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SOME PROPERTIES OF WEAK FUZZY GRAPHS

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Abstract: In this article "weak fuzzy graph" is introduced. It is proved that the set of weak fuzzy graphs is closed under union and μ −complementation. Some properties of weak fuzzy graphs and the outputs of actions of complements on weak fuzzy graphs are also discovered.

Keywords: fuzzy graph; strong fuzzy graph; complement; μ −complement.

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1. INTRODUCTION

L A Zadeh in 1965[1] introduced fuzzy sets to describe the vagueness phenomena in real world problems. In 1975[2], fuzzy graph is introduced by Azriel Rosenfeld. The complement of a fuzzy graph is introduced by J N Mordeson in 1994. Sunitha M S and Vijayakumar A modified the complement in 2002[3]. A Nagoor Gani and Chandrasekaran introduced μ −complement of a fuzzy graph in 2006[4]. In this article "weak fuzzy graph" is introduced. It is proved that the set of weak fuzzy graphs is closed under union and μ −complementation. Some properties of weak

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fuzzy graphs and the outputs of actions of complements on weak fuzzy graphs is also discovered.

2. PRELIMINARIES

A mapping $m : N \to [0,1]$ from a non empty set N is a fuzzy subset of N. A fuzzy relation r on the fuzzy subset m , is a fuzzy subset of $N \times N$. The fuzzy relation r is assumed as symmetric and N is assumed as finite non empty set. Following definitions are from [1-4].

Definition 2.1: Suppose N is the underlying set. A fuzzy graph is a pair of functions F: (m, r) where fuzzy subset $m : N \to [0,1]$, the fuzzy relation r on m is denoted by $r : N \times N \to [0,1]$, such that for all $a, b \in N$, we have $r(a,b) \leq m(a) \wedge m(b)$ where $\wedge$ stands minimum.

F^*: (m^*, r^*) denotes the underlying crisp graph of a fuzzy graph F : (m, r) where $m^* = \{a \in N / m(a) > 0\}$ and $r^* = \{(a,b) \in N \times N / r(a,b) > 0\}$. The nodes a and b are known as neighbours if $r(a,b) > 0$.

Throughout this article F :(m, r) is taken to be fuzzy graph F with underlying non empty set N.

Definition 2.2.: A fuzzy graph F :(m, r) is a strong fuzzy graph if $r(a,b) = m(a) \wedge m(b)$ for all $(a,b) \in r^*$ and F :(m, r) is called complete if $r(a,b) = m(a) \wedge m(b)$ for all a,b in m^*.

Definition 2.3: Let F :(m, r) be a fuzzy graph, Its complement is defined as $\bar{F}: (\bar{m}, \bar{r})$where $\bar{m} = m$ and $\bar{r}(a,b) = m(a) \wedge m(b) - r(a,b)$ for all $a, b \in N$.

Definition 2.4: The μ −complement of a fuzzy graph F : (m, r) is a fuzzy graph $F^{\mu}: (m, r^{\mu})$ where r^{μ} is defined as $r^{\mu}(a,b) = 0$ if $r(a,b) = 0$ and
$r^{\mu}(a,b) = m(a) \wedge m(b) - r(a,b)$ if $r(a,b) > 0$ for all $a, b \in m^*$.

3. WEAK FUZZY GRAPHS

Definition 3.1: A fuzzy graph F :(m, r) is a weak fuzzy graph if $r(a,b) < m(a) \wedge m(b)$, for all $(a,b) \in r^*$.

The definition of weak fuzzy graph ensures that the “flows” or “capacities” through the links is always less than the minimum “flows” or “capacities” in the respective nodes. The set of fuzzy graphs is considered as the union of mutually disjoint three subsets – (i) The set of strong fuzzy graphs,(ii) The set of weak fuzzy graphs and(iii) The set of neither strong nor weak fuzzy graphs.

Example 3.1: The following graph illustrates a weak fuzzy graph.

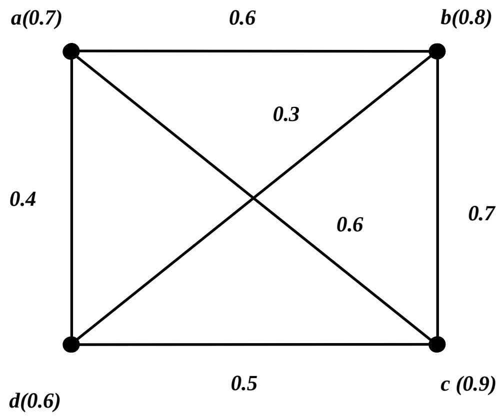

Figure 3.1

4. SOME PROPERTIES OF WEAK FUZZY GRAPHS

1. The union of two weak fuzzy graphs is a weak fuzzy graph.

2. The spanning subgraph of a weak fuzzy graph is a weak fuzzy graph.

3. A weak fuzzy subgraph can be formed from every strong fuzzy graph $H:(m,\tau)$.

Theorem 4.1. The union of two weak fuzzy graphs is a weak fuzzy graph.

