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ON COMPLETELY HOMOGENEOUS L-TOPOLOGICAL SPACES

PINKY[1,*], T.P. JOHNSON[2]

[1]Department of Mathematics, Cochin University of Science and Technology, Cochin-682022, India
[2]Applied Science and Humanities Divison, School of Engineering, Cochin University of Science and Technology, Cochin-682022, India

Abstract. In this paper, we study completely homogeneous L-topological spaces on a non-empty set X when membership lattice L is a complete chain.

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1. INTRODUCTION

For a topological property P and a set X, let $P(X)$ denote the collection of all topologies on X with property P. Then $P(X)$ is a partially ordered set under the natural order of set inclusion. A topological space $(X,\Im)$ with property P is minimum P (maximum P) if $P(X)$ is non-empty and $\Im$ is a minimum (maximum) element the set $P(X)$. In 1970, Roland E. Larson characterizes all minimum and maximum P spaces [4]. He proved that a topological space $(X,\Im)$ is minimum P (maximum $P^{'}$) for some topological property P ($P^{'}$) if and only if it is completely homogeneous, where completely homogeneous means that every one-to-one function of X onto itself is a homeomorphism. He then proved that the only completely homogeneous topologies on a set X

*Corresponding author
E-mail address: malikpinky1989@gmail.com
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are the indiscrete topology, the discrete topology and those topologies in which the closed sets are the space X and all subsets of X of cardinality less than m, where m is an infinite cardinal not greater than the cardinality of X.

T.P. Johnson has defined the concept of a complete homogeneous fuzzy topological space in an analogous way and studied some of its properties [2]. P. Sini et al. have characterized completely homogeneous L-topological spaces when X is a finite set and $L = \{0, a, 1\}$, where $a \neq 0, 1$ [5].

However, we consider an equivalence relation R on the set of all completely homogeneous L-topologies on a non-empty set X when membership lattice L is a complete chain and investigate all disjoint equivalence classes with respect to the relation R.

2. Preliminaries

Throughout this paper, X stands for a non-empty set, L for a complete chain with the least element 0 and the greatest element 1, $S(X)$ stands for the set of all permutations of the set X. The constant function in L^X, taking value α is denoted by $\underline{\alpha}$ and x_γ, where $\gamma (\neq 0) \in L$ denotes the L- fuzzy point defined by $x_\gamma(y) = \begin{cases} \gamma & \text{if} \quad y = x \\ 0 & \text{otherwise} \end{cases}$. Any $f \in L^X$ is called as an L-subset of X. The following are some important definition reported in $[3, 6]$:

Definition 2.1. Let δ be a non-empty subset of L^X. We call δ an L-topology on X, if δ satisfies the following conditions :

(1) $\underline{0}, \underline{1} \in \delta$.

(2) if $f, g \in \delta$, then $f \wedge g \in \delta$.

(3) if $\delta_1 \subseteq \delta$, then $\bigvee_{f \in \delta_1} f \in \delta$.

The pair (L^X, δ) is called an L-topological space. The elements of δ are said to be open L-subsets of X.

Definition 2.2. Let X and Y be two sets and $\theta : X \to Y$ be a function. Then for any L-subset g in X, $\theta(g)$ is an L-subset in Y defined by

$$\theta(g)(y) = \begin{cases} \sup\{g(z) : z \in \theta^{-1}(y)\} & \text{if} \quad \theta^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases},$$

where $\theta^{-1}(y) = \{x \in X : \theta(x) = y\}$.

For an L-subset f in Y, we define

$\theta^{-1}(f)(x) = f[\theta(x)], \forall x \in X$. Obviously $\theta^{-1}(f)$ is an L-subset in X.

Definition 2.3. Let (X,δ) and $(Y,\delta^{'})$ be two L-topological spaces. Then a function $\theta : X \to Y$ is said to be L-continuous if $\theta^{-1}(g) \in \delta$ for every $g \in \delta^{'}$ and θ is said to be open if $\theta(f) \in \delta^{'}$ for every $f \in \delta$.

Definition 2.4. Let (X,δ) and $(Y,\delta^{'})$ be two L-topological spaces. Then a bijection $\theta : X \to Y$ is said to be L-homeomorphism if both θ and θ^{-1} are L-continuous.

By an L-homeomorphism of (X,δ), we mean an L-homeomorphism from (X,δ) to itself. The set of all L-homeomorphism of an L-topological space (X,δ) onto itself is a group under composition, which is a subgroup of the group of all permutations on the set X. It is called the group of L-homeomorphisms of (X,δ).

