Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 4248-4283
https://doi.org/10.28919/jmcs/5533
ISSN: 1927-5307

COMPLEXITY OF SOME GRAPHS GENERATED BY SQUARE

MOHAMED R. ZEEN EL DEEN*, WALAA A. ABOAMER

Department of Mathematics and Computer Science, Faculty of Science, Suez University, Suez 42524, Egypt

Abstract. Complexity plays a vital and significant role when designing communication networks (graphs). The more quality and perfect the network, the greater the number of trees spanning this network, which leads to greater possibilities of connection between two vertices, and this ensures good rigidity and resistance. In this work, we present nine network designs created by a square of different average degree 4, 6 and 8, then we deduce a simpler and evident formula expressing the number of spanning trees of these networks using some basic properties of orthogonal polynomials, block matrix analysis technique, and recurrence relations. In addition, we compute the entropy of each network and determine the best by comparing these designs using network entropy. Finally, we compare the entropy of spanning trees on our networks with other triangle and Apollonian networks and observe the entropy of our networks, which is the highest among the triangle and Apollonian networks studied.

Keywords: complexity; recurrence relation; Chebyshev polynomials; duplication of graphs; entropy.
2010 AMS Subject Classification: 05C05, 05C50, 05C90.

1. INTRODUCTION

In crowded places and occasions telecommunications companies facing significant problems, when someone makes contact with another person and the lines are busy. So another empty communication channel must be provided in order for the communication to take place.

*Corresponding author
E-mail address: mohamed.zeeneldeen@suezuniv.edu.eg
Received February 10, 2021

The good company try to increase the number of communication channels, these demands to calculate the number of different tracks (the total number of spanning trees) available within each network and choosing the best design to grantee this increasing.

A graph is a formal mathematical illustration of a network since any network can be modeled by a graph G where nodes are represented by vertices $V(G)$ and links are represented by edges $E(G)$. Let $|E(G)|$ be the cardinality of $E(G)$ and $|V(G)|$ be the cardinality of $V(G)$. We deal with finite and undirected with multiple edges and loops permitted graphs. The degree of a vertex $x \in V(G)$ is the number of edges incident with the vertex, while the average degree of a graph is applied to measure the number of edges compared to the number of vertices which calculates by dividing the summation of all vertex degrees by the total number of nodes.

A spanning tree of any graph is a communication subgraph that guarantees connectivity between all vertices of the original graph with a minimum number of edges. In other words, a spanning tree ensures the existence and uniqueness of a connection between any pair of vertices. The number of spanning trees $\tau(G)$ is equal to the total number of various spanning subgraphs of G that are trees, this quantity is also known as complexity $\tau(G)$ of G.

Many new graphs can be generated from a given pair of graphs using graph operations [1]. For every vertex $x \in V(G)$, the open neighborhood set $N(x)$ is the set of all adjacent vertices to x in G. Duplication of an edge $e = xy$ in a graph G by a new vertex z creates a new graph G' such that $N(z) = \{x, y\}$. Duplication of a vertex x in a graph G by a new edge $e = uv$ creates a new graph G' such that $N(u) \cap N(v) = \{x\}$.

There are several methods of finding this number. The celebrated matrix tree theorem of Kirchhoff [2] tells us that: the complexity $\tau(G)$ of a graph G is equal any cofactor of Laplace matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix of G. $\tau(G)$ can also be calculated from the eigenvalues of the Kirchhoff matrix H. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n (= 0)$ denote the eigenvalues of H matrix. Kelmans and Chenlnokov [3], have shown that

$$\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i \,. \tag{1}$$

After that Temperley [4] , has shown that:

$$\tau(G) = \frac{1}{n^2}\, det(\, H \,+\, J\,)\,, \tag{2}$$

where J is the $n \times n$ matrix all of whose elements are unity.

From Temperley's Equation (2), it is easy to prove the following lemma.

Lemma 1.1. *Let G be a graph with n vertices and $\overline{D}$, $\overline{A}$ are the degree and adjacency matrices, respectively, of $\overline{G}$, the complement of G. Then,*

$$\tau(G\,) = \frac{1}{n^2}\, det(\, nI - \overline{D} + \overline{A}\,)\,. \tag{3}$$

The senior advantage of the formula (3) is that it directly expresses $\tau(G\,)$ as a determinant rather than in terms of co-factors or eigenvalues.

For some special classes of graphs, there are simple closed formulas that make it much easier to calculate and determine the number of corresponding spanning trees, especially when these numbers are very large. Cayley showed that a complete graph K_n has $n^{n-2}, n \geq 2$ spanning trees [2]. Another result is due to Sedlacek [5], he derived a formula for the wheel with $n+1$ vertices, W_{n+1}, has a number of spanning trees $(\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2,\ n \geq 2$. Recently, several closed formulas have been published for counting and maximizing the number of spanning trees for some families of graphs (see, [6–11]).

2. Basic Proof Tools

There exists a powerful relation concerning orthogonal polynomials, especially the Chebyshev polynomials of the first and second kinds and determinants that we use in our computations. For positive integer n, the Chebyshev polynomials of the first kind are defined by [12]:

For positive integer m, the Chebyshev polynomials of the first kind are defined by:

$$T_n(x) = \cos(n \arccos x)\,. \tag{4}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)}\,. \tag{5}$$

The Chebyshev polynomials of the second kind satisfy the recursion relation

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0\ . \tag{6}$$

Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2x & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2x & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2x & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2x \end{pmatrix}.$$

From this recursion relation and by expanding $\det A_n(x)$, one obtains:

$$U_n(x) = det(A_n(x)),\ n \geq 1\ . \tag{7}$$

Using standard methods for solving the recursion (6), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}],\ n \geq 1\ ,\ x = \pm 1\ . \tag{8}$$

Lemma 2.1. *[13] If*

$$B_n(x) = \begin{pmatrix} x & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & (x+1) & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & (x+1) & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 & (x+1) & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & x \end{pmatrix},\ \textit{then}\ \ det\ (B_n(x)) = (x-1)\, U_{n-1}(\tfrac{1+x}{2})\,.$$

Lemma 2.2. *[8] If*

$$E_n(x) = \begin{pmatrix} x & 1 & 1 & 1 & \dots & 1 \\ 1 & x & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & x & 1 \\ 1 & 1 & 1 & \dots & 1 & x \end{pmatrix},\ \textit{then}\ \ det\ (E_n(x)) = (x+n-1)\,(x-1)^{n-1}\,.$$

Lemma 2.3. Let P, Q and R be matrices of dimension $n \times n$, then

$$det \begin{pmatrix} P & Q & R & Q \\ Q & P & Q & R \\ R & Q & P & Q \\ Q & R & Q & P \end{pmatrix} = [det(P-R)]^2 \, det(P+R+2Q) \, det(P+R-2Q).$$

Proof. Using the properties of determinants and matrix row and column operations [17] yields:

$$\begin{pmatrix} P & Q & R & Q \\ Q & P & Q & R \\ R & Q & P & Q \\ Q & R & Q & P \end{pmatrix} = \begin{pmatrix} P-R & O & R & Q \\ O & P-R & Q & R \\ R-P & O & P & Q \\ O & R-P & Q & P \end{pmatrix} = \begin{pmatrix} P-R & O & R & Q \\ O & P-R & Q & R \\ O & O & P+R & 2Q \\ O & O & 2Q & P+R \end{pmatrix}, \text{ then}$$

$$\begin{aligned} det \begin{pmatrix} P & Q & R & Q \\ Q & P & Q & R \\ R & Q & P & Q \\ Q & R & Q & P \end{pmatrix} &= det \begin{pmatrix} P-R & O \\ O & P-R \end{pmatrix} \times det \begin{pmatrix} P+R & 2Q \\ 2Q & P+R \end{pmatrix} \\ &= [det(P-R)]^2 \, det(P+R+2Q) \, det(P+R-2Q) \end{aligned} \tag{9}$$

□

3. Complexity of Some Families of Graphs Generated by a Square

Suppose we have four transmission sources in a broadcasting network, and we want to reach all the nodes in that network without closing that circuit linking those nodes. Therefore, we are going to study this problem in the presence of various models linking these nodes to determine which of these models are the best to connect all nodes.

3.1. Complexity of some families of graphs generated by a square with average degree 4.

Theorem 3.1. *For $n \geq 1$, the number of the spanning trees of the family of graphs Θ_n is given by:*

$$\tau(\Theta_n) = \frac{1}{144} \left[(3+\sqrt{3})(2+\sqrt{3})^n + (3-\sqrt{3})(2-\sqrt{3})^n \right]^2 \left[(2+\sqrt{2})(3+2\sqrt{2})^n + (2-\sqrt{2})(3-2\sqrt{2})^n \right].$$

Proof. The Kirchhoff matrix associated with the graph Θ_n is,

Consider the family of graphs Θ_n, generated by a square, and constructed as shown in Fig. 1, according to the construction, the number of vertices in the graph Θ_n are $|V(\Theta_n)| = 4n+1$ and edges $|E(\Theta_n)| = 8n,\ n = 1,2,\cdots$. When n is large, the average degree of Θ_n is 4.