Proof. Let $F_1:(m_1,r_1)$ and $F_2:(m_2,r_2)$ be two weak fuzzy graphs with the underlying crisp graphs $F_1^*:(N_1,X_1)$ and $F_2^*:(N_2,X_2)$ respectively. The union $F:(m_1\cup m_2, r_1\cup r_2)$ is defined by

$(m_1\cup m_2)(a) = m_1(a)$ if $a\in N_1-N_2$.

$(m_1\cup m_2)(a) = m_2(a)$ if $a\in N_2-N_1$, and

$(m_1\cup m_2)(a) = \max\{m_1(a), m_2(a)\}$ if $a\in N_1\cap N_2$.

$(r_1\cup r_2)(a,b) = r_1(a,b)$ if $(a,b)\in X_1-X_2$.

$(r_1\cup r_2)(a,b) = r_2(a,b)$ if $(a,b)\in X_2-X_1$, and

$(r_1\cup r_2)(a,b) = \max\{r_1(a,b), r_2(a,b)\}$ if $(a,b)\in X_1\cap X_2$.

To prove that $F:(m_1\cup m_2, r_1\cup r_2)$ is a weak fuzzy graph, it is enough to prove that

$(r_1\cup r_2)(a,b) < ((m_1\cup m_2)(a)\wedge(m_1\cup m_2)(b))$ for all $(a,b)\in(r_1\cup r_2)^*$

Case 1: When $(a,b) \in X_1 - X_2$

$(r_1 \cup r_2)(a,b) = r_1(a,b) < m_1(a) \wedge m_1(b)$. Since $F_1: (m_1, r_1)$ is a weak fuzzy graph

$(r_1 \cup r_2)(a,b) = r_1(a,b) < m_1(a) \wedge m_1(b) < ((m_1 \cup m_2)(a) \wedge (m_1 \cup m_2)(b))$.

Since $m_2(a) = 0$ for all $a \in N_1 - N_2$.

Case 2: When $(a,b) \in X_2 - X_1$

$(r_1 \cup r_2)(a,b) = r_2(a,b) < m_2(a) \wedge m_2(b)$ .Since $F_2: (m_2, r_2)$ is a weak fuzzy graph

$(r_1 \cup r_2)(a,b) = r_2(a,b) < m_2(a) \wedge m_2(b) < ((m_1 \cup m_2)(a) \wedge (m_1 \cup m_2)(b))$.

Since $m_1(a) = 0$ for all $a \in N_2 - N_1$.

Case 3: When $(a,b) \in X_1 \cap X_2$

Since $F_1: (m_1, r_1)$ and $F_2: (m_2, r_2)$ are weak fuzzy graphs, $r_1(a,b) < m_1(a) \wedge m_1(b)$ and $r_2(a,b) < m_2(a) \wedge m_2(b)$.

Hence $\max\{r_1(a,b), r_2(a,b)\} < \max\{m_1(a) \wedge m_1(b), m_2(a) \wedge m_2(b)\}$

Therefore $(r_1 \cup r_2)(a,b) = \max\{r_1(a,b), r_2(a,b)\} < \max\{m_1(a) \wedge m_1(b), m_2(a) \wedge m_2(b)\}$

$< (\max\{m_1(a), m_2(a)\} \wedge \max\{m_1(b), m_2(b)\})$

In all cases $(r_1 \cup r_2)(a,b) < ((m_1 \cup m_2)(a) \wedge (m_1 \cup m_2)(b))$for all $(a,b) \in (r_1 \cup r_2)^*$.

The above theorem implies that the set of weak fuzzy graphs is closed under union.

Theorem 4.2. The spanning subgraph of a weak fuzzy graph is a weak fuzzy graph.

Proof. Let fuzzy graph H : (τ, ρ)be a spanning subgraph of a weak fuzzy graph F : (m, r).

By the definition of spanning fuzzy subgraph $\tau(u) = m(a)$for all $a \in m^*$ and

$\rho(a,b) \leq r(a,b)$ for all $a, b \in m^*$.

To prove that $\rho(a,b) < \tau(a) \wedge \tau(b)$ for all $(a,b) \in \rho^*$.

Since F:(m, r) is a weak fuzzy graph $r(a,b) < m(a) \wedge m(b)$ for all $(a,b) \in r^*$.

Now $\rho(a,b) \leq r(a,b) < m(a) \wedge m(b) = \tau(a) \wedge \tau(b)$ for all $(a,b) \in r^*$.

Therefore $\rho(a,b) < \tau(a) \wedge \tau(b)$ for all $(a,b) \in \rho^*$.