Definition 2.5. An L-topological space (X,δ) is called a completely homogeneous space if every bijection of X onto itself is an L-homeomorphism.

Notations:

- $|A|$ stands for the cardinality of a given set A.
- If (X,δ) is an L-topological space, then define
  (1) $\overline{\delta} = \delta \setminus \{\underline{0}, \underline{1}\}$.
  (2) $\Re_f = \{f(x) : x \in X\}$.
  (3) $Ł^X_{\Re_f} = \{g : X \to \Re_f\}$.
  (4) $\Re_{\overline{\delta}} = \{f(x) : x \in X \text{ and } f \in \overline{\delta}\}$.
- For any $A \subseteq L$, define $Ł^X_A = \{f : X \to A\}$.
- For any $H \subseteq L$, define $H^\star = \{\alpha \in L : \alpha = \bigvee_{\gamma \in M} \gamma$, where $M \subseteq H\}$. Then H is said to be closed with respect to arbitrary join if $H^\star \subseteq H$ i.e. if H contains all the possible join of its elements.

3. Completely Homogeneous L-Topological Spaces

Definition 3.1. Let **CHLT(X)** be the collection of all completely homogeneous L-topologies on X.

Let $\delta_1, \delta_2 \in$ **CHLT(X)** and define the relation R on the set **CHLT(X)** as :

$\delta_1 \; R \; \delta_2$ if and only if $|\Re_{\overline{\delta_1}}| = |\Re_{\overline{\delta_2}}|$.

Clearly, R is an equivalence relation.

For $0 \leq m \leq |L|$, define

$\big[m\big]$(class of m) $= \{\delta : \delta$ is a completely homogeneous L-topology on X and $|\Re_{\overline{\delta}}| = m\}$.

Definition 3.2. Let $A \subseteq L$ be any subset and $|A| > 1$. Then a subset $M \subseteq A$ is called a c-subset of A if

(i) $M^{\star} \subseteq M$.

(ii) $|M| > 1$.

(iii) if $\alpha, \beta \in M$ and $\alpha < \gamma < \beta$ for some $\gamma \in A$, then $\gamma \in M$.

Definition 3.3. Two c-subsets $\triangle_i$ and $\triangle_j$ of a subset $A \subseteq L$ are said to be distinct if $\exists$ at least one $\alpha_i \in \triangle_i$ and $\alpha_j \in \triangle_j$ such that $\alpha_i \notin \triangle_j$ and $\alpha_j \notin \triangle_i$.

4. Completely Homogeneous L-Topological Spaces When X Is a Finite Set

Throughout this section, X stands for a finite set.

Theorem 4.1. Let X be a finite set and L be a complete chain. Then (X, δ) is a completely homogeneous L-topological space if and only if $Ł^X_{\Re_f} \subset \delta, \forall f \in \delta$.

Proof. First suppose that (X, δ) is a completely homogeneous L-topological space.

Let $k_1 = \bigwedge_{k \in \Re_f} k$ and $x'_{k_i}(y) = \begin{cases} k_i & \text{if} \quad y = x \\ k_1 & \text{otherwise} \end{cases}$.

We claim that $x'_{k_i} \in \delta, \forall k_i \in \Re_f$.

Clearly, $k_1 \in \Re_f$. Since $k_1 \in \Re_f, \exists$ an element $x_0 \in X$ such that $f(x_0) = k_1$. For each $x \in X \setminus \{x_0\}$, define $f_x = f o h_x$, where $h_x : X \to X$ is defined as :

$$h_x(y) = \begin{cases} x_0 & \text{if} \quad y = x \\ x & \text{if} \quad y = x_0 \\ y & \text{otherwise} \end{cases}.$$

$$\text{Then } f_x(y) = \begin{cases} k_1 & \text{if} \quad y = x \\ f(x) & \text{if} \quad y = x_0 \\ f(y) & \text{otherwise} \end{cases}.$$

Clearly, $f_x \in \delta, \forall x \in X \setminus \{x_0\}$.

Let $k \in \Re_f$. So $\exists$ an element $z \in X$ such that $f(z) = k$.

Now $\bigwedge_{x \in X \setminus \{z\}} f_x(y) = \begin{cases} k & \text{if} \quad y = z \\ k_1 & \text{otherwise} \end{cases}$.

$\Rightarrow \bigwedge_{x \in X \setminus \{z\}} f_x(y) = x_k^{'} \in \delta$.