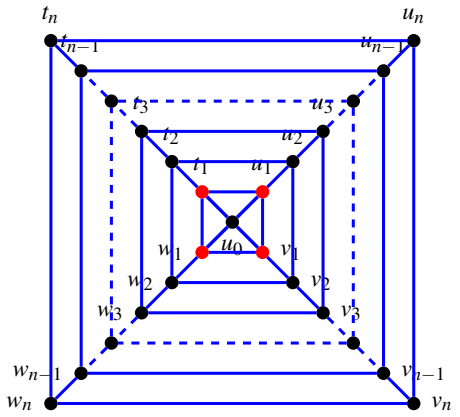

FIGURE 1. the graph Θ_n

$$H = \begin{pmatrix}
4 & -1 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 \\
-1 & 4 & -1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 \\
0 & -1 & 4 & -1 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 \\
0 & 0 & -1 & 4 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 \\
\vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 4 & -1 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\
0 & 0 & 0 & \dots & -1 & 3 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\
-1 & -1 & 0 & 0 & \dots & 0 & 4 & -1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & -1 & 0 & \dots & 0 & -1 & 4 & -1 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & -1 & \dots & 0 & 0 & -1 & 4 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\
0 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 4 & -1 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & -1 & 3 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\
-1 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & -4 & -1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & -1 & 4 & -1 & \dots & 0 & 0 & -1 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & -1 & 4 & \dots & 0 & 0 & 0 & -1 & \dots & 0 \\
\vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 4 & -1 & 0 & 0 & \dots & -1 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & -1 & 3 & 0 & 0 & \dots & 0 & -1 \\
-1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 4 & -1 & 0 & \dots & 0 \\
0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & -1 & 4 & -1 & \dots & 0 \\
0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & -1 & 4 & \dots & 0 \\
\vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\
0 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 4 & -1 \\
0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & -1 & 3
\end{pmatrix}_{4n+1.4n+1}.$$

Thus, we get:

$$\tau(\Theta_n) = det \begin{pmatrix}
4 & -1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 \\
-1 & 4 & -1 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 \\
0 & -1 & 4 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 \\
\vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\
0 & 0 & \dots & 4 & -1 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\
0 & 0 & \dots & -1 & 3 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\
-1 & 0 & 0 & \dots & 0 & 4 & -1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & -1 & 0 & \dots & 0 & -1 & 4 & -1 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & -1 & \dots & 0 & 0 & -1 & 4 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\
0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 4 & -1 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & -1 & 3 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & -4 & -1 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & -1 & 4 & -1 & \dots & 0 & 0 & -1 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & -1 & 4 & \dots & 0 & 0 & 0 & -1 & \dots & 0 \\
\vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 4 & -1 & 0 & 0 & \dots & -1 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & -1 & 3 & 0 & 0 & \dots & 0 & -1 \\
-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 4 & -1 & 0 & \dots & 0 \\
0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & -1 & 4 & -1 & \dots & 0 \\
0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & -1 & 4 & \dots & 0 \\
\vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots & \vdots & \vdots \\
0 & 0 & \dots & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 4 & -1 \\
0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & -1 & 3
\end{pmatrix}_{4n.4n}$$

$$\tau(\Theta_n) \;=\; det \begin{pmatrix} P & Q & O & Q \\ Q & P & Q & O \\ O & Q & P & Q \\ Q & O & Q & P \end{pmatrix} = det \begin{pmatrix} P & O & O & Q \\ O & P & Q & O \\ O & O & P & 2Q \\ O & O & 2Q & P \end{pmatrix}.$$

Applying Lemma (1.1), we get:

$$\tau(\Theta_n) \;=\; [\,[det(P)]^2 \;\times\; det(P+2Q) \;\times\; det(P-2Q)\,]. \tag{10}$$

$$det(P) = det \begin{pmatrix} 4 & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 4 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 4 & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 4 & -1 \\ 0 & 0 & 0 & \dots & \dots & -1 & 3 \end{pmatrix}.$$

Expanding the first seven determinants of P gives the values $3, 11, 41, 153, 571, 2131, \cdots$ for $n = 1, 2, 3, 4, 5, 6, \cdots$, these values can be written in the form

$$41 = 4 \times 11 - 3,\ 153 = 4 \times 41 - 11,$$
$$571 = 4 \times 153 - 41,\ 2131 = 4 \times 571 - 153, \cdots.$$

Consequently, we have the following homogeneous recurrence relation

$$a_{n+2} = 4\, a_{n+1} - a_n. \tag{11}$$

Its characteristic equation is $r^2 - 4r + 1 = 0$, with two roots being $2+\sqrt{3}$ and $2-\sqrt{3}$. The general solution of the Recurrence Relation (11), is $a_n = \alpha\,(2+\sqrt{3})^n + \beta\,(2-\sqrt{3})^n$. Using the initial conditions $det(P) = 3, 11$ at $n = 1, 2,$ respectively, we have:

$$3 = \alpha\,(2+\sqrt{3}) + \beta\,(2-\sqrt{3}),$$
$$11 = \alpha\,(2+\sqrt{3})^2 + \beta\,(2-\sqrt{3})^2.$$

Solving these equations, we get: $\alpha = \dfrac{3+\sqrt{3}}{6}$ and $\beta = \dfrac{3-\sqrt{3}}{6}$. Therefore

$$det(P) = \left[\left(\frac{3+\sqrt{3}}{6}\right)(2+\sqrt{3})^n + \left(\frac{3-\sqrt{3}}{6}\right)(2-\sqrt{3})^n \right]. \tag{12}$$

Using induction and some properties of determinants, we obtain,

$$det(P+2Q) = det \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 2 & -1 \\ 0 & 0 & \dots & \dots & \dots & -1 & 1 \end{pmatrix} = 1. \tag{13}$$

$$det(P-2Q) = det \begin{pmatrix} 6 & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 6 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 6 & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 6 & -1 \\ 0 & 0 & 0 & \dots & \dots & -1 & 5 \end{pmatrix}$$

Expanding the first seven determinants of $(P-2Q)$, gives the values $5, 29, 169, 985, 5741, 33461, \cdots$ for $n = 1,2,3,4,5,6, \cdots$, these values can be written in the form,

$$169 = 6 \times 29 - 5,\ 985 = 6 \times 169 - 29,$$
$$5741 = 6 \times 985 - 169,\ 33461 = 6 \times 5741 - 985\ , \cdots$$

Consequently, we have the following homogeneous recurrence relation

$$a_{n+2} = 6\, a_{n+1} - a_n. \tag{14}$$

Its characteristic equation is $r^2 - 6r + 1 = 0$ with two roots being $3+2\sqrt{2}$ and $3-2\sqrt{2}$. The general solution of the Recurrence Relation (14), is $a_n = \alpha\,(3+2\sqrt{2})^n + \beta(3-2\sqrt{2})^n$. Using the initial conditions, we get: $\alpha = \dfrac{2+\sqrt{2}}{4}$ and $\beta = \dfrac{2-\sqrt{2}}{4}$. Therefore:

$$det(P-2Q) = \ [\,(\frac{2+\sqrt{2}}{4})(3+2\sqrt{2})^n + (\frac{2-\sqrt{2}}{4})(3-2\sqrt{2}\,)^n\,]. \tag{15}$$

From Eq.(12), Eq.(13) and Eq.(15) in Eq.(10), we get:

$$\tau(\Theta_n) = \frac{1}{144}\,[\,(3+\sqrt{3})(2+\sqrt{3})^n + (3-\sqrt{3})(2-\sqrt{3}\,)^n\,]^2\,[\,(2+\sqrt{2})(3+2\sqrt{2})^n + (2-\sqrt{2})(3-2\sqrt{2}\,)^n\,].$$

□

Consider the family of graphs Δ_n , generated by a square, and constructed as the cross product $\Delta_n = C_4 \square P_n$ of a square C_4 and the path P_n [1], see Fig. 2. The graph Δ_n has the number of vertices $|V(\Delta_n)| = 4n$ and the edges $|E(\Delta_n)| = 8n-4,\ n = 1,2,\cdots$. When n is large, the average degree of Δ_n is 4 .

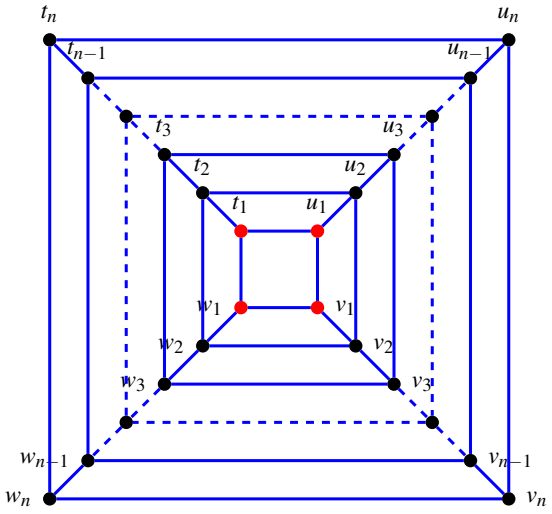

FIGURE 2. The graph Δ_n.

Theorem 3.2. *For $n \geq 1$, the number of the spanning trees of the graph $\Delta_n = C_4 \times P_n$ is given by*

$$\tau(\Delta_n) = \frac{1}{12\sqrt{2}}\,[\,(2+\sqrt{3})^n - (2-\sqrt{3})^n\,]^2\,[(3+2\sqrt{2})^n - (3-2\sqrt{2}\,)^n]\,.$$

Proof. Applying Lemma (1.1) and Lemma(2.3), we have:

$$\tau(\Delta_n) = \frac{1}{(4n)^2}\,det\,[\,4nI - \overline{D} + \overline{A}\,]\,.$$

$$(16)\qquad \tau(\Delta_n) = \frac{1}{(4n)^2} det \begin{pmatrix} P & Q & J & Q \\ Q & P & Q & J \\ J & Q & P & Q \\ Q & J & Q & P \end{pmatrix} = \frac{1}{(4n)^2}[\,det(P-J)]^2 \times det(P+J-2Q) \times det(P+J+2Q)\,]$$

$$\text{where } P=\begin{pmatrix} 4 & 0 & 1 & \dots & 1 & 1 \\ 0 & 5 & 0 & 1 & \dots & 1 \\ 1 & 0 & 5 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 5 & 0 \\ 1 & 1 & \dots & \dots & 0 & 4 \end{pmatrix}_{n.n}, Q=\begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 1 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}_{n.n},$$

$$\tau(\Delta_n) = \frac{1}{(4n)^2} \left(det \begin{pmatrix} 3 & -1 & 0 & \dots & 0 \\ -1 & 4 & -1 & \dots & 0 \\ 0 & -1 & 4 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 4 & -1 \\ 0 & 0 & \dots & -1 & 3 \end{pmatrix} \right)^2 \times det \begin{pmatrix} 5 & -1 & 0 & \dots & 0 \\ -1 & 6 & -1 & \dots & 0 \\ 0 & -1 & 6 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & 0 & \dots & 6 & -1 \\ 0 & 0 & \dots & -1 & 5 \end{pmatrix} \times$$

$$det \begin{pmatrix} 5 & 3 & 4 & \dots & 4 \\ 3 & 6 & 3 & \dots & 4 \\ 4 & 3 & 6 & \dots & 4 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 4 & 4 & \dots & 6 & 3 \\ 4 & 4 & \dots & 3 & 5 \end{pmatrix}.$$

