)} \varepsilon^i \right] \qquad (24)$$

$$\Omega^{(2)}(\varepsilon) = \det\left[J_2^{(2)} \varepsilon^{2+r} - J_1^{(2)} \varepsilon^{1+r} - L_2 \sum_{i=1}^{2} E_i^{(2)} \varepsilon^i \right] \qquad (25)$$

$$\Omega^{(3)}(\varepsilon) = \det\left[J_2^{(3)} \varepsilon^{2+r} - J_1^{(3)} \varepsilon^{1+r} - L_2 \sum_{i=1}^{2} E_i^{(3)} \varepsilon^i \right] \qquad (26)$$

Making use of Maple 18 and MATLAB, the region of U - and W -stability for (9), (11) and (13) are shown in Fig.1 to 6.
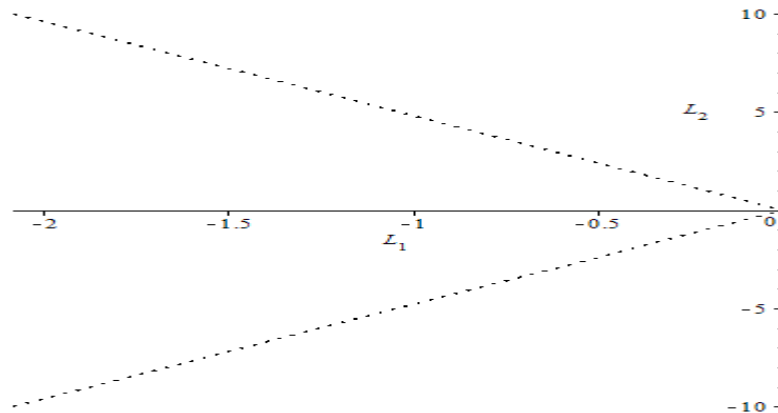

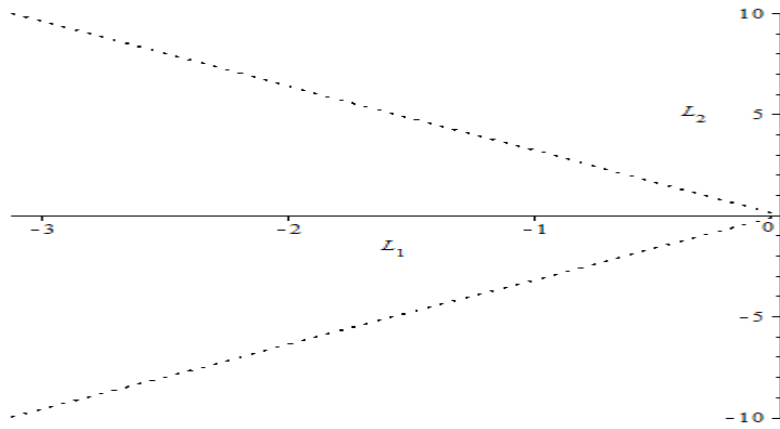Fig.1.Region of U -stability (HEBBDFM) in (9)

Fig.2.Region of U-stability (HEBBDFM) in (11)