Hence $Ł^X_{\Re_f} \subset \delta, \forall f \in \delta$.

Conversely, suppose that $Ł^X_{\Re_f} \subset \delta, \forall f \in \delta$. Then $foh \in \delta, \forall f \in \delta$ and $\forall h \in S(X)$. Hence δ is a completely homogeneous L-topology on X.

Remark 4.2. It can be checked that following are the disjoint equivalence classes with respect to the relation R when X is a finite set :

- $[0]$ contains only one completely homogeneous L-topology $\{\underline{0}, \underline{1}\}$.
- $[1]$ contains only one type of completely homogeneous L-topologies $\{\underline{0}, \underline{1}, \underline{\alpha}\}$, where $\alpha \in L \setminus \{0, 1\}$ i.e.

  $[1] = \big\{\{\underline{0}, \underline{1}, \underline{\alpha}\} : \alpha \in L \setminus \{0, 1\}\big\}$.
- $[2]$ contains following two types of completely homogeneous L-topologies :

  (i) $\big\{\{\underline{0}, \underline{1}, \underline{\alpha_1}, \underline{\alpha_2}\} : \alpha_1, \alpha_2 \in L \setminus \{0, 1\}\big\}$.

  (ii) $\{\underline{0}, \underline{1}, g : g \in Ł^X_H$, where $H \subseteq L$ and $|H| = 2\}$.
- For $m \geq 3$, $[m]$ contains following three types of completely homogeneous L-topologies :

  (i) $\{\underline{0}, \underline{1}, \underline{\alpha} : \alpha \in H_1$, where $H_1 \subseteq L \setminus \{0, 1\}$ such that $H_1^\star \subseteq H_1$ and $|H_1| = m\}$.

  (ii) $\{\underline{0}, \underline{1}, g : g \in Ł^X_{H_2}$, where $H_2 \subseteq L$ such that $H_2^\star \subseteq H_2$ and $|H_2| = m\}$.

  (iii) Let $H \subseteq L$ be any subset such that $H^\star \subseteq H$ and $|H| = m$. Consider a family $\triangle_i, i \in \Omega$ of distinct c-subsets of H.

  Let $\mathbb{E} = \{\beta \in H : \beta \notin \bigcup_{i \in \Omega} \triangle_i\}$.

  The L-topologies of the form $\{\underline{0}, \underline{1}, \underline{\beta}, f : \beta \in \mathbb{E}$ and $f \in \bigcup_{i \in \Omega} Ł^X_{\triangle_i}\}$.

Theorem 4.3. Let X be a finite set and L be a complete chain. If F is a completely homogeneous L-topology on X, then F is equal to one of the L-topologies defined in the remark 4.2.

Proof. Let $|\Re_{\overline{F}}| = m$. If $m = 0$, then clearly $F = \{\underline{0}, \underline{1}\}$. So, assume that $m > 0$.

Case 1 : If F contains only constant L-subsets, then $F = \{\underline{0}, \underline{1}, \underline{\alpha} : \alpha \in \Re_{\overline{F}}\}$.

Case 2 : If $Ł^X_{\overline{F}} \subseteq F$, then $F = \{\underline{0}, \underline{1}, g : g \in Ł^X_{\overline{F}}\}$.

Case 3 : Suppose Case 1 and Case 2 do not hold.

Let $f \in F$ be any non-constant L-subset.

We claim that $f \in Ł^X_{\triangle}$ for some c-subset $\triangle \subset \Re_{\overline{F}}$.

Let $\triangle \subset \Re_{\overline{F}}$ be a subset such that

(i) $\Re_f \subseteq \triangle$.

(ii) $Ł^X_{\triangle} \subset F$.

(iii) $\triangle^{\star} \subseteq \triangle$.

(iv) $\triangle$ is not properly contained in any proper subset of $\Re_{\overline{F}}$ satisfying above three properties.

Let $\alpha, \beta \in \triangle$ and $\gamma \in \Re_{\overline{F}}$ such that $\alpha < \gamma < \beta$.

$\gamma \in \Re_{\overline{F}}$ and $Ł^X_{\Re_g} \subseteq F, \forall g \in F \Rightarrow \underline{\gamma} \in F$.

Let $\gamma \notin \triangle$. Since L is a chain and $Ł^X_{\triangle} \subset F$, it is easy to see that $T = \triangle \cup \{\gamma\}$ satisfies properties (i)-(iii) and $\triangle \subset T$, a contradiction $\Rightarrow \gamma \in \triangle \Rightarrow \triangle$ is a c-subset.