From Lemma (2.1), we obtain:

(17)

$$det(P-J)=det \begin{pmatrix} 3 & -1 & 0 & \dots & \dots & 0 \\ -1 & 4 & -1 & \dots & \dots & 0 \\ 0 & -1 & 4 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 4 & -1 \\ 0 & 0 & \dots & \dots & -1 & 3 \end{pmatrix} = 2\,U_{n-1}(2) = \frac{1}{\sqrt{3}}[\,(2+\sqrt{3})^n-(2-\sqrt{3}\,)^n].$$

(18)

$$det(P+J-2Q) = det \begin{pmatrix} 5 & -1 & 0 & \dots & \dots & 0 \\ -1 & 6 & -1 & \dots & \dots & 0 \\ 0 & -1 & 6 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 6 & -1 \\ 0 & 0 & \dots & \dots & -1 & 5 \end{pmatrix} = 4U_{n-1}(3) = \frac{1}{\sqrt{2}}[(3+2\sqrt{2})^n - (3-2\sqrt{2})^n]$$

$$det(P+J+2Q) = det \begin{pmatrix} 5 & 3 & 4 & 4 & \dots & \dots & 4 & 4 \\ 3 & 6 & 3 & 4 & \dots & \dots & 4 & 4 \\ \vdots & \vdots & \dots & \dots & \vdots & \ddots & \ddots & \vdots \\ 4 & 4 & \dots & \dots & \dots & 3 & 6 & 3 \\ 4 & 4 & \dots & \dots & \dots & 4 & 3 & 5 \end{pmatrix} = 4\, det \begin{pmatrix} 2 & 0 & 1 & 1 & \dots & \dots & 1 & 1 \\ 0 & 3 & 0 & 1 & \dots & \dots & 1 & 1 \\ \vdots & \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & \dots & 0 & 3 & 0 \\ 1 & 1 & \dots & \dots & \dots & 1 & 0 & 2 \end{pmatrix}.$$

Using the induction and the properties of determinants, we obtain the value

(19) $$det(P+J+2Q) = 4\, n^2\ .$$

From Eq. (17) , Eq. (18) and Eq. (19) in Eq. (16), we get:

$$\tau(\Delta_n) = \frac{1}{12\sqrt{2}}\,[\,(2+\sqrt{3})^n - (2-\sqrt{3})^n\,]^2\,[(3+2\sqrt{2})^n - (3-2\sqrt{2}\,)^n].$$ □

Consider the family of graphs Λ_n, generated by a square, and constructed starting with a square and finding its line graph [1], one gets an inner square. Repeating this process to the new interior square given the graph shown in Fig. 3, the graph Λ_n has the number of vertices $|V(\Lambda_n)| = 4n$ and the edges $|E(\Lambda_n)| = 8n - 4$, $n = 2,3,\cdots$. When n is large, the average degree of Λ_n is 4.

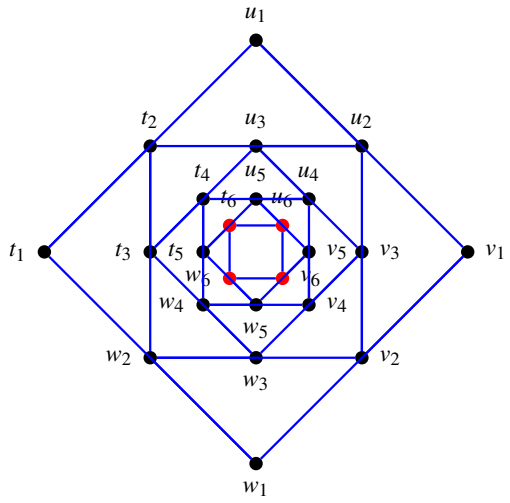

FIGURE 3. The graph Λ_6.

Theorem 3.3. *For $n \geq 2$, the number of the spanning trees of the graph Λ_n is given by*

$$\tau(\Lambda_n) = 3\,2^{3n-7}\,[\,(6+4\sqrt{2})^n + (6-4\sqrt{2}\,)^n + 2^{n+1}\,]\,.$$

Proof. Applying Lemma 1.1, we have: $\tau(\Lambda_n) = \dfrac{1}{(4n)^2}\, det\,[\,4nI - \overline{D} + \overline{A}\,]\,.$

$$\tau(\Lambda_n) = \frac{1}{(4n)^2} det \begin{pmatrix} P & Q & J & Q^t \\ Q^t & P & Q & J \\ J & Q^t & P & Q \\ Q & J & Q^t & P \end{pmatrix} = \frac{1}{(4n)^2} det \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n)^2}[det(L+M) \times det(L-M)] \tag{20}$$

where $P = \begin{pmatrix} 3 & 0 & 1 & \dots & 1 & 1 \\ 0 & 5 & 0 & 1 & \dots & 1 \\ 1 & 0 & 5 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 5 & 0 \\ 1 & 1 & \dots & \dots & 0 & 5 \end{pmatrix}_{n.n}$, $Q = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 1 & \dots & \dots & 0 & 0 \end{pmatrix}_{n.n}$,

$$det(L+M) = det \begin{pmatrix} 4 & 1 & 2 & \dots & \dots & 2 & 2 & 1 & 2 & \dots & \dots & 2 \\ 1 & 6 & 1 & 2 & \dots & 2 & 1 & 2 & 1 & 2 & \dots & 2 \\ 2 & 1 & 6 & 1 & \dots & 2 & 2 & 1 & 2 & 1 & \dots & 2 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & \dots & 6 & 1 & 2 & 2 & \dots & \dots & 2 & 1 \\ 2 & 2 & \dots & \dots & 1 & 6 & 2 & 2 & \dots & \dots & 1 & 0 \\ 2 & 1 & 2 & \dots & \dots & 2 & 4 & 1 & 2 & \dots & \dots & 2 \\ 1 & 2 & 1 & 2 & \dots & 2 & 1 & 6 & 1 & 2 & \dots & 2 \\ 2 & 1 & 2 & 1 & \dots & 2 & 2 & 1 & 6 & 1 & \dots & 2 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & \dots & 2 & 1 & 2 & 2 & \dots & \dots & 6 & 1 \\ 2 & 2 & \dots & \dots & 1 & 0 & 2 & 2 & \dots & \dots & 1 & 6 \end{pmatrix} = det \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$$

$$det(L+M) = det \begin{pmatrix} 6 & 2 & 4 & 4 & \dots & \dots & 4 \\ 2 & 8 & 2 & 4 & \dots & \dots & 4 \\ 4 & 2 & 8 & 2 & \dots & \dots & 4 \\ 4 & 4 & 2 & 8 & \dots & \dots & 4 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 4 & 4 & 4 & \dots & \dots & 8 & 2 \\ 4 & 4 & 4 & \dots & \dots & 2 & 6 \end{pmatrix} \times det \begin{pmatrix} 2 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 4 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 4 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 4 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 4 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 6 \end{pmatrix}.$$

Using the properties of the determinants, we arrive at:

$$det(L+M) = 2^n det \begin{pmatrix} 3 & 1 & 2 & 2 & \dots & \dots & 2 \\ 1 & 4 & 1 & 2 & \dots & \dots & 2 \\ 2 & 1 & 4 & 1 & \dots & \dots & 2 \\ 2 & 2 & 1 & 4 & \dots & \dots & 2 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & \dots & 4 & 1 \\ 2 & 2 & 2 & \dots & \dots & 1 & 3 \end{pmatrix} \times 3\, 2^{2n-2} = 3\, n^2\, 2^{3n-1}. \tag{21}$$

$$det(L-M) = det \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & \dots & 0 \\ -1 & 4 & -1 & 0 & \dots & 0 & -1 & 0 & -1 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & \dots & 0 & 0 & 1 & 0 & 1 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 4 & -1 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & -1 & 4 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 & 2 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & -1 & 4 & -1 & 0 & \dots & 0 \\ 0 & -1 & 0 & -1 & \dots & 0 & 0 & -1 & 4 & -1 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 & 1 & 0 & 0 & \dots & \dots & 4 & -1 \\ 0 & 0 & \dots & \dots & -1 & 0 & 0 & 0 & \dots & \dots & -1 & 4 \end{pmatrix}.$$

Expanding the first seven determinants of $(L - M)$, gives the values $36, 400, 4624, 53824, 627264, 7311616, 85229824 \cdots$ for $n = 2,3,4,5,6,7,8, \cdots$. These values can be written in the form:

$53824 = 12 \times 4624 - 4 \times 400 - 64$, $627264 = 12 \times 53824 - 4 \times 4624 - 128$,

$7311616 = 12 \times 627264 - 4 \times 53824 - 256$,

$85229824 = 12 \times 7311616 - 4 \times 627264 - 512$.

Consequently, we have the following non-homogeneous recurrence relation

$$a_{n+2} = 12\, a_{n+1} - 4\, a_n - 8\, 2^n. \tag{22}$$

Let $a_n = b_n + d\, 2^n$ be the solution of the non-homogeneous Recurrence Relation Eq. (22), where b_n is the solution of the homogeneous recurrence relation $b_{n+2} - 12\, b_{n+1} + 4b_n = 0$. Substituting in Eq. (22), we get, $d = \frac{1}{2}$. The characteristic equation of homogeneous recurrence relation is $r^2 - 12r + 4 = 0$ with two roots being $6 + 4\sqrt{2}$ and $6 - 4\sqrt{2}$,then

$b_n = \alpha\, (6 + 4\sqrt{2})^n + \beta (6 - 2\sqrt{2})^n$. The general solution of the non-homogeneous Recurrence Relation Eq. (22), is:

$$a_n = \alpha\, (6 + 4\sqrt{2})^n + \beta (6 - 4\sqrt{2})^n + 2^{n-1}\, .$$

Using the initial conditions $det(L - M) = 400$, 4624 to $n = 3,4,$ respectively, we get $\alpha = \beta = \dfrac{1}{4}$. Therefore

$$det(L - M) = a_n = \frac{1}{4}\, [\, (6 + 4\sqrt{2})^n + (6 - 4\sqrt{2})^n\,] + 2^{n-1}. \tag{23}$$

From Eq. (21) and Eq.(23) in Eq. (37), we get:

$$\tau(\Lambda_n) = 3\, 2^{3n-7}\, [\, (6 + 4\sqrt{2})^n + (6 - 4\sqrt{2}\,)^n] + 2^{n+1}\, .$$ □

Consider the family of graphs Ψ_n, generated by a square, and constructed by duplicating each vertex u of the square by vertices u_i' $i = 2,3,4,\cdots,n$ such that $N(u) = N(u_i')$, see Fig. 4. Based on construction, the total number of vertices $|V(\Psi_n)|$ and edges $|E(\Psi_n)|$ are $|V(\Psi_n| = 4n$ and $|E(\Psi_n)| = 8n-4$, $n = 2,3,4,\cdots$. It is notice that, when n is large, the average degree of Ψ_n is 4.