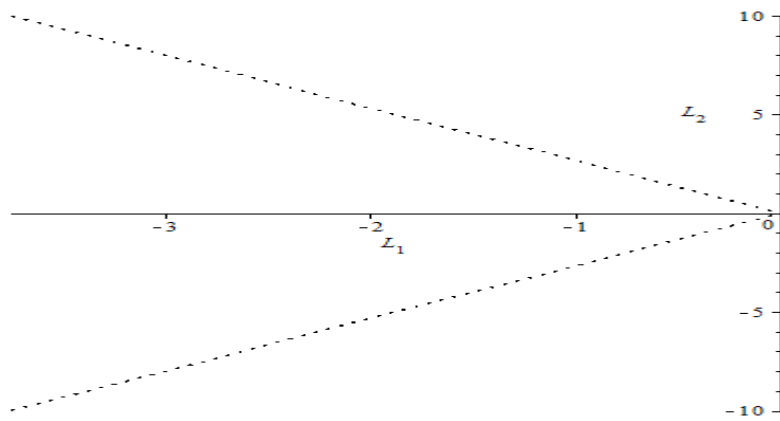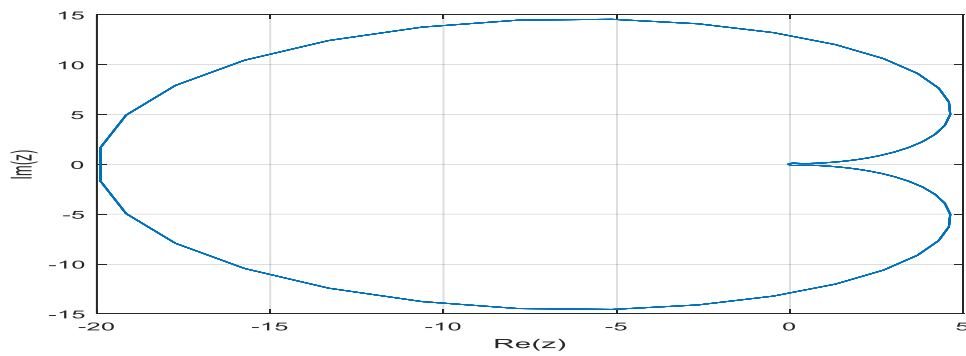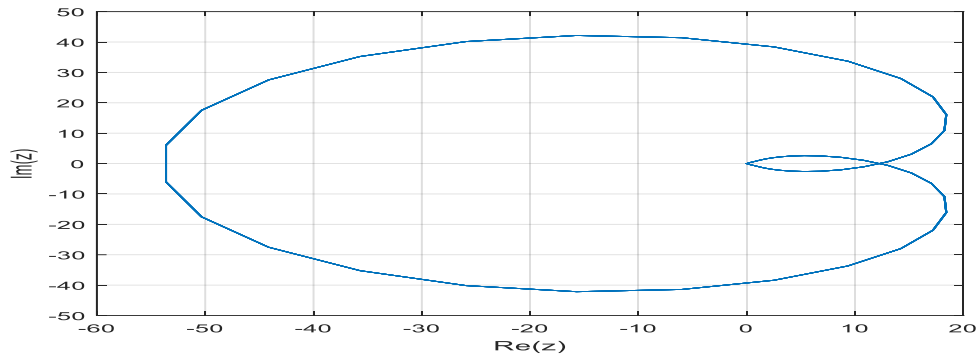Fig.3.Region of U-stability (HEBBDFM) in (13)

Fig.4. Region of W-stability (HEBBDFM) in (9)

Fig.5. Region of W -stability (HEBBDFM) in (11)
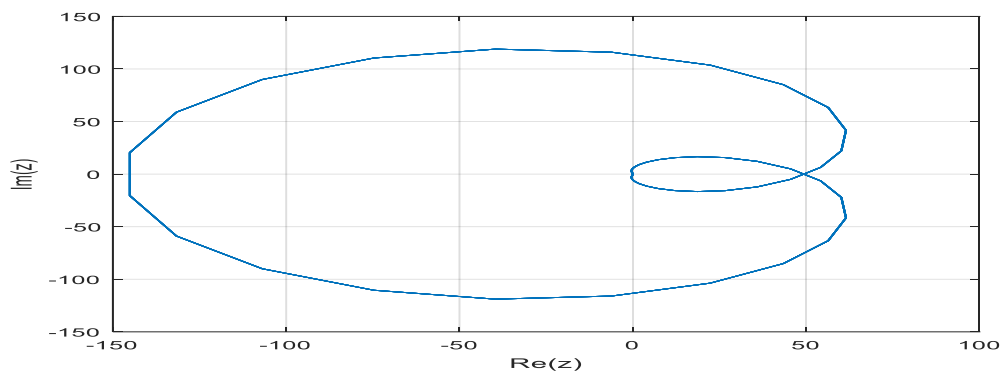
Fig.6. Region of W -stability (HEBBDFM) in (13)

The U -stability regions in Figs 1 to 3 lie inside the open-ended region while the W -stability regions in Figs 4 to 6 lie inside the enclosed region.

4. NUMERICAL COMPUTATIONS

In this section, some first-order delay differential equations shall be solved using (9), (11) and (13) of the discrete schemes been constructed. The delay argument shall be evaluated using the idea of sequence developed by Sirisena et al. (2019).