Therefore, corresponding to every L-subset g of F, $\exists$ a c-subset $\nabla \subset \Re_F$ such that $g \in Ł^X_{\nabla} \subseteq F$.

Let $\triangle_i, i \in \Omega$ be the collection of those distinct c-subsets of $\Re_{\overline{F}}$ such that $Ł^X_{\triangle_i} \subset F, \forall i \in \Omega$ and for every non-constant L-subset $h \in F, h \in Ł^X_{\triangle_i}$ for at-least one $i \in \Omega$.

Let $\mathbb{E} = \{\beta \in \Re_{\overline{F}} : \beta \notin \bigcup_{i \in \Omega} \triangle_i\}$.

Thus $F = \{\underline{0}, \underline{1}, \underline{\beta}, f : \beta \in \mathbb{E}$ and $f \in \bigcup_{i \in \Omega} Ł^X_{\triangle_i}\}$.

$\Rightarrow$ If F is a completely homogeneous L-topology on a finite set X, then F is equal to one of the L-topologies defined in remark 4.2.

5. Completely Homogeneous L-Topological Spaces When X Is a Countable Set

Throughout this section, X stands for a countable set.

Remark 5.1. It can be checked that following are the disjoint equivalence classes with respect to the relation R when X is a countable set :

- $[0]$ contains only one completely homogeneous L-topology $\{\underline{0}, \underline{1}\}$.

- $\left[1\right]$ contains only one type of completely homogeneous L-topologies $\{\underline{0},\underline{1},\underline{\alpha}\}$, where $\alpha \in L\setminus\{0,1\}$ i.e.

  $\left[1\right] = \left\{\{\underline{0},\underline{1},\underline{\alpha}\} : \alpha \in L\setminus\{0,1\}\right\}$.

- $\left[2\right]$ contains four types of completely homogeneous L-topologies :

  (i) $\left\{\{\underline{0},\underline{1},\underline{\alpha},\underline{\beta}\} : \alpha,\beta \in L\setminus\{0,1\}\right\}$.

  (ii) $\{\underline{0},\underline{1},g : g \in Ł^X_H$, where $H \subseteq L$ and $|H| = 2\}$.

  Let $\alpha_1,\alpha_2 \in L$ be two arbitrary elements such that $\alpha_1 < \alpha_2$ and $g \in L^X$ be defined by

  $$g(x) = \begin{cases} \alpha_1 \text{ for at-most finitely many } x \in X \\ \alpha_2 \text{ otherwise} \end{cases}.$$

  (iii) L-topologies generated by the sets of the form $\{\underline{0},\underline{1},goh : h \in S(X)\}$.

  (iv) L-topologies generated by the sets of the form $\{\underline{0},\underline{1},\underline{\alpha_1},goh : h \in S(X)\}$.

- For $m \geq 3$, $\left[m\right]$ contains following types of completely homogeneous L-topologies :

  (i) $\{\underline{0},\underline{1},\underline{\alpha} : \alpha \in H_1$, where $H_1 \subseteq L\setminus\{0,1\}$ such that $H_1^\star \subseteq H_1$ and $|H_1| = m\}$.

  (ii) $\{\underline{0},\underline{1},g : g \in Ł^X_H$, where $H \subseteq L$ such that $H^\star \subseteq H$ and $|H| = m\}$.

  (iii) Let $H \subseteq L$ be any subset such that $H^\star \subseteq H$ and $|H| = m$.

  Consider a c-subset $\triangle \subseteq H$, choose an arbitrary element $\gamma \in \triangle$ and define :

  $\mathbb{P}_1 = \{\alpha \in \triangle : \alpha < (\leq)\gamma\}$,

  $\mathbb{P}_2 = \{\beta \in \triangle : \gamma \leq (<)\beta\}$,

  $$\mathbb{L}_\triangle = \begin{cases} f \in L^X : \quad f(x) \in \mathbb{P}_1 \text{ for finitely many } x \in X \\ \qquad\qquad\quad f(x) \in \mathbb{P}_2 \text{ otherwise} \end{cases}$$

  and $\mathbb{L}_{\triangle,\mathbb{C}} = \mathbb{L}_\triangle \cup \{\underline{\alpha} : \alpha \in \mathbb{C} \subseteq \mathbb{P}_1\}$.