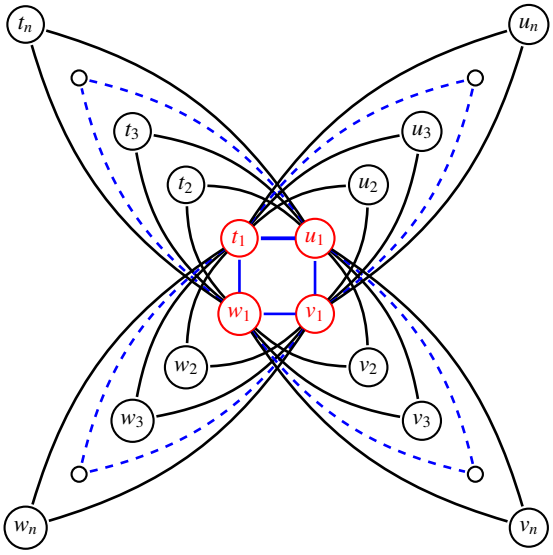

FIGURE 4. The graph Ψ_n .

Theorem 3.4. *For $n \geq 2$, the number of the spanning trees of the graph Ψ_n is given by $\tau(\Psi_n) = n^2\, 2^{4n-2}$.*

Proof. Applying Lemma 1.1, we have:

$$\tau(\Psi_n) = \frac{1}{(4n)^2}\, det\,[\,4nI - \overline{D} + \overline{A}\,] = \frac{1}{(4n)^2}\, det \begin{pmatrix} P & Q & J & Q \\ Q & P & Q & J \\ J & Q & P & Q \\ Q & J & Q & P \end{pmatrix}.$$

Applying Lemma 2.3, with the matrix $R = J$ where J is the unit matrix, we obtain:

$$\tau(\Psi_n) = \frac{1}{(4n)^2}[det(P-J)]^2 \times det(P+J+2Q) \times det(P+J-2Q)],$$

where $P = \begin{pmatrix} 2n+1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 3 & 1 & 1 & \dots & 1 \\ 1 & 1 & 3 & 1 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 3 & 1 \\ 1 & 1 & \dots & \dots & 1 & 3 \end{pmatrix}_{n.n}$, $Q = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & 1 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \dots & \dots & 1 & 1 \\ 0 & 1 & \dots & \dots & 1 & 1 \end{pmatrix}_{n.n}$,

2

$$\tau(\Psi_n) \;=\; \frac{1}{(4n)^2} \qquad \left(det \begin{pmatrix} 2n & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 2 & 0 \\ 0 & 0 & \dots & 0 & 2 \end{pmatrix} \right) \;\times\; det \begin{pmatrix} 2n+2 & 2 & 2 & \dots & 2 \\ 2 & 6 & 4 & \dots & 4 \\ 2 & 4 & 4 & \dots & 4 \\ \vdots & \dots & \dots & \dots & \vdots \\ 2 & 4 & \dots & 6 & 4 \\ 2 & 4 & \dots & 4 & 6 \end{pmatrix} \times$$

$$det \begin{pmatrix} 2n+2 & 2 & 2 & \dots & 2 \\ 2 & 2 & 0 & \dots & 0 \\ 2 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 2 & 0 & \dots & 2 & 0 \\ 2 & 0 & \dots & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{(4n)^2} \;\; (\; n\, 2^n \;)^2 \times \;\; 4n \; det \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 1 & 4 & 2 & \dots & \dots & 2 \\ 1 & 2 & 4 & \dots & \dots & 2 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & 2 & \dots & 2 & 4 & 2 \\ 1 & 2 & \dots & \dots & 2 & 4 \end{pmatrix}_{n.n} \qquad \times 2^{n+1} \;,$$

$$= \frac{1}{(4n)^2} \;\; (\; n\, 2^n \;)^2 \times \;\; 4n \; 2^{n-1} det \begin{pmatrix} 2 & 1 & 1 & \dots & \dots & 1 \\ 1 & 2 & 1 & \dots & \dots & 1 \\ 1 & 1 & 2 & \dots & \dots & 1 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 2 & 1 \\ 1 & 1 & \dots & \dots & 1 & 2 \end{pmatrix}_{n-1.n-1} \qquad \times 2^{n+1} \;.$$

Applying Lemma (2.2), with $x = 2$, we get:

$$\tau(\Psi_n) = \frac{1}{(4n)^2} \; (\; n\, 2^n \;)^2 \;\times\; 2^{n+1}\, n^2 \;\times\; 2^{n+1} \;=\; n^2\, 2^{4n-2}. \qquad \square$$

Consider the family of graphs Γ_n, generated by a square and constructed as shown in Fig. 5. It is noticed that according to the construction, the number of total vertices $|V(\Gamma_n)|$ are $|V(\Gamma_n)| = 4n$ and edges $|E(\Gamma_n)| = 8n-4$, $n = 1,2,3,4,\cdots$. When n is large , the average degree of Γ_n is 4.

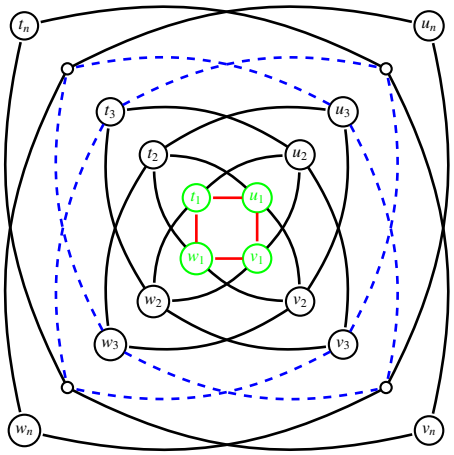

FIGURE 5. The graph Γ_n .

Theorem 3.5. *For $n \geq 1$, the number of the spanning trees of the graph Γ_n is given by:* $\tau(\Gamma_n) = 2^{6n-4}$.

Proof. Applying Lemma (1.1), we have:

$$\tau(\Gamma_n) = \frac{1}{(4n)^2}\, det\,[\,4nI - \overline{D} + \overline{A}\,] = \frac{1}{(4n)^2}\, det \begin{pmatrix} P & Q & J & Q \\ Q & P & Q & J \\ J & Q & P & Q \\ Q & J & Q & P \end{pmatrix}.$$

$$\tau(\Gamma_n) = \frac{1}{(4n)^2} det \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n)^2}[det(L+M) \times det(L-M)]. \tag{24}$$

where $P = \begin{pmatrix} 5 & 1 & 1 & \dots & 1 & 1 \\ 1 & 5 & 1 & 1 & \dots & 1 \\ 1 & 1 & 5 & 1 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 5 & 1 \\ 1 & 1 & \dots & \dots & 1 & 3 \end{pmatrix}_{n.n}$, $Q = \begin{pmatrix} 0 & 0 & 1 & \dots & 1 & 1 \\ 0 & 1 & 0 & 1 & \dots & 1 \\ 1 & 0 & 1 & 0 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 & 0 \\ 1 & 1 & \dots & \dots & 0 & 1 \end{pmatrix}_{n.n}$,

$$det(L+M) = det \begin{pmatrix} 6 & 2 & \dots & \dots & 2 & 0 & 0 & 2 & \dots & 2 \\ 2 & 6 & 2 & \dots & 2 & 0 & 2 & 0 & \dots & 2 \\ \vdots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & 6 & 2 & 2 & 2 & \dots & 2 & 0 \\ 2 & 2 & \dots & 2 & 4 & 2 & 2 & \dots & 0 & 2 \\ 0 & 0 & 2 & \dots & 2 & 6 & 2 & \dots & \dots & 2 \\ 0 & 2 & 0 & \dots & 2 & 2 & 6 & 2 & \dots & 2 \\ \vdots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & 2 & 0 & 2 & 2 & \dots & 6 & 2 \\ 2 & 2 & \dots & 0 & 2 & 2 & 2 & \dots & 2 & 4 \end{pmatrix} = det \begin{pmatrix} X & Y \\ Y & X \end{pmatrix},$$

$$= det(X+Y)\, det(X-Y) \; = det \begin{pmatrix} 6 & 2 & 4 & \dots & \dots & 4 \\ 2 & 8 & 2 & \dots & \dots & 4 \\ 4 & 2 & 8 & 2 & \dots & 4 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 4 & 4 & \dots & \dots & 8 & 2 \\ 4 & 4 & \dots & \dots & 2 & 6 \end{pmatrix} \times det \begin{pmatrix} 6 & 2 & 0 & \dots & \dots & 0 \\ 2 & 4 & 2 & 0 & \dots & 0 \\ 0 & 2 & 4 & 2 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 4 & 2 \\ 0 & 0 & \dots & \dots & 2 & 2 \end{pmatrix},$$

$$(25) \qquad = 2^{n+1}\, det \begin{pmatrix} 2 & 0 & 1 & \dots & \dots & 1 \\ 0 & 3 & 0 & \dots & \dots & 1 \\ 1 & 0 & 3 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 3 & 0 \\ 1 & 1 & \dots & \dots & 0 & 2 \end{pmatrix} \times 2^{n+1} = \; 2^{n+1}\, n^2 \; \times 2^{n+1} = \; n^2\, 2^{2n+2}.$$

$$(26)\quad det(L-M) = det \begin{pmatrix} 4 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 4 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 4 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 4 & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 2 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 & 4 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 4 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 4 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & \dots & 0 & 4 & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 2 \end{pmatrix}_{2n.2n} = 4^{2n-1}.$$

From Eq. (25) and Eq. (26) in Eq. (24), we get:

$\tau(\Gamma_n) = \frac{1}{(4n)^2}\, 4^{2n-1}\, n^2\, 2^{2n+2} = 2^{6n-4}$. □

3.2. Complexity of some families of graphs generated by a square with average degree 6.

Consider the family of graphs Υ_n, generated by a square, and constructed as shown in Fig. 6. It is noticed that according to the construction, the total number of vertices and edges are $|V(\Upsilon_n)| = 4n$ and $|E(\Upsilon_n)| = 12n-8$, $n = 1,2,3,4,\cdots$. When n is large , the average degree of Υ_n is 6.