4.1 Implementation of Numerical Problems

Problem 1

$$x^{'}(t) = -1000x(t) + x(t - (\ln(1000 - 1))), \ 0 \le t \le 3$$

$$x(t) = e^{-t}, t \le 0$$

Exact solution $x(t) = e^{-t}$

Problem 2

$$x'(t) = -1000x(t) + 997e^{-3}x(t-1) + (1000 - 997e^{-3}), \ 0 \leq t \leq 3$$

$$x(t) = 1 + e^{-3t}, t \leq 0$$

Exact solution $x(t) = 1 + e^{-3t}$

Table 4.1.1: Numerical Solution of HEBBDF Method with Integrated Off-grid Extended Future Points and One Extended Future Point for Problem 1

t	Exact Solution	k = 2 Computational Solution	k = 3 Computational Solution	k = 4 Computational Solution
0.01	0.990049834	0.99005027	0.990049781	0.990049833
0.02	0.980198673	0.980198556	0.980198692	0.980198674
0.03	0.970445534	0.970445443	0.970445525	0.970445533
0.04	0.960789439	0.960788262	0.960789427	0.960789441
0.05	0.951229425	0.951229548	0.951229458	0.951229421
0.06	0.941764534	0.941764395	0.941764521	0.941764536
0.07	0.93239382	0.932394212	0.932393827	0.932393818
0.08	0.923116346	0.923116285	0.923116342	0.923116347
0.09	0.913931185	0.913931228	0.913931178	0.913931185
0.1	0.904837418	0.90483857	0.904837424	0.90483742
0.11	0.895834135	0.895834011	0.895834135	0.895834134
0.12	0.886920437	0.886920445	0.88692043	0.886920437
0.13	0.878095431	0.878098402	0.878095404	0.878095431
0.14	0.869358235	0.869357868	0.869358248	0.869358236
0.15	0.860707976	0.860708091	0.860707972	0.860707975
0.16	0.852143789	0.852143413	0.852143785	0.85214379
0.17	0.843664817	0.843664858	0.843664803	0.843664817
0.18	0.835270211	0.835270198	0.83527022	0.835270212
0.19	0.826959134	0.82695957	0.826959131	0.826959134
0.2	0.818730753	0.818730689	0.818730761	0.818730756
0.21	0.810584246	0.81058418	0.810584233	0.810584246
0.22	0.802518798	0.802517968	0.802518805	0.802518798
0.23	0.794533603	0.794533712	0.7945336	0.794533602
0.24	0.786627861	0.786627844	0.786627867	0.786627861
0.25	0.778800783	0.778801357	0.778800767	0.77880078
0.26	0.771051586	0.771051512	0.771051587	0.771051587
0.27	0.763379494	0.763379506	0.763379491	0.763379494
0.28	0.755783741	0.755783528	0.755783739	0.755783741
0.29	0.748263568	0.748263592	0.74826351	0.748263568
0.3	0.740818221	0.740818177	0.740818238	0.740818223

Table 4.1.2: Numerical Solution of HEBBDF Method with Integrated Off-grid Extended Future Points and One Extended Future Point for Problem 2

T	Exact Solution	k = 2 Computational Solution	k = 3 Computational Solution	k = 4 Computational Solution
0.01	1.970445534	1.970448149	1.970445527	1.970445534
0.02	1.941764534	1.941764203	1.941764534	1.941764533
0.03	1.913931185	1.913931359	1.913931187	1.913931184
0.04	1.886920437	1.886922874	1.88692042	1.886920436
0.05	1.860707976	1.860707673	1.860707969	1.860707979
0.06	1.835270211	1.835270327	1.835270211	1.835270216
0.07	1.810584246	1.810587578	1.810584245	1.810584244
0.08	1.786627861	1.786627445	1.786627853	1.786627861
0.09	1.763379494	1.763379634	1.763379448	1.763379492
0.1	1.740818221	1.74082047	1.740818235	1.740818229
0.11	1.718923733	1.718923448	1.718923725	1.718923731
0.12	1.697676326	1.697676407	1.697676323	1.697676327
0.13	1.677056874	1.677059151	1.677056776	1.677056872
0.14	1.65704682	1.657046528	1.657046854	1.657046821
0.15	1.637628152	1.637628263	1.637628136	1.63762815
0.16	1.618783392	1.618785777	1.61878337	1.618783396
0.17	1.600495579	1.600495256	1.600495497	1.600495577
0.18	1.582748252	1.582748366	1.58274828	1.582748252
0.19	1.565525439	1.565527012	1.565525423	1.565525438
0.2	1.548811636	1.548811434	1.548811622	1.548811642
0.21	1.532591801	1.53259188	1.53259175	1.532591799
0.22	1.516851334	1.516853106	1.516851348	1.516851334
0.23	1.501576069	1.501575851	1.501576061	1.501576069
0.24	1.486752256	1.486752409	1.486752254	1.486752257
0.25	1.472366553	1.472367915	1.472366548	1.472366551
0.26	1.458406011	1.458405815	1.458406012	1.458406012
0.27	1.444858066	1.444858137	1.444858066	1.444858066
0.28	1.431710523	1.431712328	1.431710522	1.431710524
0.29	1.418951549	1.418951321	1.418951511	1.41895155
0.3	1.40656966	1.406569696	1.406569675	1.406569662