Theorem 4.3. A weak fuzzy subgraph can be formed from every strong fuzzy graph $H: (m, \tau)$.

Proof. Let $H: (m, \tau)$ be a strong fuzzy graph. Control the flow through the links such that $r(a,b) = \tau(a,b) - \varepsilon$, for every $\tau(a,b) > 0$,where ε is fixed real number such that

$\varepsilon \in (0, min\{\tau(a,b) \,/\, \tau(a,b) > 0\})$ .Let F: (m,r) be the resulting subgraph. In F: (m,r) $r(a,b) < m(a) \wedge m(b)$ for all $(a,b) \in r^*$ .Now F: (m,r) is a weak fuzzy subgraph of $H: (m,\tau)$.

5. OUTPUTS OF ACTIONS OF COMPLEMENTS ON WEAK FUZZY GRAPHS

1. The action of complement on a weak fuzzy graph gives a weak fuzzy graph iff every pair of nodes are neighbours.
2. If every pair of nodes of a fuzzy graph are not neighbours then the action of complement on a weak fuzzy graph gives either a strong fuzzy graph or a fuzzy graph which is neither strong nor weak.
3. The action of complement on a complete fuzzy graph gives a weak fuzzy graph where no pair of nodes are neighbors.
4. The fuzzy graph F : (m,r)such that $m(a,b) = \frac{1}{2}(m\ (a) \wedge m\ (b))$ for all for all a,b in m^* is a weak self complementary fuzzy graph.
5. The action of μ −complement on a weak fuzzy graph gives a weak fuzzy graph.
6. Law of double complementation. If F is a weak fuzzy graph then $(F^{\mu})^{\mu} = F$.
7. The action of μ −complement on a strong fuzzy graph gives a weak fuzzy graph having isolated nodes.

Theorem 5.1. The action of complement on a weak fuzzy graph gives a weak fuzzy graph iff every pair of nodes are neighbours.

Proof. Suppose the complement of a weak fuzzy graph is a weak fuzzy graph.

We have to show that $r(a,b) > 0$ for all a,b in m^*.Since the complement of the weak fuzzy graph is weak, $\overline{r\ (a,b)} < m\ (a) \wedge m(b)$ for all a,b in m^*.

Therefore $r(a,b) = \overline{\left(\overline{r(a,b)}\right)} = m(a) \wedge m(b) - \overline{r(a.b)} > 0$, for all a,b in m^*.

If every pair of nodes are not neighbours, there will be some links with $r(a,b) = 0$, $m(a) = \varepsilon$, $m(b) = \delta$ where ε , $\delta \in (0,1]$. Now as per the definition of complement $\overline{r\ (a,b)} = \min\{\varepsilon,\delta\} - 0 = \min\{\varepsilon,\delta\}$which is either $m(a)$ or $m(b)$.Hence the complement fails to become a weak fuzzy graph.

Theorem 5.2. If every pair of nodes of a fuzzy graph are not neighbours then the action of complement on a weak fuzzy graph gives either a strong fuzzy graph or a fuzzy graph which is neither strong nor weak.

Proof. Suppose every pair of nodes of a fuzzy graph are not neighbours. Then there exists three types of links.

Case 1: Absent links. i.e, links are such that $r(a,b) = 0$.

In this case $\overline{r\ (a,b)}$= min $\{\varepsilon,\delta\}$- 0 = min$\{\varepsilon,\delta\}$ which is either $m(a)$ or $m(b)$ for all a,b in $m*$. In this case the complement is a strong fuzzy graph.

Case 2: Some absent and some active links. i.e, the links are such that

2.1 $r(a,b) = 0$ for some a,b in $m*$ and

2.2 $r(a,b) > 0$ for some a,b in $m*$

In the case 2.1, By the definition of complement, $\overline{r(a,b)} = m(a) \wedge m\ (b)$

In the case 2.2, Let $r(a,b) = \varepsilon,$ where $\varepsilon \in (0, m(a) \wedge m\ (b))$.

$\overline{r\ (a,b)} = m(a) \wedge m\ (b)$ - $\varepsilon < m(a) \wedge m(b)$. In Case 2, flow allowed in various links shows that the complement is neither a strong fuzzy graph nor a weak fuzzy graph.

Theorem 5.3. The action of complement on a complete fuzzy graph gives a weak fuzzy graph where no pair of nodes are neighbors.

Proof. If the fuzzy graph is complete then $r(a,b) = m(a) \wedge m(b)$ for all a,b in $m*$, then

$\overline{r\ (a,b)} = m(a) \wedge m(b)$ - $r(a,b) = m(a) \wedge m(b)$ - $m(a) \wedge m(b)$ = 0 for all a,b in $m*$.