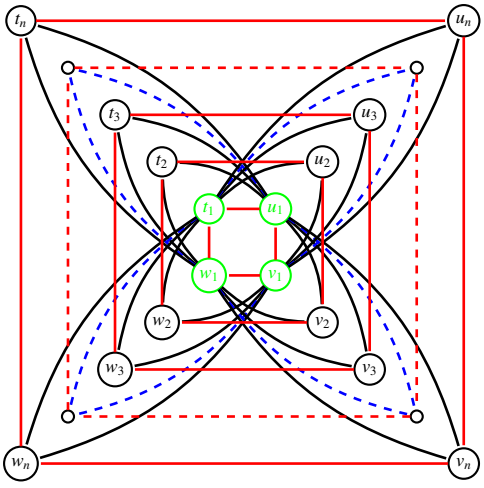

FIGURE 6. The graph Υ_n .

Theorem 3.6. *For $n \geq 1$, the number of the spanning trees of the family of graphs Υ_n is given by:* $\tau(\Upsilon_n) = n^2\,(2n+4)\,3^{n-2}\,2^{6n-5}$.

Proof. Applying Lemma 1.1, we have:

$$\tau(\Upsilon_n) = \frac{1}{(4n)^2}\, det\,[\,4nI - \overline{D} + \overline{A}\,] = \frac{1}{(4n)^2} det \begin{pmatrix} P & Q & J & Q \\ Q & P & Q & J \\ J & Q & P & Q \\ Q & J & Q & P \end{pmatrix}.$$

$$\tau(\Upsilon_n) = \frac{1}{(4n)^2}[det(P-J)]^2 \times det(P+J+2Q) \times det(P+J-2Q)]\,. \tag{27}$$

where $P = \begin{pmatrix} 2n+1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 5 & 1 & 1 & \dots & 1 \\ 1 & 1 & 5 & 1 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 5 & 1 \\ 1 & 1 & \dots & \dots & 1 & 5 \end{pmatrix}_{n.n}$, $Q = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \dots & \dots & 0 & 1 \\ 0 & 1 & \dots & \dots & 1 & 0 \end{pmatrix}_{n.n}$,

$$\tau(\Upsilon_n) = \frac{1}{(4n)^2} \left(det \begin{pmatrix} 2n & 0 & \dots & \dots & 0 \\ 0 & 4 & 0 & \dots & 0 \\ 0 & 0 & 4 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 4 & 0 \\ 0 & 0 & \dots & 0 & 4 \end{pmatrix} \right) \times det \begin{pmatrix} 2n+2 & 2 & \dots & \dots & 2 \\ 2 & 6 & 4 & \dots & 4 \\ 2 & 4 & 6 & \dots & 4 \\ \vdots & \dots & \dots & \dots & \vdots \\ 2 & 4 & \dots & 6 & 4 \\ 2 & 4 & \dots & 4 & 6 \end{pmatrix} \times$$

$$det \begin{pmatrix} 2n+2 & 2 & 2 & \dots & 2 \\ 2 & 6 & 0 & \dots & 0 \\ 2 & 0 & 6 & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots \\ 2 & 0 & \dots & 6 & 0 \\ 2 & 0 & \dots & 0 & 6 \end{pmatrix},$$

$$= \frac{1}{(4n)^2}\ (\ n\ 2^{2n-1}\ )^2 \times\ 4n\ det \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 \\ 1 & 4 & 2 & \dots & \dots & 2 \\ 1 & 2 & 4 & \dots & \dots & 2 \\ 1 & 2 & 2 & \dots & \dots & 2 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 2 & \dots & 2 & 4 & 2 \\ 1 & 2 & 2 & \dots & 2 & 4 \end{pmatrix}_{n.n} \times 2^n\ 3^{n-2}\ (2n+4)\ ,$$

$$= \frac{1}{(4n)^2}\ (\ n\ 2^{2n-1}\ )^2 \times\ 4n\ 2^{n-1} det \begin{pmatrix} 2 & 1 & 1 & \dots & \dots & 1 \\ 1 & 2 & 1 & \dots & \dots & 1 \\ 1 & 1 & 2 & \dots & \dots & 1 \\ 1 & 1 & 1 & \dots & \dots & 2 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 2 & 1 \\ 1 & 1 & \dots & \dots & 1 & 2 \end{pmatrix}_{n-1.n-1} \times 2^n\ 3^{n-2}\ (2n+4)\ .$$

From Lemma 2.2, with $x = 2$, we obtain:

$$\tau(\Upsilon_n) = \frac{1}{(4n)^2}\ (\ n\ 2^{2n-1}\ )^2\ \times\ 2^{n+1}\ n^2\ \times\ 2^n\ 3^{n-2}\ (2n+4)\ =\ n^2(2n+4)\ 2^{6n-5}\ 3^{n-2}\ .\quad \square$$

Consider the family of graphs Π_n, generated by a square, and constructed as shown in Fig. 7. The graph Π_n has a number of vertices and edges are $|V(\Pi_n)| = 4n$ and $|E(\Pi_n)| = 12n-8,\ n = 1,2,\cdots$. When n is large, the average degree of Π_n is 6.

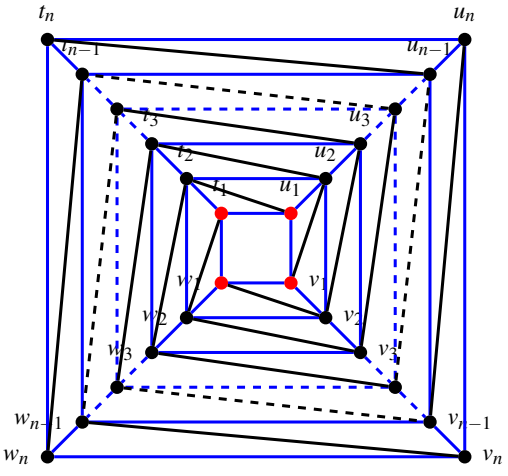

FIGURE 7. The graph Π_n.

Theorem 3.7. *For $n \geq 2$, the number of the spanning trees of the family of graphs Π_n is given by:*

$$\tau(\Pi_n) = \frac{9\times\ 2^{4n-8}}{7}\ [(16-5\sqrt{7}\ )\ (16+6\sqrt{7})^n + \ (16+5\sqrt{7}\ )(16-6\sqrt{7})^n - 9\ \times 2^{n+1}\ ]\ .$$

Proof. Applying Lemma 1.1, we have:

$$\tau(\Pi_n)\ = \frac{1}{(4n)^2}\ det\ [\ 4nI-\overline{D}\ +\overline{A}\ ] = \frac{1}{(4n)^2}\ det\ \begin{pmatrix} P & Q & J & Q^t \\ Q^t & P & Q & J \\ J & Q^t & P & Q \\ Q & J & Q^t & P \end{pmatrix}.$$

$$\tau(\Pi_n) = \frac{1}{(4n)^2}\ det\ \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n)^2}[det(L+M)\times det(L-M)] \tag{28}$$

where $P = \begin{pmatrix} 5 & 0 & 1 & \dots & 1 & 1 \\ 0 & 7 & 0 & 1 & \dots & 1 \\ 1 & 0 & 7 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 7 & 0 \\ 1 & 1 & \dots & \dots & 0 & 5 \end{pmatrix}_{n.n}$, $Q = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & 1 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 1 \\ 0 & 1 & \dots & \dots & 0 & 0 \end{pmatrix}_{n.n}$,

$$det(L+M) = det\ \begin{pmatrix} 6 & 1 & 2 & 2 & \dots & 2 & 0 & 1 & 2 & 2 & \dots & 2 \\ 1 & 8 & 1 & 2 & \dots & 2 & 1 & 0 & 1 & 2 & \dots & 2 \\ 2 & 1 & 8 & 1 & \dots & 2 & 2 & 1 & 0 & 1 & \dots & 2 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & \dots & 8 & 1 & 2 & 2 & \dots & \dots & 0 & 1 \\ 2 & 2 & \dots & \dots & 1 & 6 & 2 & 2 & \dots & \dots & 1 & 0 \\ 0 & 1 & 2 & 2 & \dots & 2 & 6 & 1 & 2 & 2 & \dots & 2 \\ 1 & 0 & 1 & 2 & \dots & 2 & 1 & 8 & 1 & 2 & \dots & 2 \\ 2 & 1 & 0 & 1 & \dots & 2 & 2 & 1 & 8 & 1 & \dots & 2 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & \dots & 0 & 1 & 2 & 2 & \dots & \dots & 8 & 1 \\ 2 & 2 & \dots & \dots & 1 & 0 & 2 & 2 & \dots & \dots & 1 & 6 \end{pmatrix} = det\ \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$$

$$det(L+M) = det \begin{pmatrix} 6 & 2 & 4 & 4 & \dots & \dots & 4 & 4 \\ 2 & 8 & 2 & 4 & \dots & \dots & 4 & 4 \\ 4 & 2 & 8 & 2 & \dots & \dots & 4 & 4 \\ 4 & 4 & 2 & & \dots & \dots & 4 & 4 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 4 & 4 & 4 & 4 & \dots & 2 & 8 & 2 \\ 4 & 4 & 4 & 4 & \dots & 4 & 2 & 6 \end{pmatrix} \times det \begin{pmatrix} 6 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 8 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 8 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 8 & \dots & \dots & 0 & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 6 \end{pmatrix}.$$