5. RESULTS AND DISCUSSIONS

Here, the solutions of the schemes derived in (9), (11) and (13), shall be examined in solving the two problems above by computing their absolute errors.

5.1 Analysis of Results

The analysis of results is obtained by determining absolute differences of the exact solutions and the numerical solutions. The results are presented in the tables 5.1.1 to 5.1.2,

Table 5.1.1: Absolute Error of HEBBDF Method with Integrated Off-grid Extended Future Points and One Extended Future Point for Problem 1

T	k = 2 Error	k = 3 Error	k = 4 Error
0.01	4.36051E-07	5.27492E-08	8.49168E-10
0.02	1.17007E-07	1.83932E-08	5.93245E-10
0.03	9.08485E-08	9.04851E-09	8.48508E-10
0.04	1.17705E-06	1.23523E-08	2.24768E-09
0.05	1.23099E-07	3.35993E-08	3.90071E-09
0.06	1.38184E-07	1.28842E-08	2.41575E-09
0.07	3.91694E-07	7.19405E-09	1.70595E-09
0.08	6.11866E-08	3.98664E-09	4.13364E-10
0.09	4.24288E-08	6.87123E-09	6.71228E-10
0.1	1.15176E-06	6.26404E-09	1.96404E-09
0.11	1.23997E-07	3.47178E-12	8.96528E-10
0.12	8.18284E-09	7.11716E-09	5.82843E-10
0.13	2.97138E-06	2.69206E-08	2.79439E-10
0.14	3.67599E-07	1.25012E-08	4.01194E-10
0.15	1.14775E-07	4.42506E-09	1.72506E-09
0.16	3.75966E-07	3.56621E-09	1.03379E-09
0.17	4.09036E-08	1.33964E-08	3.03616E-10
0.18	1.31113E-08	8.58873E-09	3.88728E-10
0.19	4.36257E-07	2.84336E-09	4.43362E-10
0.2	6.4078E-08	7.92202E-09	2.42202E-09
0.21	6.60702E-08	1.32702E-08	7.01871E-11
0.22	8.30062E-07	7.43752E-09	4.37521E-10
0.23	1.09797E-07	2.20333E-09	4.03334E-10
0.24	1.72666E-08	5.73345E-09	3.33446E-10
0.25	5.73629E-07	1.62714E-08	3.0714E-09
0.26	7.38036E-08	7.96434E-10	7.96434E-10
0.27	1.18631E-08	3.23685E-09	6.31468E-11
0.28	2.13756E-07	2.15573E-09	4.55725E-10
0.29	2.47214E-08	5.73786E-08	2.21435E-10
0.3	4.39817E-08	1.68183E-08	1.91828E-09

Table 5.1.2: Absolute Error of HEBBDF Method with Integrated Off-grid Extended Future Points and One Extended Future Point for Problem 2