  Consider a family $\triangle_i, i \in \Omega$ of distinct c-subsets of H and corresponding to each c-subset $\triangle_i, i \in \Omega$, choose exactly one set from the set $\left\{Ł^X_{\triangle_i},\mathbb{L}_{\triangle_i},\mathbb{L}_{\triangle_i,\mathbb{C}}\right\}$ and denote that set by $\Sigma_{\triangle_i}$.

  Let $\mathbb{E} = \{\beta \in H : \beta \notin \bigcup_{i\in\Omega}\triangle_i\}$.

  The L-topologies of the form $\left\{\underline{0},\underline{1},\underline{\beta},f : \beta \in \mathbb{E} \text{ and } f \in \Sigma_{\triangle_i}, i \in \Omega\right\}$.

Theorem 5.2. Let X be a countable set and L be a complete chain. If F is a completely homogeneous L-topology on X, then F is equal to one of the L-topologies defined in the remark 5.1.

Proof. Let $|\Re_{\overline{F}}| = m$. If $m = 0$, then clearly $F = \{\underline{0}, \underline{1}\}$. So, assume that $m > 0$.

Case 1: If F contains only constant L-subsets, then $F = \{\underline{0}, \underline{1}, \underline{\alpha} : \alpha \in \Re_{\overline{F}}\}$.

Case 2: If $Ł^X_{\overline{F}} \subseteq F$, then $F = \{\underline{0}, \underline{1}, g : g \in Ł^X_{\overline{F}}\}$.

Case 3: Suppose Case 1 and Case 2 do not hold.

Let $f \in F$ be any non-constant L-subset.

Let $\triangle \subset \Re_{\overline{F}}$ be a subset such that

(i) $\Re_f \subseteq \triangle$.

(ii) $\triangle^\star \subseteq \triangle$.

(iii) for any two elements $\alpha, \beta \in \triangle, \exists$ an L-subset $g \in F$ such that $g(x) = \alpha, g(y) = \beta$ for some $x, y \in X$.

(iv) $\triangle$ is not properly contained in any proper subset of $\Re_{\overline{F}}$ satisfying above three properties.

Let $\alpha, \beta \in \triangle$ and $\gamma \in \Re_{\overline{F}}$ such that $\alpha < \gamma < \beta$.

Let $\gamma \notin \triangle$. $\gamma \in \Re_{\overline{F}}$ and F is a completely homogeneous L-topological space so $\exists$ an L-subset $h_1 \in F$ such that $h_1(x) = h_1(y) = \gamma$ for some $x, y \in X$.

Since $\alpha, \beta \in \triangle \Rightarrow \exists$ an L-subset $h_2 \in F$ such that $h_2(x) = \alpha, h_2(y) = \beta$.

Then $(h_1 \wedge h_2)(x) = \alpha$ and $(h_1 \wedge h_2)(y) = \gamma$.

$(h_1 \vee h_2)(x) = \gamma$ and $(h_1 \vee h_2)(y) = \beta$.

In the same way, it can be shown that for any two elements $\eta_1, \eta_2 \in T = \triangle \cup \{\gamma\}, \exists$ an L-subset $g \in F$ such that $g(x) = \eta_1, g(y) = \eta_2$ for some $x, y \in X$ and $\Re_f \subset T$, a contradiction $\Rightarrow \gamma \in \triangle \Rightarrow \triangle$ is a c-subset.

Case (i) : If $Ł^X_\triangle \subset F$, then $f \in Ł^X_\triangle$.

Case (ii) : Let $Ł^X_\triangle \not\subseteq F$.

Let $\mathbb{D} = \{h \in F : \Re_h \subseteq \triangle\}$.

$Ł^X_\triangle \not\subseteq F \Rightarrow \exists$ some element(s) $\lambda \in \triangle$ such that if $h \in \mathbb{D}$ and $h(x) = \lambda$ for some $x \in X$, then $h(y) = \lambda$ for at-most finitely many $y \in X$.

Let $\mathbb{P} = \{\lambda \in \triangle :$ if $\lambda \in \Re_h$ for some $h \in \mathbb{D}$, then $h(x) = \lambda$ for at-most finitely many $x \in X\}$.

It can be checked that

(i) if $\alpha, \beta \in \mathbb{P}$ and $\eta \in \triangle$ such that $\alpha < \eta < \beta$, then $\eta \in \mathbb{P}$.