Using properties of determinants and straightforward induction, we get:

$$det(L+M) = 2^{n+1}\, n^2 \times 3^2\, 2^{3n-4} = 3^2\, n^2\, 2^{4n-3}. \tag{29}$$

$$det(L-M) = det \begin{pmatrix} 4 & -1 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 6 & -1 & 0 & \dots & 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ 0 & -1 & 6 & -1 & \dots & 0 & 0 & -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 6 & -1 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -1 & 4 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 & 4 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & \dots & 0 & -1 & 6 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & -1 & \dots & 0 & 0 & -1 & 6 & -1 & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \dots & \vdots & \dots & \dots & \vdots & \dots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & 0 & \dots & 6 & -1 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & \dots & -1 & 4 \end{pmatrix}.$$

Expanding the first six determinants of $(L - M)$ gives the values 196, 6400, 204304, 6512704, 207590464, 6616846336$\cdots$ for $n = 2,3,4,5,6,7,\cdots$. These values can be written in the form

$$204304 = 32 \times 6400 - 4 \times 196 + 288,$$

$$6512704 = 32 \times 204304 - 4 \times 6400 + 576,$$

$$207590464 = 32 \times 6512704 - 4 \times 204304 + 1152,$$

$$6616846336 = 32 \times 207590464 - 4 \times 6512704 + 2304.$$

Consequently, we have the following non-homogeneous recurrence relation

$$a_{n+2} = 32\, a_{n+1} - 4\, a_n + 9\, 2^{n+3}. \tag{30}$$

Let $a_n = b_n + d\ 2^{n+3}$ be the solution of the non-homogeneous recurrence relation, where b_n is the solution of homogeneous recurrence relation $b_{n+2} - 32\ b_{n+1} + 4b_n = 0$. Substituting in Eq.(30), we get $d = \frac{-9}{56}$. The characteristic equation of homogeneous recurrence relation is $r^2 - 32r + 4 = 0$ with two roots being $(16 + 6\sqrt{7}\,)$ and $(\,16 - 6\sqrt{7}\,)$, then

$b_n = \alpha\ (16 + 6\sqrt{7})^n + \beta\ (16 - 6\sqrt{7})^n$. The general solution of the non-homogeneous recurrence relation is

$$a_n = \alpha\ (16 + 6\sqrt{7})^n + \beta (16 - 6\sqrt{7})^n - \frac{9}{56}\ 2^{n+3}.$$

Using the initial conditions $det(L - M) = 196\ , 6400$ at $n = 2, 3,$ respectively, we have:

$$196 = \alpha\ (16 + 6\sqrt{7})^2 + \beta (16 - 6\sqrt{7})^2 - \frac{36}{7}\ ,$$
$$6400 = \alpha\ (16 + 6\sqrt{7})^3 + \beta (16 - 6\sqrt{7})^3 - \frac{72}{7}\ .$$

Solving these equations, we get $\alpha = \frac{1}{14}\ [16 - 5\sqrt{7}\,]\ ,\ \beta = \frac{1}{14}\ [16 + 5\sqrt{7}\,]$. Therefore

$$det(L - M) = a_n = \frac{1}{14}\ [(16 - 5\sqrt{7}\,)\ (16 + 6\sqrt{7})^n + (16 + 5\sqrt{7}\,)(16 - 6\sqrt{7})^n - 9\ 2^{n+1}\,] \tag{31}$$

From Eq. (29) and Eq. (31) in Eq. (28), we get:

$$\tau(\Pi_n) = \frac{9 \times 2^{4n-8}}{7}\ [(16 - 5\sqrt{7}\,)\ (16 + 6\sqrt{7})^n + (16 + 5\sqrt{7}\,)(16 - 6\sqrt{7})^n - 9 \times 2^{n+1}\,]\ . \quad \square$$

Consider the family of graphs Φ_n generated by a square and constructed as shown in Fig. 8, the number of total vertices $|V(\Phi_n)|$ and edges $|E(\Phi_n)|$ are $|V(\Phi_n| = 4n$ and $|E(\Phi_n)| = 12\ n-8,\ n = 3,4,\cdots$ according to the construction. Note that for $n = 2$ the graph Φ_n is isomorphic with the graph Υ_n. When n is large , the average degree of Φ_n is 6.

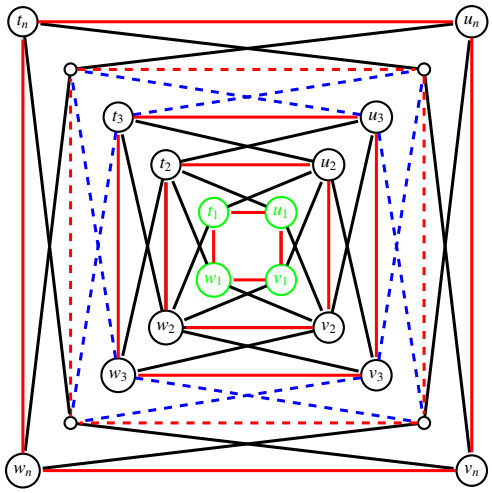

FIGURE 8. The graph Φ_n .

Theorem 3.8. *For $n \geq 3$, the number of the spanning trees of the graph Φ_n is given by:*

$$\tau(\Phi_n) = 3^{2n-5}\ 2^{4n+1}\ [\ (45+26\sqrt{3})(2+\sqrt{3})^{n-3}\ +\ (45-26\sqrt{3})(2-\sqrt{3}\ )^{n-3}\ ]\ .$$

Proof. let us applying Lemma 1.1, we have: $\tau(\ \Phi_n)\ = \dfrac{1}{(4n)^2}\ det\ [\ 4nI-\overline{D}\ +\overline{A}\ ].$

(32)

$$\tau(\ \Phi_n) = \frac{1}{(4n)^2}\ det\ \begin{pmatrix} P & Q & J & Q \\ Q & P & Q & J \\ J & Q & P & Q \\ Q & J & Q & P \end{pmatrix} = \frac{1}{(4n)^2}\ [det(P-J)\ ]^2\ det(P+J+2Q)det(P+J-2Q).$$

$$\text{where } P=\begin{pmatrix} 5 & 1 & 1 & \dots & 1 & 1 \\ 1 & 7 & 1 & 1 & \dots & 1 \\ 1 & 1 & 7 & 1 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 7 & 1 \\ 1 & 1 & \dots & \dots & 1 & 5 \end{pmatrix}_{n.n}, Q=\begin{pmatrix} 0 & 0 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & 1 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 0 \\ 1 & 1 & \dots & \dots & 0 & 0 \end{pmatrix}_{n.n},$$

$$det(P-J)=\ det\begin{pmatrix} 4 & 0 & 0 & 0 & \dots & 0 \\ 0 & 6 & 0 & 0 & \dots & 0 \\ 0 & 0 & 6 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 6 & 0 \\ 0 & 0 & 0 & \dots & 0 & 4 \end{pmatrix}=2^{n+2}\,3^{n-2}\,. \tag{33}$$

(34)

$$det(P+J+2Q)=det\begin{pmatrix} 6 & 2 & 4 & \dots & \dots & 4 \\ 2 & 8 & 2 & 4 & \dots & 4 \\ 4 & 2 & 8 & 2 & \dots & 4 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 4 & 4 & \dots & \dots & 8 & 2 \\ 4 & 4 & \dots & \dots & 2 & 6 \end{pmatrix}=2^{n+1}\,det\begin{pmatrix} 2 & 0 & 1 & \dots & \dots & 1 \\ 0 & 3 & 0 & 1 & \dots & 1 \\ 2 & 0 & 3 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 3 & 0 \\ 1 & 1 & \dots & \dots & 0 & 2 \end{pmatrix}=2^{n+1}n^2.$$

$$det(P+J-2Q)=det\begin{pmatrix} 6 & 2 & 0 & 0 & \dots & \dots & 0 \\ 2 & 8 & 2 & 0 & \dots & \dots & 0 \\ 0 & 2 & 8 & 2 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 8 & 2 \\ 0 & 0 & 0 & \dots & \dots & 2 & 6 \end{pmatrix}=2^n\,det\begin{pmatrix} 3 & 1 & 0 & 0 & \dots & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 4 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 4 & 1 \\ 0 & 0 & 0 & \dots & \dots & 1 & 3 \end{pmatrix}=2^n\,det(K)\,.$$

By expanding the first seven determinants of K, gives the values $30, 112, 418, 1560, 5822, \cdots$ for $n = 3, 4, 5, 6, 7, \cdots$, these values can be written in the form

$$418 = 4 \times 112 - 30,$$

$$1560 = 4 \times 418 - 112,$$
$$5822 = 4 \times 1560 - 418.$$

Consequently, we have the following homogeneous recurrence relation

$$a_{n+2} = 4\, a_{n+1} - a_n \,. \tag{35}$$

The general solution of the recurrence relation (35), is

$$a_n = \alpha\, (\, 2+\sqrt{3}\,)^{n-3} + \beta\, (\, 2-\sqrt{3}\,)^{n-3} \,.$$

Using the initial conditions $det(\ k) = 30\ , 112$ at $n = 3,4,$ respectively, we get $\alpha = \dfrac{45+26\sqrt{3}}{3}$ and $\beta = \dfrac{45-26\sqrt{3}}{3}$. Therefore,

$$det(P+J-2Q) = 2^n\, det(\,K) = 2^n\ [\,(\frac{45+26\sqrt{3}}{3})(2+\sqrt{3})^{n-3} + (\frac{45-26\sqrt{3}}{3})(2-\sqrt{3}\,)^{n-3}\,]\,. \tag{36}$$

From Eq. (33) , Eq. (34) and Eq. (36) in Eq. (32), we get:

$$\tau(\Phi_n) = 3^{2n-5}\, 2^{4n+1}\ [\,(45+26\sqrt{3})(2+\sqrt{3})^{n-3} + (45-26\sqrt{3})(2-\sqrt{3}\,)^{n-3}\,]$$

- For $n = 2$, the graph Φ_2 is isomorphic to the graph Υ_2 and $\tau(\Phi_2) = \tau(\Upsilon_2) = 4096$.