Theorem 5.4. The fuzzy graph $F: (m,r)$ such that $r(a,b) = \frac{1}{2}(m(a) \wedge m(b))$ for all for all a,b in $m*$ is a weak self complementary fuzzy graph.

Proof. If $F: (m,r)$is such that $r(a,b) = \frac{1}{2}(m(a) \wedge m(b))$ for all for all a,b in $m*$ then

$r(a,b) = \frac{1}{2}(m(a) \wedge m(b)) < m(a) \wedge m(b)$ for all a,b in $m*$.So F is a weak fuzzy graph.

Now $\overline{r\ (a,b)} = m(a) \wedge m(b)$ - $\frac{1}{2}\left(m(a) \wedge m(b)\right) = \frac{1}{2}\left(m(a) \wedge m(b)\right) = r(a,b)$.

Therefore F is self complementary fuzzy graph.

Theorem 5.5. The action of μ −complement on a weak fuzzy graph gives a weak fuzzy graph.

Proof. Let $F^{\mu}:(m,r^{\mu})$be the μ −complement of a weak fuzzy graph F: (m,r) where r^{μ} is defined as.

$r^{\mu}(a,b)=m(a)\wedge m(b)-r(a,b)$ if $r(a,b)>0$ for all $a,b\in m^{*}\rightarrow(1)$

and $r^{\mu}(a,b)=0$ if $r(a,b)=0\rightarrow(2)$

Since F:(m,r) is a weak fuzzy graph $r(a,b)<m(a)\wedge m(b)$ for all $(a,b)\in r^{*}\rightarrow(3)$

To prove that $F^{\mu}:(m,r^{\mu})$ is a weak fuzzy graph, it is enough to prove

$r^{\mu}(a,b)<m(a)\wedge m(b)$ for all $(a,b)\in r^{\mu *}$

Case 1. Let $(a,b)\in r^{\mu *}$, such that $r(a,b)>0$ .Then by the equations (1) and(3)

$$r^{\mu}(a,b)=m(a)\wedge m(b)-r(a,b)<m(a)\wedge m(b)$$

Case 2. Let $(a,b)\in r^{\mu *}$, such that $r(a,b)=0$ .Then by the equation (2), $r^{\mu}(a,b)=0$

Therefore in both cases $r^{\mu}(a,b)<m(a)\wedge m(b)$for all $(a,b)\in r^{\mu *}$. Hence the proof.

The above theorem implies that the set of weak fuzzy graphs is closed under μ −complementation.

Theorem 5.6. If F is a weak fuzzy graph then $(F^{\mu})^{\mu}=F$.

Proof. Let $F^{\mu}:(m,r^{\mu})$ be the μ −complement of F: (m,r).

Since F: (m,r) is a weak fuzzy graph,$r(a,b)<m(a)\wedge m(b)$ for all $(a,b)\in r^{*}$.

We shall consider two cases.

Case 1. For all $(a,b)\in r^{*}$. As per definition of r^{*}, $r(a,b)>0$.

In this case $r^{\mu}(a,b)<m(a)\wedge m(b)-r(a,b)$.

$$(r^{\mu})^{\mu}(a,b)=m(a)\wedge m(b)-r^{\mu}(a,b)=m(a)\wedge m(b)-(m(a)\wedge m(b)-r(a,b))$$
$$=r(a,b)$$

Case 2. For all $(a,b)\notin r^{*}$. As per definition of r^{*}, $r(a,b)=0$.

Here $r^{\mu}(a,b)=0$ and $(r^{\mu})^{\mu}(a,b)=0$.

The above theorem implies that the set of weak fuzzy graphs hold the law of double complementation with respect to μ −complement.

Theorem 5.7. The action of μ −complement on a strong fuzzy graph gives a weak fuzzy graph having isolated nodes.

Proof. Let $F^{\mu}:(m, r^{\mu})$ be the $\mu-$ complement of strong fuzzy graph $F:\ (m, r)$.

Since $F{:}(m, r)$ is a strong fuzzy graph, $r(a, b) = m(a) \wedge m(b)$ for all $(a, b) \in r^*$.

Then for all $(a, b) \in r^*$,

$r^{\mu}(a, b) = m(a) \wedge m(b) - r(a, b) = m(a) \wedge m(b) - m(a) \wedge m(b) = 0$. For all $(a, b) \notin r^*$, $r(a, b) = 0$. Here $r^{\mu}(a, b) = 0$ and $(r^{\mu})^{\mu}(a, b) = 0$. This implies that in the resulting fuzzy graph all the links are absent and so it is a weak fuzzy graph having isolated nodes.

6. CONCLUSION

In this article "weak fuzzy graph" is introduced. It is proved that the set of weak fuzzy graphs is closed under union and $\mu-$complementation. Some properties of weak fuzzy graphs and the outputs of actions of complements on weak fuzzy graphs is also discovered.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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