(ii) if $\alpha \in \mathbb{P}$ and $\eta \in \triangle$ such that $\eta < \alpha$, then $\eta \in \mathbb{P}$.

$$\mathbb{L}_{\triangle} = \begin{cases} g \in L^X : & g(x) \in \mathbb{P} \text{ for at-most finitely many } x \in X \\ & g(x) \in \triangle \setminus \mathbb{P} \text{ otherwise .} \end{cases}.$$

Now two cases arise:

Case (a) : when $\underline{\alpha} \notin F, \forall \alpha \in \mathbb{P}$.

Then $\mathbb{L}_{\triangle} \subseteq F$.

Case (b) : when $\underline{\alpha} \in F$ for all / some $\alpha \in \mathbb{P}$.

Let $\mathbb{C} = \{\alpha \in \mathbb{P} : \underline{\alpha} \in F\}$ and $\mathbb{L}_{\triangle,\mathbb{C}} = \mathbb{L}_{\triangle} \cup \mathbb{C} \subseteq F$.

Thus either $f \in \mathbb{L}_{\triangle}$ or $f \in \mathbb{L}_{\triangle,\mathbb{C}}$.

Let $\triangle_i, i \in \Omega$ be the collection of those distinct c-subsets of $\Re_{\overline{F}}$ such that corresponding to each c-subset $\triangle_i, i \in \Omega$, exactly one set from the set $\left\{Ł^X_{\triangle_i}, \mathbb{L}_{\triangle_i}, \mathbb{L}_{\triangle_i,\mathbb{C}}\right\}$ denoted by $\Sigma_{\triangle_i} \subset F, \forall i \in \Omega$ and for every non-constant L-subset $h \in F, h \in \Sigma_{\triangle_i}$ for at-least one $i \in \Omega$.

Let $\mathbb{E} = \{\beta \in \Re_{\overline{F}} : \beta \notin \bigcup_{i \in \Omega} \triangle_i\}$.

Thus $F = \left\{\underline{0}, \underline{1}, \underline{\beta}, f : \beta \in \mathbb{E} \text{ and } f \in \Sigma_{\triangle_i}, i \in \Omega\right\}$.

$\Rightarrow$ If F is a completely homogeneous L-topology on a countable set X, then F is equal to one of the L-topologies defined in the remark 5.1.

6. Completely Homogeneous L-Topological Spaces When X Is an Uncountable Set

Throughout this section, X stands for an uncountable set.

Remark 6.1. It can be checked that following are the disjoint equivalence classes with respect to the relation R when X is an uncountable set :

- $[0]$ contains only one completely homogeneous L-topology $\{\underline{0}, \underline{1}\}$.
- $[1]$ contains only one type of completely homogeneous L-topologies $\{\underline{0}, \underline{1}, \underline{\alpha}\}$, where $\alpha \in L \setminus \{0,1\}$ i.e.

  $[1] = \{\{\underline{0}, \underline{1}, \underline{\alpha}\} : \alpha \in L \setminus \{0,1\}\}$.
- $[2]$ contains following types of completely homogeneous L-topologies :

(i) $\{\{\underline{0},\underline{1},\underline{\alpha},\underline{\beta}\} : \alpha,\beta \in L\setminus\{0,1\}\}$.

(ii) $\{\underline{0},\underline{1},g : g \in Ł_H^X$, where $H \subseteq L$ and $|H| = 2\}$.

Let $\alpha_1,\alpha_2 \in L$ be two arbitrary elements such that $\alpha_1 < \alpha_2$ and $g_1,g_2 \in L^X$ be defined by

$$g_1(x) = \begin{cases} \alpha_1 \text{ for at-most finitely many } x \in X \\ \alpha_2 \text{ otherwise} \end{cases}$$

$$\text{and } g_2(x) = \begin{cases} \alpha_1 \text{ for at-most countably many } x \in X \\ \alpha_2 \text{ otherwise} \end{cases}.$$

(iii) L-topologies generated by the sets of the form $\{\underline{0},\underline{1},g_1oh : h \in S(X)\}$.

(iv) L-topologies generated by the sets of the form $\{\underline{0},\underline{1},\underline{\alpha_1},g_1oh : h \in S(X)\}$.

(v) L-topologies generated by the sets of the form $\{\underline{0},\underline{1},g_2oh : h \in S(X)\}$.

(vi) L-topologies generated by the sets of the form $\{\underline{0},\underline{1},\underline{\alpha_1},g_2oh : h \in S(X)\}$.