T	k = 2 Error	k = 3 Error	k = 4 Error
0.01	2.61545E-06	6.54851E-09	4.51492E-10
0.02	3.30584E-07	4.15751E-10	5.84249E-10
0.03	1.73729E-07	1.72877E-09	1.27123E-09
0.04	2.43728E-06	1.67172E-08	7.17157E-10
0.05	3.03425E-07	7.42506E-09	2.57494E-09
0.06	1.15589E-07	4.11272E-10	4.58873E-09
0.07	3.33203E-06	9.70187E-10	1.97019E-09
0.08	4.16067E-07	8.06655E-09	6.65534E-11
0.09	1.39663E-07	4.63369E-08	2.33685E-09
0.1	2.24932E-06	1.43183E-08	8.31828E-09
0.11	2.85432E-07	8.43193E-09	2.43193E-09
0.12	8.0929E-08	3.07103E-09	9.28969E-10
0.13	2.2765E-06	9.84982E-08	2.49816E-09
0.14	2.91815E-07	3.41849E-08	1.18494E-09
0.15	1.11378E-07	1.56218E-08	1.62177E-09
0.16	2.38519E-06	2.18061E-08	4.19386E-09
0.17	3.22812E-07	8.18123E-08	1.81227E-09
0.18	1.13626E-07	2.7626E-08	3.7399E-10
0.19	1.5733E-06	1.56995E-08	6.99537E-10
0.2	2.02094E-07	1.4094E-08	5.90597E-09
0.21	7.89931E-08	5.10069E-08	2.0069E-09
0.22	1.77151E-06	1.35083E-08	4.91699E-10
0.23	2.18066E-07	8.06606E-09	6.60556E-11
0.24	1.5304E-07	1.95997E-09	1.04003E-09
0.25	1.36226E-06	4.74101E-09	1.74101E-09
0.26	1.96305E-07	6.94776E-10	6.94776E-10
0.27	7.07771E-08	2.22941E-10	2.22941E-10
0.28	1.80457E-06	1.42908E-09	5.7092E-10
0.29	2.28248E-07	3.82476E-08	7.52361E-10
0.3	3.62594E-08	1.52594E-08	2.2594E-09

The notations used in the table below are stated as;

HEBBDFM = Hybrid Extended Block Backward Differentiation Formulae Methods for step numbers $k = 2, 3$ and 4.

EBBDFM = Extended Block Backward Differentiation Formulae Methods for step numbers

$k = 2,3$ and 4.

MAXE = Maximum Error.

Table 5.1.3: Comparison Between the Maximum Absolute Errors of HEBBDFM and the EBBDF been studied by Chibuisi et al. (2020) on Numerical Solutions of first order DDEs for constant step size h = 0.01 for step numbers $k = 2,3$ and 4 using Problem 1

Numerical Method	MAXE
HEBBDFM MAXE for k = 2	2.97E-06
HEBBDFM MAXE for k = 3	5.74E-08
HEBBDFM MAXE for k = 4	3.90E-09
EBBDFM MAXE for k = 2	2.72E-09
EBBDFM MAXE for k = 3	1.61E-09
EBBDFM MAXE for k = 4	1.96E-09

Table 5.1.4: Comparison Between the Maximum Absolute Errors of HEBBDFM and the EBBDF been studied by Chibuisi et al. (2020) on Numerical Solutions of first order DDEs for constant step size h = 0.01 for step numbers $k = 2,3$ and 4 using Problem 2

Numerical Method	MAXE
HEBBDFM MAXE for k = 2	3.33E-06
HEBBDFM MAXE for k = 3	9.85E-08
HEBBDFM MAXE for k = 4	8.32E-09
EBBDFM MAXE for k = 2	7.55E-09
EBBDFM MAXE for k = 3	4.34E-09
EBBDFM MAXE for k = 4	5.42E-09

