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RIESZ BASES IN SPACES WITH INDEFINITE METRIC

FERRER OSMIN[1,*], SIERRA ARLEY[2], CARRILLO DIEGO[3]

[1]Departamento de Matemáticas, Universidad de Sucre, Sincelejo, Colombia
[2]Centro de Ciencias Matemáticas, Universidad Nacional Autonóma de México, Morelia, México
[3]Departamento de Ciencias Básicas, Corporacion Universitaria del Caribe, Sincelejo, Colombia

Abstract. We introduce Riesz bases in Krein spaces from a fundamental decomposition and we study some basic properties. Among these, the non-dependence of the bases with the fundamental sitmetry and the projections from one space to another stands out.

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1. Introduction

Let $\mathfrak{K}$ be a vector space over $\mathbb{C}$ and consider a sesquilinear form $[\cdot,\cdot]:\mathfrak{K}\times\mathfrak{K}\longrightarrow\mathbb{C}$. The vector space $(\mathfrak{K},[\cdot,\cdot])$ is a Krein space if $\mathfrak{K}=\mathfrak{K}^+\oplus\mathfrak{K}^-$, where $(\mathfrak{K}^+,[\cdot,\cdot])$, $(\mathfrak{K}^-,-[\cdot,\cdot])$ are Hilbert spaces, and $\mathfrak{K}^+$, $\mathfrak{K}^-$ are orthogonal with respect to $[\cdot,\cdot]$.

In $\mathfrak{K}$ the following inner product is defined:

$$(x_1,x_2)=[{x_1}^+,{x_2}^+]-[{x_1}^-,{x_2}^-],\{{x_i}^\pm\in\mathfrak{K}^\pm,\ x_i={x_i}^++{x_i}^-.$$

*Corresponding author
E-mail address: osmin.ferrer@unisucre.co
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This inner product defines the Hilbert space $(\mathfrak{K}, (\cdot,\cdot))$, which is called the Hilbert space associated to $\mathfrak{K}$. In addition, there are unique orthogonal projections on $\mathfrak{K}^+$ and $\mathfrak{K}^-$ which are denoted by P^+ and P^- respectively. The linear, bounded and invertible operator $J = P^+ - P^-$ is called fundamental symmetry and it satisfies

$$[x,y]_J := [Jx,y] = (x,y), \quad \|x\|_J := \sqrt{[x,y]_J},\ x,y \in \mathfrak{K} \tag{1.1}$$

It is easy to prove that J is self-adjoining, isometric, J-isometric in the sense that $\|Jx\|_J = \|x\|_J$ and that the following equalities are true:

$$J^2 = I_{\mathfrak{K}},\ P^+ = \frac{1}{2}(J + I_{\mathfrak{K}}),\ P^- = \frac{1}{2}(I_{\mathfrak{K}} - J)$$

The Hilbert space $(\mathfrak{K}, [\cdot,\cdot]_J)$ is used to study linear operators that act on the Krein space $(\mathfrak{K}, [\cdot,\cdot])$. Topological concepts such as continuity, lock of linear operators, spectral theory, among others, refer to the topology induced by the J -norm given in (1.1). Therefore, we can apply the same definitions as in the theory of operators in Hilbert spaces. For example, the adjoint of an operator T in Krein spaces $T^{[*]}$ satisfies $[T(x),y] = [x, T^{[*]}(y)]$ but we must bear in mind that T also has an adjoint operator in the Hilbert space $(\mathfrak{K}, [\cdot,\cdot]_J)$, denoted by T^{*J}, where J is the fundamental symmetry in $\mathfrak{K}$. The relationship between $T^{[*]}$ y T^{*J} is $T^{[*]} = JT^{*J}J$. Also, let $\mathfrak{K}$ and $\mathfrak{K}'$ be Krein spaces with fundamental symmetries $J_{\mathfrak{K}}$ y $J_{\mathfrak{K}'}$ respectively. If $T \in \mathscr{B}(\mathfrak{K}, \mathfrak{K}')$ then $T^{[*]} = J_{\mathfrak{K}} T^{*J_{\mathfrak{K}}} J_{\mathfrak{K}'}$. An operator $T \in \mathscr{B}(\mathfrak{K})$is said self-adjoint if $T = T^{[*]}$ and J-self-adjoint if $T = T^{*J}$. Also, a linear operator T is said to be invertible if its range and domain are all space. (See K. Esmeral, O. Ferrer, Lora, B. [3], K. Esmeral, O. Ferrer, E. Wagner, [6])

Orthonormal systems give a criterion to generalize to spaces with indefinite inner product, the orthogonal expansion given by the theorem 5 in terms of a Riesz basis. For this, it is necessary to be clear about the concept of biortogonality in these spaces and give a definition of Riez bases in spaces of indefinite metric according to what is established in Hilbert spaces.