- For $m \geq 3$, $[m]$ contains following types of completely homogeneous L-topologies :

(i) $\{\underline{0},\underline{1},\underline{\alpha} : \alpha \in H_1$, where $H_1 \subseteq L\setminus\{0,1\}$ such that $H_1^\star \subseteq H_1$ and $|H_1| = m\}$.

(ii) $\{\underline{0},\underline{1},g : g \in Ł_H^X$, where $H \subseteq L$ such that $H^\star \subseteq H$ and $|H| = m\}$.

(iii) Let $H \subseteq L$ be any subset such that $H^\star \subseteq H$ and $|H| = m$.

Consider a c-subset $\triangle \subseteq H$, choose an arbitrary element $\gamma \in \triangle$ and define :

$\mathbb{P}_1 = \{\alpha \in \triangle : \alpha < (\leq)\gamma\}$,

$\mathbb{P}_2 = \{\beta \in \triangle : \gamma \leq (<)\beta\}$,

$\mathbb{P}_1^\star = \mathbb{P}_1 \setminus \{\gamma\}$,

$\mathbb{P}_2^\star = \mathbb{P}_2 \setminus \{\gamma\}$,

$$\mathbb{L}_\triangle^1 = \begin{cases} f \in L^X : \quad f(x) \in \mathbb{P}_1 \text{ for at-most finitely many } x \in X \\ \qquad\qquad f(x) \in \mathbb{P}_2 \text{ otherwise} \end{cases},$$

$$\mathbb{L}^2_{\triangle} = \begin{cases} f \in L^X : & f(x) \in \mathbb{P}_1 \text{ for at-most countably many } x \in X \\ & f(x) \in \mathbb{P}_2 \text{ otherwise} \end{cases},$$

$$\mathbb{L}^3_{\triangle} = \begin{cases} f \in L^X : & f(x) \in \mathbb{P}^\star_1 \text{ for at-most finitely many } x \in X \\ & f(x) = \gamma \text{ for at-most countably many } x \in X, \\ & f(x) \in \mathbb{P}^\star_2 \text{ otherwise} \end{cases}$$

$\mathbb{L}^1_{\triangle,\mathbb{C}} = \mathbb{L}^1_{\triangle} \cup \{\underline{\alpha} : \alpha \in \mathbb{C} \subseteq \mathbb{P}_1\}$,

$\mathbb{L}^2_{\triangle,\mathbb{C}} = \mathbb{L}^2_{\triangle} \cup \{\underline{\alpha} : \alpha \in \mathbb{C} \subseteq \mathbb{P}_1\}$,

$\mathbb{L}^3_{\triangle,\mathbb{C}} = \mathbb{L}^3_{\triangle} \cup \{\underline{\alpha} : \alpha \in \mathbb{C} \subseteq \mathbb{P}^\star_1 \cup \{\gamma\}\}$.

Consider a family $\triangle_i, i \in \Omega$ of distinct c-subsets of H and corresponding to each c-subset $\triangle_i, i \in \Omega$, choose exactly one set from the set $\left\{Ł^X_{\triangle_i}, \mathbb{L}^k_{\triangle_i}, \mathbb{L}^k_{\triangle_i,\mathbb{C}} : k = 1,2,3\right\}$ and denote that set by $\Sigma_{\triangle_i}$.

Let $\mathbb{E} = \{\beta \in H : \beta \notin \bigcup_{i\in\Omega} \triangle_i\}$.

The L-topologies of the form $\left\{\underline{0}, \underline{1}, \underline{\beta}, f : \beta \in \mathbb{E} \text{ and } f \in \Sigma_{\triangle_i}, i \in \Omega\right\}$.

Theorem 6.2. Let X be an uncountable set and L be a complete chain. If F is a completely homogeneous L-topology on X, then F is equal to one of the L-topologies defined in remark 6.1.

Proof. Let $|\mathfrak{R}_{\overline{F}}| = m$. If $m = 0$, then clearly $F = \{\underline{0}, \underline{1}\}$. So, assume that $m > 0$.

Case 1: If F contains only constant L-subsets, then $F = \{\underline{0}, \underline{1}, \underline{\alpha} : \alpha \in \mathfrak{R}_{\overline{F}}\}$.

Case 2: If $Ł^X_{\overline{F}} \subseteq F$, then $F = \{\underline{0}, \underline{1}, g : g \in Ł^X_{\overline{F}}\}$.

Case 3: Suppose Case 1 and Case 2 do not hold.

Let $f \in F$ be any non-constant L-subset.