Using Microsoft Excel, the MAXE for HEBBDFM and EBBDFM for Problem 1 and 2 are displayed as;
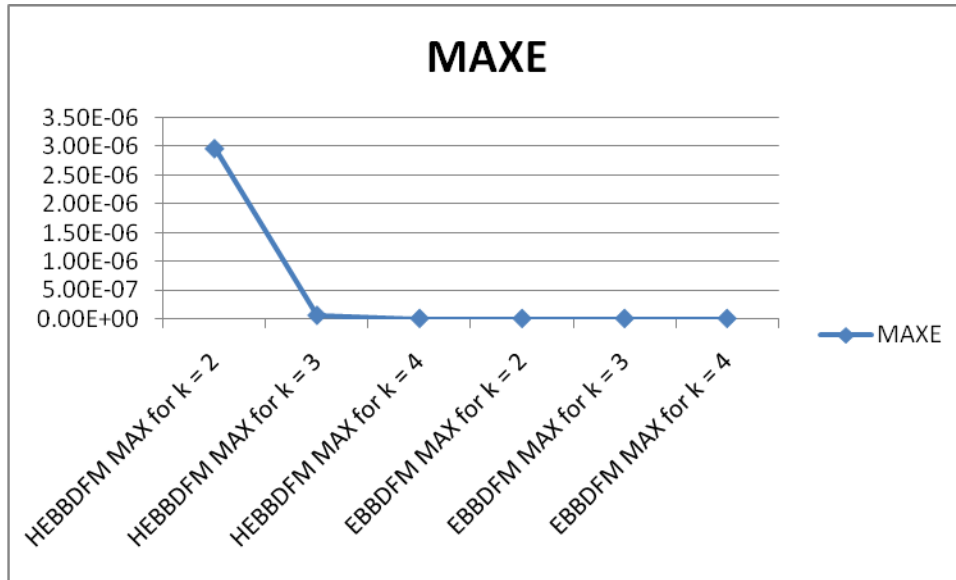
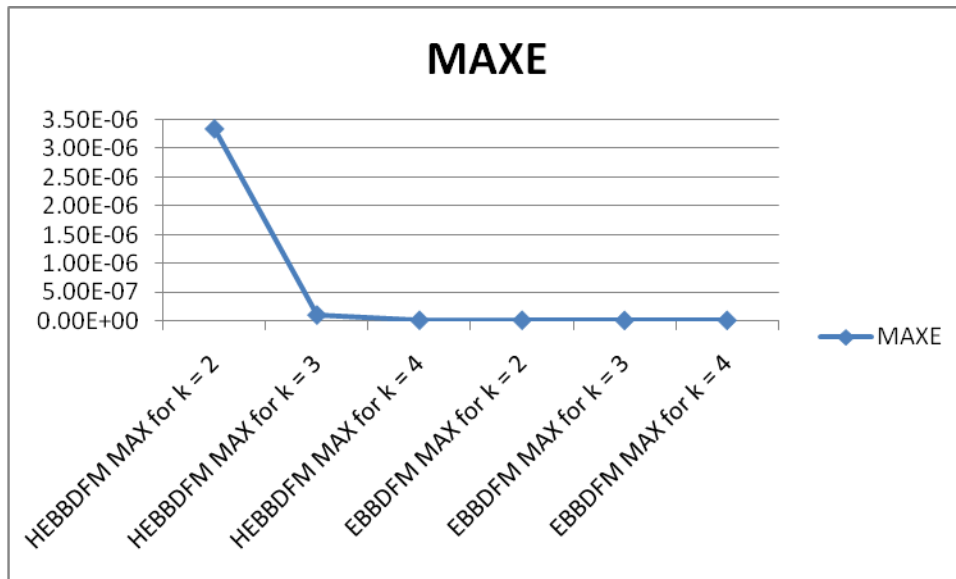Fig.7. MAXE for HEBBDFM and EBBDFM (Chibuisi et al, 2020) for Problem 1

Fig.8. MAXE for HEBBDFM and EBBDFM (Chibuisi et al, 2020) for Problem 2

5.2 Conclusions

In conclusion, the discrete schemes of (9), (11) and (13), were worked-out from their individual continuous form and were revealed to be convergent, U - and W -stable. Also, it was revealed in

tables 5.1.1 to 5.1.2 that the lower step number of HEBBDF method with integrated off-grid extended future points and one extended future point performed better than the higher step numbers of HEBBDF method with integrated off-grid extended future points and one extended future point when compared with the exact solutions. Most Importantly, this method performed better in terms of efficiency, accuracy, consistency, convergence and region of absolute stability at constant step width a when compared with other existing method as presented in table 5.1.3, 5.1.4 and figure 7 and 8. Therefore, it is recommended that HEBBDFM schemes for step numbers $k = 2, 3,$ and 4 are suitable for solving DDEs. Further studies should be investigated for step numbers $k = 5, 6, 7...$ on the development of discrete schemes of HEBBDFM for solutions of DDEs without the implementation of interpolation techniques in examining their delay arguments.

CONFLICT OF INTERESTS

The author(s) declares that there is no conflict of interests.