Definition 1. Let $\mathscr{V}$ be a closed subspace of $\mathfrak{K}$. The subspace:

$$\mathscr{V}^{[\perp]} = \{x \in \mathfrak{K} : [x,y] = 0, \forall y \in \mathscr{V}\}$$

is called the J-ortogonal complement of $\mathscr{V}$ with respect to $[\cdot,\cdot]$ (or simply J -ortogonal complement of $\mathscr{V}$).

Definition 2. (*Bi-orthogonality*) Let $(\mathfrak{K},[\cdot,\cdot])$ be a Krein space. We say that $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ y $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ are biortogonal, if it is satisfied that

$$[x_k, y_n] = \pm\delta_{kn}$$

Remark 3. If $(\mathscr{F},[\cdot,\cdot])$ is a space with inner product, the following sets are defined:

$$\mathscr{B}^0 = \{x \in \mathscr{F} : [x,x] = 0\} \quad and \quad \mathscr{B}^{00} = \{x \in \mathscr{F} : [x,x] = 0,\ x \neq 0\}$$

$$\mathscr{B}^{+} = \{x \in \mathscr{F} : [x,x] \geq 0\} \quad and \quad \mathscr{B}^{++} = \{x \in \mathscr{F} : [x,x] > 0,\ \vee\, x = 0\}$$

$$\mathscr{B}^{-} = \{x \in \mathscr{F} : [x,x] \leq 0\} \quad and \quad \mathscr{B}^{--} = \{x \in \mathscr{F} : [x,x] < 0,\ \vee\, x = 0\}$$

Theorem 4. [1, Theorem 7.19] *Let $(\mathscr{F},[\cdot,\cdot])$ be a space with inner product, not degenerate and decomposable. Let*

$$\mathscr{F} = \mathscr{F}_1^{+} \oplus \mathscr{F}_1^{-},\ \mathscr{F}_1^{+} \subseteq \mathscr{B}^{++},\ \mathscr{F}_1^{-} \subseteq \mathscr{B}^{--}$$

and

$$\mathscr{F} = \mathscr{F}_2^{+} \oplus \mathscr{F}_2^{-},\ \mathscr{F}_2^{+} \subseteq \mathscr{B}^{++},\ \mathscr{F}_2^{-} \subseteq \mathscr{B}^{--}$$

be two fundamental decompositions of $\mathscr{F}$. If $(\mathscr{F}_1^{+},[\cdot,\cdot])$, $(\mathscr{F}_1^{-},-[\cdot,\cdot])$ are Hilbert spaces, then $(\mathscr{F}_2^{+},[\cdot,\cdot])$, $(\mathscr{F}_2^{-},-[\cdot,\cdot])$ they are also Hilbert spaces and the Hilbertian norms induced by both decompositions are equivalent.

Taking into account the fundamental decomposition of a Krein space, it is possible to guarantee the existence of these bases in spaces of indefinite metric taking into account the Riesz bases for the subspaces corresponding to the decomposition.

Remark 5. By virtue of [4, Theorem 3.4] If $(\mathscr{F},[\cdot,\cdot])$ is a space with inner product and if $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal system in $\mathscr{F}$, then following conditions are equivalent:

$a)$ We have to:

$$\sum_{j\in\mathbb{N}}[x,e_j]^2<\infty, \qquad x\in\mathscr{F} \tag{1.2}$$

and

$$-\sum_{[e_j,e_j]=-1}[x,e_j]^2\leq[x,x]\leq\sum_{[e_j,e_j]=1}[x,e_j]^2,\ (x\in\mathscr{F})$$

$b)$ Are fulfilled (1.2) and

$$[x,y]=\sum_{j\in\mathbb{N}}[e_j,e_j][x,e_j][e_j,y],\ (x,y\in\mathscr{F})$$

$c)$ The function

$$\|x\|=\left(\sum_{j\in\mathbb{N}}[x,e_j]^2\right)^{\frac{1}{2}},\ (x\in\mathscr{F})$$

is a quadratic norm in $\mathscr{F}$ and it defines a larger topology. With respect to said topology, it is true that:

$$x=\sum_{j\in\mathbb{N}}[e_j,e_j][x,e_j]e_j,\ (x\in\mathscr{F})$$

Lema 6. *Let* $(\mathfrak{K},[\cdot,\cdot])$ *a Krein space with associated fundamental symmetry* J *and* P *an orthogonal projection that commutes with* J*, then the spaces* $P\mathfrak{K}$ *y* $(I-P)\mathfrak{K}$ *they are Krein spaces with fundamental symmetries* PJ *and* $(I-P)J$ *respectively.*