□

3.3. Complexity of a graph generated by a square with average degree 8.

Consider the family of graphs Ω_n , generated by a square, and constructed as a composition $C_4[P_n]$ of the graphs C_4 and P_n as shown in Fig. 9, the graph Ω_n has number of vertices $|V(\Omega_n)| = 4n$ and edges $|E(\Omega_n)| = 4(4n-3)$, $n = 2,3,\cdots$. When n is large, the average degree of Ω_n is 8.

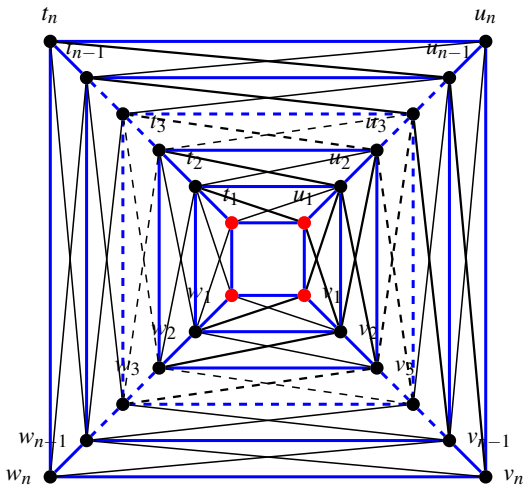

FIGURE 9. The graph Ω_n.

Theorem 3.9. *For $n \geq 2$, the number of the spanning trees of the graph Ω_n is given by:*

$\tau[\Omega_n] = (\frac{3^{n-3}\sqrt{15}}{500})[(4+\sqrt{15})^n - (4-\sqrt{15})^n][(-60+17\sqrt{15})(4+\sqrt{15})^n + (-60-17\sqrt{15})(4-\sqrt{15})^n]^2$.

Proof. let us applying Lemma (1.1), we have: $\tau(\Omega_n)) = \frac{1}{(4n)^2}\, det\,[\,4nI - \overline{D} + \overline{A}\,]$.

(37)

$$\tau[\Omega_n] = \frac{1}{(4n)^2}\, det \begin{pmatrix} P & Q & Q & J \\ Q & P & J & Q \\ Q & J & P & Q \\ J & Q & Q & P \end{pmatrix} = \frac{1}{(4n)^2}\, det \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n)^2}[\, det(L+M) \times det(L-M)\,].$$

where $P = \begin{pmatrix} 6 & 0 & 1 & \dots & 1 & 1 \\ 0 & 9 & 0 & 1 & \dots & 1 \\ 1 & 0 & 9 & 0 & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 9 & 0 \\ 1 & 1 & \dots & \dots & 0 & 6 \end{pmatrix}_{n.n}$, $Q = \begin{pmatrix} 0 & 0 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & 1 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 0 & 0 \\ 1 & 1 & \dots & \dots & 0 & 0 \end{pmatrix}_{n.n}$,

$$det(L+M) = det \begin{pmatrix} 6 & 0 & 2 & 2 & \dots & 2 & 1 & 1 & 2 & 2 & \dots & 2 \\ 0 & 9 & 0 & 2 & \dots & 2 & 1 & 1 & 1 & 2 & \dots & 2 \\ 2 & 0 & 9 & 0 & \dots & 2 & 2 & 1 & 1 & 1 & \dots & 2 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & \dots & 9 & 0 & 2 & 2 & \dots & \dots & 1 & 1 \\ 2 & 2 & \dots & \dots & 0 & 6 & 2 & 2 & \dots & \dots & 1 & 1 \\ 1 & 1 & 2 & 2 & \dots & 2 & 6 & 0 & 2 & 2 & \dots & 2 \\ 1 & 1 & 1 & 2 & \dots & 2 & 0 & 9 & 0 & 2 & \dots & 2 \\ 2 & 1 & 1 & 1 & \dots & 2 & 2 & 0 & 9 & 0 & \dots & 2 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 2 & 2 & \dots & \dots & 1 & 1 & 2 & 2 & \dots & \dots & 9 & 0 \\ 2 & 2 & \dots & \dots & 1 & 1 & 2 & 2 & \dots & \dots & 0 & 6 \end{pmatrix} = det \begin{pmatrix} X & Y \\ Y & X \end{pmatrix},$$

$$= det \begin{pmatrix} 7 & 1 & 4 & 4 & \dots & \dots & 4 \\ 1 & 10 & 1 & 4 & \dots & \dots & 4 \\ 4 & 1 & 10 & 1 & \dots & \dots & 4 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 4 & 4 & 4 & \dots & \dots & 10 & 1 \\ 4 & 4 & 4 & \dots & \dots & 1 & 7 \end{pmatrix} \times det \begin{pmatrix} 5 & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 8 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 8 & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 8 & -1 \\ 0 & 0 & 0 & \dots & \dots & -1 & 5 \end{pmatrix},$$

$$= 4\,3^{n-1}\ det \begin{pmatrix} 2 & 0 & 1 & 1 & \dots & \dots & 1 \\ 0 & 3 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 3 & 0 & \dots & \dots & 1 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & \dots & 3 & 0 \\ 1 & 1 & 1 & 2 & \dots & 0 & 2 \end{pmatrix} \times det \begin{pmatrix} 5 & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 8 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 8 & -1 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 8 & -1 \\ 0 & 0 & 0 & \dots & \dots & -1 & 5 \end{pmatrix},$$

$$= 4\,3^{n-1}\,n^2 \times det(X-Y)\,.$$

Expanding the first seven determinants of $(X - Y)$ yields the values $24, 190, 1496, 11778, 92728, 730046, 5747640\cdots$ for $n = 2,3,4,5,6,7,8,\cdots$, these values have the following recurrence relation $a_{n+2} = 8a_{n+1} - a_n$. Its characteristic equation is $r^2 - 8r + 1 = 0$, with the roots being $4+\sqrt{15}$ and $4-\sqrt{15}$.

The general solution of the recurrence relation is $a_n = \alpha\,(4+\sqrt{15})^n + \beta\,(4-\sqrt{15})^n$. Using the initial conditions $a_n = 24, 190$ at $n = 2,3$, respectively, we get: $\alpha = \dfrac{-60+17\sqrt{15}}{15}$ and $\beta = \dfrac{-60-17\sqrt{15}}{15}$.

Therefore, $det(X-Y) = a_n = (\dfrac{-60+17\sqrt{15}}{15})(4+\sqrt{15})^n + (\dfrac{-60-17\sqrt{15}}{15})(4-\sqrt{15})^n$,

$$(38)\quad det(L+M) = 4\,n^2\,3^{n-1}\,[\,(\frac{-60+17\sqrt{15}}{15})(4+\sqrt{15})^n + (\frac{-60-17\sqrt{15}}{15})(4-\sqrt{15})^n\,]\,.$$

$$det(L-M) = det \begin{pmatrix} 6 & 0 & \dots & \dots & \dots & 0 & -1 & -1 & 0 & \dots & \dots & 0 \\ 0 & 9 & 0 & \dots & \dots & 0 & -1 & -1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 9 & 0 & \dots & 0 & 0 & -1 & -1 & -1 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 9 & 0 & 0 & 0 & 0 & \dots & -1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 6 & 0 & \dots & \dots & \dots & -1 & -1 \\ -1 & -1 & 0 & 0 & \dots & 0 & 6 & 0 & \dots & \dots & \dots & 0 \\ -1 & -1 & -1 & 0 & \dots & 0 & 0 & 9 & 0 & \dots & \dots & 0 \\ 0 & -1 & -1 & -1 & \dots & 0 & 0 & 0 & 9 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & -1 & -1 & 0 & 0 & \dots & \dots & 9 & 0 \\ 0 & 0 & \dots & \dots & -1 & -1 & 0 & 0 & \dots & \dots & 0 & 6 \end{pmatrix}$$

$$= det \begin{pmatrix} K & H \\ H & K \end{pmatrix} = det(K+H) \times det(K-H) ,$$

$$= det \begin{pmatrix} 5 & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 8 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 8 & -1 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 8 & -1 \\ 0 & 0 & 0 & \dots & \dots & -1 & 5 \end{pmatrix} \times det \begin{pmatrix} 7 & 1 & 0 & 0 & \dots & \dots & 0 \\ 1 & 8 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 8 & 1 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 8 & 1 \\ 0 & 0 & 0 & \dots & \dots & 1 & 7 \end{pmatrix} .$$

Expanding the first seven determinants of $(K - H)$ yields the values $48, 378, 2976, 23430, 184464, 1452282, 11433792 \cdots$ for $n = 2,3,4,5,6,7,8, \cdots$, these values have the recurrence relation

$a_{n+2} = 8a_{n+1} - a_n$. Its characteristic equation is $r^2 - 8r + 1 = 0$, with the roots being $4+\sqrt{15}$ and $4-\sqrt{15}$.

The general solution of the recurrence relation is $a_n = \alpha\ (4+\sqrt{15})^n + \beta\ (4-\sqrt{15})^n$.

Using the initial conditions $a_n = 48\ , 378$ at $n = 2,3,$ respectively, we get: $\alpha = \frac{\sqrt{15}}{5}$ and $\beta = \frac{-\sqrt{15}}{5}$. Therefore, $det(K-H) = a_n = (\frac{\sqrt{15}}{5})\ [\ (4+\sqrt{15})^n - (4-\sqrt{15})^n\ ]$.

(39)
$$det(L-M) = (\frac{\sqrt{15}}{5})\ [\ (4+\sqrt{15})^n - \ (4-\sqrt{15})^n\ ][\ (\frac{-60+17\sqrt{15}}{15})(4+\sqrt{15})^n + \ (\frac{-60-17\sqrt{15}}{15})(4-\sqrt{15})^n\ ]\ .$$

From Eq. (38) and Eq.(39) in Eq. (37), we get:
$$\tau[\Omega_n] = (\frac{3^{n-3}\ \sqrt{15}}{500})[(4+\sqrt{15})^n - (4-\sqrt{15})^n][(-60+17\sqrt{15})(4+\sqrt{15})^n + (-60-17\sqrt{15})(4-\sqrt{15})^n\ ]^2.$$

□

4. NUMERICAL RESULTS

In the following table 1 we enumerate some of the values of $\tau(G)$, the number of spanning trees, of our graphs Ψ_n , Γ_n , Λ_n, Δ_n , Θ_n , Υ_n , Π_n , Φ_n and Ω_n.