Let $\triangle \subset \mathfrak{R}_{\overline{F}}$ be a subset such that

(i) $\mathfrak{R}_f \subseteq \triangle$.

(ii) $\triangle^\star \subseteq \triangle$.

(iii) for any two elements $\alpha, \beta \in \triangle, \exists$ an L-subset $g \in F$ such that $g(x) = \alpha, g(y) = \beta$ for some $x, y \in X$.

(iv) $\triangle$ is not properly contained in any proper subset of $\mathfrak{R}_{\overline{F}}$ satisfying above three properties.

In the same way, as in theorem 5.2, it can be shown that $\triangle$ is a c-subset.

Case (i) : If $Ł^X_\triangle \subset F$, then $f \in Ł^X_\triangle$.

Case (ii) : Let $Ł^X_\triangle \nsubseteq F$.

Let $\mathbb{D} = \{h \in F : \Re_h \subseteq \triangle\}$.

$Ł^X_\triangle \nsubseteq F \Rightarrow \exists$ some element(s) $\lambda \in \triangle$ such that if $h \in \mathbb{D}$ and $h(x) = \lambda$ for some $x \in X$, then $h(y) = \lambda$ for at-most finitely/countably many $x \in X$.

Let $\mathbb{P}_1 = \{\lambda \in \triangle :$ if $\lambda \in \Re_h$ for some $h \in \mathbb{D}$, then $h(x) = \lambda$ for at-most finitely many $x \in X\}$.

and $\mathbb{P}_2 = \{\eta \in \triangle :$ if $\eta \in \Re_g$ for some $g \in \mathbb{D}$, then $g(x) = \eta$ for at-most countably many $x \in X\}$. Clearly, $\mathbb{P}_1 \subseteq \mathbb{P}_2$.

It can be checked that

(i) if $\alpha, \beta \in \mathbb{P}_1(\mathbb{P}_2)$ and $\eta \in \triangle$ such that $\alpha < \eta < \beta$, then $\eta \in \mathbb{P}_1(\mathbb{P}_2)$.

(ii) if $\alpha \in \mathbb{P}_1$ and $\eta \in \triangle$ such that $\eta < \alpha$, then $\eta \in \mathbb{P}_1$.

$$\text{Let } \mathbb{L}_\triangle = \begin{cases} g \in L^X : & g(x) \in \mathbb{P}_1 \text{ for at-most finitely many } x \in X \\ & g(x) \in \mathbb{P}_2 \setminus \mathbb{P}_1 \text{ for at-most countably many } x \in X \\ & g(x) \in \triangle \setminus \{\mathbb{P}_2\} \text{ otherwise} \end{cases}$$

Now two cases arise:

Case (a) : when $\underline{\alpha} \notin F, \forall \alpha \in \mathbb{P}_2$.

Then $\mathbb{L}_\triangle \subseteq F$.

Case (b) : when $\underline{\alpha} \in F$ for all / some $\alpha \in \mathbb{P}_2$.

Let $\mathbb{C} = \{\alpha \in \mathbb{P}_2 : \underline{\alpha} \in F\}$ and $\mathbb{L}_{\triangle,\mathbb{C}} = \mathbb{L}_\triangle \cup \mathbb{C}$.

Thus either $f \in \mathbb{L}_\triangle$ or $f \in \mathbb{L}_{\triangle,\mathbb{C}}$.

Let $\triangle_i, i \in \Omega$ be the collection of those distinct c-subsets of $\Re_{\overline{F}}$ such that corresponding to each c-subset $\triangle_i, i \in \Omega$, exactly one set from the set $\left\{Ł^X_{\triangle_i}, \mathbb{L}_{\triangle_i}, \mathbb{L}_{\triangle_i,\mathbb{C}}\right\}$ denoted by $\Sigma_{\triangle_i} \subset F, \forall i \in \Omega$ and for every non-constant L-subset $h \in F, h \in \Sigma_{\triangle_i}$ for at-least one $i \in \Omega$.

Let $\mathbb{E} = \{\beta \in \Re_{\overline{F}} : \beta \notin \bigcup_{i\in\Omega} \triangle_i\}$.

Thus $F = \left\{\underline{0}, \underline{1}, \underline{\beta}, f : \beta \in \mathbb{E} \text{ and } f \in \Sigma_{\triangle_i}, i \in \Omega\right\}$.

$\Rightarrow$ If F is a completely homogeneous L-topology on an uncountable set X, then F is equal to one of the L-topologies defined in the remark 6.1.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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