Definition 7. Let $\mathscr{H}$ be a Hilbert space. It is said that the succession $\{x_n\}_{n\in\mathbb{N}}\subseteq\mathscr{H}$ is a Riesz basis for $\mathscr{H}$ if there are $T:\mathscr{H}\longrightarrow\mathscr{H}$ linear, continuous and invertible operator and $\{e_n\}_{n\in\mathbb{N}}\subseteq\mathscr{H}$ orthonormal basis of $\mathscr{H}$ such that $Te_n=x_n$ for each $n\in\mathbb{N}$.

The results about Riesz bases in Hilbert spaces and their elemental properties can be seen in C. Heil [2] and R. Young [5].

In the next section we will show the main results of this work as a generalization of the Riesz bases for Hilbert spaces to spaces with indefinite metrics.

2. MAIN RESULTS

2.1. Riesz bases in Krein spaces.

Proposition 8. *Let* $(\mathfrak{K},[\cdot,\cdot])$ *be a Krein space with fundamental symmetry* J *associated with the decomposition* $\mathfrak{K}=\mathfrak{K}^+\oplus\mathfrak{K}^-$. *If* $T\in\mathscr{B}(\mathfrak{K})$ *then* $\mathfrak{K}^+$ *is* T, $T^{[*]}$*-invariant if and only if* $TJ=JT$.

Proof. $\Rightarrow$) Suppose $T\mathfrak{K}^+\subseteq\mathfrak{K}^+$ and $T^{[*]}\mathfrak{K}^+\subseteq\mathfrak{K}^+$, we must show that $T\mathfrak{K}^-\subseteq\mathfrak{K}^-$. Note that if $y\in\mathfrak{K}^-$ and $z\in\mathfrak{K}^+$ then

$$[Ty,z]=[y,T^{[*]}z]=0 \tag{2.1}$$

and since $z\in\mathfrak{K}^+$ is arbitrary it follows that $Ty\in\mathfrak{K}^-$, this is, $T\mathfrak{K}^-\subseteq\mathfrak{K}^-$. Now, let $x\in\mathfrak{K}$, $x=x^++x^-$, then as $Jx^+=x^+$ and $Jx^-=-x^-$ we have following:

$$\begin{aligned} TJx &= TJ(x^++x^-) = T(x^+-x^-) \\ &= Tx^+-Tx^- = JTx^+-JTx^- \\ &= J(Tx^++Tx^-) = JTx \end{aligned}$$

$\Leftarrow$) Suppose $TJ=JT$. Note that:

$$\begin{aligned} 0 &= JT-TJ = JTI_{\mathfrak{K}}-I_{\mathfrak{K}}TJ \\ &= (P^+-P^-)T(P^++P^-)-(P^++P^-)T(P^+-P^-) \\ &= (P^+T-P^-T)(P^++P^-)-(P^+T+P^-T)(P^+-P^-) \\ &= 2(P^+TP^--P^-TP^+). \end{aligned}$$

Then $P^+TP^-=P^-TP^+$ and since $P^-P^+=0=P^+P^-$, we conclude that $T\mathfrak{K}^+\subseteq\mathfrak{K}^+$. In addition, with a ragument similar to the equation (2.1) it follows $T^{[*]}\mathfrak{K}^+\subseteq\mathfrak{K}^+$. $\square$

Remark 9. Let $(\mathfrak{K},[\cdot,\cdot])$ be a Krein space with fundamental symmetry J. If $T\in\mathscr{B}(\mathfrak{K})$ is invertible and commutes with J, then T^{-1} also commutes with J since $TJ=JT$ implies $TJT^{-1}=J$ and therefore

$$T^{-1}J=T^{-1}(TJT^{-1})=(T^{-1}T)(JT^{-1})=JT^{-1}$$

Definition 10. (*Riesz bases in Krein spaces*) Let $(\mathfrak{K},[\cdot,\cdot])$ be a Krein space with fundamental symmetry J associated with the decomposition $\mathfrak{K}=\mathfrak{K}^{+}\oplus\mathfrak{K}^{-}$. We say that $\{x_n\}_{n\in\mathbb{N}}\subseteq\mathfrak{K}$ is a Riesz basis for $\mathfrak{K}$ with respect to J if there are $T:\mathfrak{K}\longrightarrow\mathfrak{K}$ linear, continuous and inverse operator and $\{e_n\}_{n\in\mathbb{N}}\subseteq\mathfrak{K}$ orthonormal basis of $\mathfrak{K}$ such that $TJe_n=Jx_n$ for each $n\in\mathbb{N}$.