	$\tau(\Psi_n)$	$\tau(\Gamma_n)$	$\tau(\Lambda_n)$	$\tau(\Delta_n)$	$\tau(\Theta_n)$	$\tau(\Upsilon_n)$	$\tau(\Pi_n)$	$\tau(\Phi_n)$	$\tau(\Omega_n)$
n=1	4	4	4	4	45	4	4	4	4
n=2	256	256	216	384	3509	4096	3528	4096	20736
n=3	9216	16384	19200	31500	284089	2211840	1843200	2211840	30703050
n=4	262144	1048576	1775616	2558976	23057865	905969664	941432832	1189085184	44957265408
n=5	6553600	67108864	165347328	207746836	1871801381	317089382400	$2^{19}\times 3^2\times 101761$	$2^{22}\times 3^6\times 209$	$3^7\times 3009731890$

TABLE 1. Some values of spanning trees of our graphs.

According to the results in table 1, it is easily seen that:

(i) Of all the graphs of average degree 4, the graph Θ_n has a number of spanning trees greater than the other four graphs Ψ_n , Γ_n , Λ_n, and Δ_n.

(ii) Of all the graphs of average degree 6, the graph Φ_n has a number of spanning trees greater than the other two graphs Υ_n , and π_n.

(iii) The number of the spanning trees of the graph Ω_n, of average degree 8 is the largest among all the graphs at different values of $n \geq 2$.

5. ENTROPY OF OUR NETWORKS

Because the complexity of a network $\tau(G)$ increases exponentially with the number of vertices, there exists a constant $\rho(G)$, called the **entropy of spanning trees** [15], described by this relation:

$$\rho(G) = \lim_{n\to\infty} \frac{ln\ \tau(G)}{|V(G)|}\ . \tag{40}$$

By calculating the entropy of graphs (networks), we hope to determine the best one. The entropy of spanning trees of a network is a quantitative measure of the number of spanning trees to evaluate the goodness and the resistance of a network and to describe its structure. The most goodness and resistance network is the network that has the highest spanning-tree entropy. According to the definition of the entropy of spanning trees of a network, the bigger the entropy value, the more the number of spanning trees, so there are more possibilities of connections between two vertices.

$$\rho\ (\Psi_n) = \lim_{n\to\infty} \frac{ln\ [\tau(\Psi_n)\ ]}{4n} = \lim_{n\to\infty} \frac{ln\ n^2\ 2^{4n-2}}{4n} = \ ln\ 2\ = \ 0.693147\ .$$

$$\rho\ (\Gamma_n) = \lim_{n\to\infty} \frac{ln\ [\tau(\Gamma_n)\ ]}{4n} = \lim_{n\to\infty} \frac{ln\ 2^{6n-4}}{4n} = \frac{3\ ln\ 2}{2} = \ 1.03972\ .$$

$$\rho\ (\Lambda_n) = \lim_{n\to\infty} \frac{ln\ \{3\ 2^{3n-7}\ [\ (6+4\sqrt{2})^n + (6-4\sqrt{2}\ )^n + \ 2^{n+1}\ ]\ \}}{4n} \ = \ \frac{7\ ln\ 2}{4} = \ 1.213007\ .$$

$$\rho\ (\Upsilon_n) = \lim_{n\to\infty} \frac{ln\ [\ n^2\ (2n+4)\ 2^{6n-5}\ 3^{n-2}\ ]}{4n} = \frac{ln\ 3\ +6\ ln\ 2}{4} = \ 1.314373\ .$$

$$\rho\ (\Pi_n) = \lim_{n\to\infty} \frac{ln\ [\tau(\Pi_n)\ ]}{4n} \ = \ \frac{4\ ln2 + ln(\ 16 + \ 6\ \sqrt{7})}{4} = \ 1.558598\ .$$

$$\rho\ (\Phi_n) = \lim_{n\to\infty} \frac{ln\ [\tau(\Phi_n)\ ]}{4n} \ = \ \frac{2\ ln\ 3\ +4\ ln2 + ln(2+\sqrt{3})}{4} = \ 1.571692\ .$$

$$\rho(\Omega_n) = \lim_{n\to\infty} \frac{ln\ [\tau(\Omega_n)\ ]}{4n} \ = \ \frac{ln\ 3\ \times (\ 3\ ln\ (4+\sqrt{15}\ )/ln\ 3+1)}{4} = \ 1.82223\ .$$

$$\rho\ (\Delta_n) = \lim_{n\to\infty} \frac{ln\ [\ \frac{1}{12\sqrt{2}}\ [\ (2+\sqrt{3})^n - (2-\sqrt{3})^n\ ]^2\ [(3+2\sqrt{2})^n - (3-2\sqrt{2}\ )^n]\ ]}{4n}$$

$$= \frac{ln\ (\ 2\ +\sqrt{3}\ )}{2} = \ 1.866025\ .$$

$$\rho(\Theta_n) = \lim_{n\to\infty} \frac{ln\,[\tau(\Theta_n)\,]}{4n+1} = \frac{ln\,(\,2+\sqrt{3}\,)}{2} = 1.866025\,.$$

The following table 2 illustrates the values of entropy and average degree of studying networks.

Graphs	Ψ_n	Γ_n	Λ_n	Δ_n	Θ_n	Υ_n	Π_n	Φ_n	Ω_n
$\|V(G)\|$	4n	4n	4n	4n	4n+1	4n	4n	4n	4n
$\|E(G)\|$	8n-4	8n-4	8n-4	8n-4	8n	12n-8	12n-8	12n-8	16n-12
Average degree	4	4	4	4	4	6	6	6	8
$\rho(G)$	0.693147	1.03972	1.213007	1.866025	1.866025	1.314373	1.558598	1.571692	1.82223

TABLE 2. The entropy of the studied graphs .

Comparing the value of entropy in our design networks (graphs), we have found that:

(i) Of all our different design networks (graphs) of average degree 4 , the entropy of the graphs Δ_n and Θ_n is the largest.

(ii) Among all our different design networks (graphs) of average degree 6 , the entropy of the graph Φ_n is the largest.

(iii) The entropy of the graphs Δ_n and Θ_n of average degree 4 is larger than the entropy of all graphs of average degrees 6 and 8.

Now we compare the value of entropy in our design networks (graphs) with other networks:

In 2013, Zhang et all [16] proved that the Apollonian graph of average degree 6 has entropy 1.354 . In 2019, Daoud [17] introduced some networks generated by a triangle with average degree 4 and 6, and proved that the graph Y_n of average degree 6 has entropy 1.514280 . It is clear that the entropy of our studied networks Π_n and Φ_n with the same average degree 6 is larger than the entropy of the Apollonian graph, and the graphs generated by a triangle with the same average degree 6. Finally, the entropy of spanning trees of our studied (graphs) networks Θ_n and Δ_n of average degree 4 are the highest among these networks.

6. CONCLUSIONS

The complexity which is the number of spanning trees in the networks is a significant invariant. The enumerating of this number is not only helpful from a combinatorial standpoint, but it is also an important measure of the network reliability and electrical circuit design. In this paper, we introduce nine network designs that are created by a square of average degree 4, 6 and 8 , then we obtain a simple and evident expression for the number of spanning trees of these networks using some basic properties of orthogonal polynomials, block matrix analysis technique, and recurrence relations. Finally, we compute the entropy of these networks and compare the entropy of spanning trees on our networks with the other triangle and Apollonian networks. We deduce that the entropy of our networks is the highest among the studied triangle and Apollonian networks and the networks Θ_n and Δ_n of average degree 4 are the highest among these networks.

ACKNOWLEDGMENTS

The authors wish to express their thanks to the editor for his cooperation.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] D.B. West, Introduction to Graph Theory, 2 Eds., Prentice-Hall of India, New Delhi, 2003.

[2] N.L. Biggs, Algebraic Graph Theory, 2 Eds., Cambridge University Press, Cambridge, 1993.

[3] A.K. Kelmans, V.M. Chelnokov, A certain polynomials of a graph and graphs with an external number of trees, J. Comb. Theory (B), 16 (1974), 197-214.

[4] H.N.V. Temperley, Graph theory and applications, Halsted Press, New York, 1981.

[5] J. Sedlacek, Lucas number in graph theory Mathematics (Geometry and Graph Theory) (Chech), 111–115, Univ. Karlova, Prague, 1970.

[6] L. Clark, On the enumeration of multipartite spanning trees of the complete graph, Bull. ICA, 38 (2003), 50-60.

[7] S.N. Daoud, Generating formulas of the number of spanning trees of some special graphs, Eur. Phys. J. Plus. 129 (2014), 146.

[8] S. N. Daoud, K. Mohamed, The complexity of some families of cycle-related graphs, J. Taibah Univ. Sci. 11 (2017), 205-228.

[9] J.-B. Liu, S.N. Daoud, Number of Spanning Trees in the Sequence of Some Graphs, Complexity. 2019 (2019), 4271783.

[10] S.N. Daoud, W. Saleh, Complexity trees of the sequence of some nonahedral graphs generated by triangle, Heliyon. 6 (2020), e04786.

[11] M.R.Z. El Deen, W.A. Aboamer, Complexity of Some Duplicating Graphs, IEEE Access. 9 (2021), 56736-56756.

[12] J.C. Mason, D.C. Handscomb, Chebyshev polynomials, Chapman & Hall/CRC, Boca Raton, 2003.

[13] Y. Zhang, X. Yong, M.J. Golin, Chebyshev polynomials and spanning tree formulas for circulant and related graphs, Discrete Math. 298 (2005), 334–364.

[14] R.B. Bapat, Graphs and Matrices, 1st ed. Springer, New York, 2010.

[15] R. Mokhlissi, D. Lotfi, J. Debnath, M. El Marraki, N. EL Khattabi, The evaluation of the number and the entropy of spanning trees on generalized small-world networks, J. Appl. Math. 2018 (2018), 1017308.

[16] J. Zhang, W. Sun, G. Xu, Enumeration of spanning trees on Apollonian networks, J. Stat. Mech. 2013 (2013), P09015.

[17] S.N. Daoud, Number of spanning trees of some families of graphs generated by a triangle, J. Taibah Univ. Sci. 13 (2019), 731-739.