REFERENCES

[1] A.N. Al-Mutib, Numerical methods for solving delay differential equations. Ph.D. thesis, University of Manchester, UK (1977).

[2] A. Bellen, M. Zennaro, Numerical solution of delay differential equations by uniform corrections to an implicit Runge-Kutta method, Numer. Math. 47 (1985), 301–316.

[3] G.A. Bocharov, G.I. Marchuk, A.A. Romanyukha, Numerical solution by LMMs of stiff delay differential systems modelling an immune response, Numer. Math. 73 (1996), 131–148.

[4] C. Chibuisi, B.O. Osu, C.P. Ogbogbo, Solving first order delay differential equations using block Simpson's methods. Int. J. Basic Sci. Technol. 6 (2020), 76 – 86.

[5] C. Chibuisi, B.O. Osu, S.O. Edeki, G.O. Akinlabi, Numerical treatment of first order delay differential equations using extended block backward differentiation formulae, J. Phys.: Conf. Ser. 1734 (2021), 012023.

[6] C. Chibuisi, B.O. Osu, S. Amaraihu, Solving first order delay differential equations using multiple off-grid hybrid block Simpson's methods, FUW Trends Sci. Technol. J. 5 (2020), 856 – 870.

[7] S.O. Edeki, G.O. Akinlabi, N. Hinov, Zhou method for the solutions of system of proportional delay

differential equations, MATEC Web Conf. 125 (2017), 02001.

[8] S.O. Edeki, G.O. Akinlabi, S.A. Adeosun, Analytical solutions of a time-fractional system of proportional delay differential equations, in: 2017 2nd International Conference on Knowledge Engineering and Applications (ICKEA), IEEE, London, 2017: pp. 142–145.

[9] D.J. Evans, K.R. Raslan, The Adomian decomposition method for solving delay differential equation, Int. J. Computer Math. 82 (2005), 49–54.

[10] S.C. Heng, Z.B. Ibrahim, M.B. Suleiman, F. Ismail, Solving delay differential equations using implicit 2-point block backward differential Formulae. Pentica J. Sci. Technol. 21 (2013), 37-44.

[11] F. Ishak, M.B. Suleiman, Z. Omar, Two-point predictor-corrector block method for solving delay differential equations. Matematika, 24 (2008), 131-140.

[12] R.M. Jena, S. Chakraverty, S.O. Edeki, O.M. Ofuyatan, Shifted legendre polynomial based galerkin and collocation methods for solving fractional order delay differential equations, J. Theor. Appl. Inform. Technol. 98 (2020), 535-547.

[13] Z.A. Majid, H.M. Radzi, F. Ismail, Solving delay differential equations by the five-point one-step block method using Neville's interpolation, Int. J. Computer Math. 90 (2013), 1459–1470.

[14] H.J. Oberle, H.J. Pesh, Numerical treatment of delay differential equations by Hermite interpolation. Numer. Math, 37 (1981), 235–255.

[15] B.O. Osu, C. Chibuisi, N.N. Okwuchukwu, C. Olunkwa, N.A. Okore, Implementation of third derivative block backward differentiation formulae for solving first order delay differential equations without interpolation techniques, Asian J. Math. Computer Res. 27 (2020), 1-26,

[16] H.Y. Seong, Z.A. Majid, Solving second order delay differential equations using direct two-point block method, Ain Shams Eng. J. 8 (2015), 59-66.

[17] U.W. Sirisena, A reformulation of the continuous general linear multistep method by matrix inversion for the first order initial value problems. Ph.D. Thesis, University of Ilorin, Nigeria, (1997).

[18] U.W. Sirisena, S.Y. Yakubu, Solving delay differential equation using reformulated backward differentiation methods, J. Adv. Math. Computer Sci. 32 (2019), 1-15, JAMCS.48662.

[19] E. Tziperman, L. Stone, M.A. Cane, H. Jarosh, El Nino Chaos: Overlapping of resonances between the seasonal cycle and the Pacific Ocean-atmosphere oscillator, Science, 264 (1994), 72–74.

[20] S.Y. Yakubu, C. Chibuisi, A class of fifth order block hybrid adams moulton's method for solving stiff initial value problems. Int. J. Phys. Math. 2 (2020),1-18.