The previous concept generalizes the Riesz bases given in the definition 7 since in Hilbert spaces the fundamental symmetry J is the identity operator. From now on, when talking about continuity in a Krein space, it will be with respect to the J -norm.

Theorem 11. *Let $(\mathfrak{K},[\cdot,\cdot])$ be a Krein space with fundamental decomposition given by $\mathfrak{K}=\mathfrak{K}_1^{+}\oplus\mathfrak{K}_1^{-}$, J_1 the associated fundamental symmetry and $\{x_n\}_{n\in\mathbb{N}}$ a Riesz basis for $\mathfrak{K}$ with respect a J_1. If J_2 is the fundamental symmetry associated with the decomposition $\mathfrak{K}=\mathfrak{K}_2^{+}\oplus\mathfrak{K}_2^{-}$, then $\{x_n\}_{n\in\mathbb{N}}$ also is a Riesz basis for $\mathfrak{K}$ with respect to J_2.*

Proof. Let T be the linear, continuous and invertible operator that makes the sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq\mathfrak{K}$ a Riesz basis with respect to symmetry fundamental J_1 associated with the decomposition $\mathfrak{K}=\mathfrak{K}_1^{+}\oplus\mathfrak{K}_1^{-}$. Then, there is $\{e_n\}_{n\in\mathbb{N}}$ orthonormal basis of the Krein space $\mathfrak{K}$ that satisfies $TJ_1e_n=J_1x_n$. Well, if J_2 is the fundamental symmetry associated with the decomposition $\mathfrak{K}=\mathfrak{K}_2^{+}\oplus\mathfrak{K}_2^{-}$ y $\Gamma:\mathfrak{K}\to\mathfrak{K}$ is the operator defined by

$$\Gamma := J_2J_1TJ_1J_2$$

so Γ is also linear, continuous and invertible. Finally, it is shown that Γ makes $\{x_n\}_{n\in\mathbb{N}}\subseteq\mathfrak{K}$ a Riesz basis with respect to the fundamental symmetry J_2. Indeed, considering the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ we obtain

$$\begin{aligned}\Gamma J_2e_n &= (J_2J_1TJ_1J_2)J_2e_n = (J_2J_1TJ_1)J_2^2e_n = (J_2J_1TJ_1)e_n \\ &= (J_2J_1)TJ_1e_n = (J_2J_1)J_1x_n = J_2J_1^2x_n = J_2x_n\end{aligned}$$

□

The previous result shows that the Riesz basis condition $TJe_n=Jx_n$ for each $n\in\mathbb{N}$ associated with the T operator, is independent of the fundamental symmetry J. Therefore, in the definition

10 we can simply speak of the Riesz basis for the Krein space $\mathfrak{K}$.

Theorem 12. *Let $(\mathfrak{K}, [\cdot,\cdot])$ be a Krein space with fundamental symmetry J associated with the decomposition $\mathfrak{K} = \mathfrak{K}^+ \oplus \mathfrak{K}^-$ y T the operator that makes the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ a Riesz basis for $\mathfrak{K}$. If $\mathfrak{K}^+$ is T, $T^{[*]}$ - invariant, then there is a unique succession $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ and an orthonormal base $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ such that*

$$k = \sum_{n\in\mathbb{N}} [z_n, z_n][k, y_n]x_n$$

for each $k \in \mathfrak{K}$ and also, $\{y_n\}_{n\in\mathbb{N}}$ is biortogonal to $\{x_n\}_{n\in\mathbb{N}}$.

Proof. If $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ is a Riesz basis for $\mathfrak{K}$ with respect to J, there are $T : \mathfrak{K} \longrightarrow \mathfrak{K}$ linear operator, continuous (with respect to the J-norm) and invertible and $\{e_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ orthonormal basis of $\mathfrak{K}$ such that $TJe_n = Jx_n$ for each $n \in \mathbb{N}$. Thus, being $\{e_n\}_{n\in\mathbb{N}}$ orthonormal basis of $\mathfrak{K}$, due to [4, Theorem 3.4] it is true that:

$$T^{-1}k = \sum_{n\in\mathbb{N}} [e_n, e_n][T^{-1}k, e_n]e_n$$

Also, if $\mathfrak{K}^+$ is T, $T^{[*]}$ - invariant then T commutes with J due to the statement 8. Then,

$$\begin{aligned} k &= TT^{-1}k = T\left(\sum_{n\in\mathbb{N}} [e_n, e_n][T^{-1}k, e_n]e_n\right) = \sum_{n\in\mathbb{N}} [e_n, e_n][T^{-1}k, e_n]Te_n \\ &= \sum_{n\in\mathbb{N}} [e_n, e_n][T^{-1}k, e_n]TJJe_n = \sum_{n\in\mathbb{N}} [e_n, e_n][T^{-1}k, e_n]JTJe_n \\ &= \sum_{n\in\mathbb{N}} [e_n, e_n][T^{-1}k, e_n]JJx_n = \sum_{n\in\mathbb{N}} [e_n, e_n][T^{-1}k, e_n]x_n \\ &= \sum_{n\in\mathbb{N}} [e_n, e_n]\left[k, \left(T^{-1}\right)^{[*]} e_n\right] x_n = \sum_{n\in\mathbb{N}} [z_n, z_n][k, y_n]x_n \end{aligned}$$

Where $y_n := (T^{-1})^{[*]}e_n$ y $z_n := e_n$. Also, taking into account the observation 9 it is concluded

$$\begin{aligned} [x_k, y_n] &= [x_k, \left(T^{-1}\right)^{[*]} e_n] = [T^{-1}x_k, e_n] = [T^{-1}JJx_k, e_n] \\ &= [JT^{-1}Jx_k, e_n] = [T^{-1}Jx_k, Je_n] \\ &= [Je_k, Je_n] = [e_k, e_n] = \pm\delta_{kn} \end{aligned}$$

Therefore, $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ they are biortogonal. $\square$

Remark 13. Since every Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$, it can be seen as a Krein space with fundamental decomposition $\mathscr{H} = \mathscr{H} \oplus \{0\}$, where $(\mathscr{H}^+ = \mathscr{H}, \langle\cdot,\cdot\rangle)$ is positive definite, and $(\mathscr{H}^- = \{0\}, \langle\cdot,\cdot\rangle)$ is negative definite. Then if $\{x_n\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathscr{H}$, the Theorem 2.2 is the already known result for the Riesz bases in Hilbert spaces, that is, we have that:

$$h = \sum_{n\in\mathbb{N}} \langle z_n, z_n\rangle \langle h, y_n\rangle x_n = \sum_{n\in\mathbb{N}} \langle h, y_n\rangle x_n, \forall h \in \mathscr{H}$$

since it is known that $\{z_n\}_{n\in\mathscr{N}}$ is the orthonormal basis of $\mathscr{H}$ and that in this case the invariance of $\mathscr{H}^+$ is easy to check.

Proposition 14. *Let $(\mathfrak{K}, [\cdot,\cdot])$ be a Krein space with fundamental symmetry J associated with the decomposition $\mathfrak{K} = \mathfrak{K}^+ \oplus \mathfrak{K}^-$. If $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ is a Riesz basis for $\mathfrak{K}$, then $\{Jx_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ is also a Riesz basis for $\mathfrak{K}$.*

Proof. We have that there are $T : \mathfrak{K} \longrightarrow \mathfrak{K}$ linear operator, continuous (with respect to the J norm) and invertible and $\{e_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ orthonormal basis of $\mathfrak{K}$ such that $TJe_n = Jx_n$ for each $n \in \mathbb{N}$. Well, since J fulfills the same properties mentioned, it is concluded that the operator JT is also linear, continuous and invertible. Therefore, since

$$(JT)Je_n = J(TJe_n) = J(Jx_n)\ \forall n \in \mathbb{N}$$

we get that $\{Jx_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ is a Riesz basis for $\mathfrak{K}$ $\square$

Proposition 15. *Let $(\mathfrak{K}, [\cdot,\cdot])$ be a Krein space with fundamental symmetry J associated with the decomposition $\mathfrak{K} = \mathfrak{K}^+ \oplus \mathfrak{K}^-$. Si $\{x_n{}^+\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}^+$ y $\{y_n{}^-\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}^-$ are Riesz bases for $\mathfrak{K}^+$ and $\mathfrak{K}^-$ respectively, then $\left\{\frac{\sqrt{2}}{2}(x_n{}^+ + y_n{}^-)\right\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ is a Riesz base for $(\mathfrak{K}, [\cdot,\cdot]_J)$.*

Proof. By hypothesis there are $T^+ : \mathfrak{K}^+ \longrightarrow \mathfrak{K}^+$ and $T^- : \mathfrak{K}^- \longrightarrow \mathfrak{K}^-$ linear, continuous and invertible operators and $\{t_n{}^+\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_+$, $\{z_n{}^-\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_-$ orthonormal bases for $\mathfrak{K}_+$ and $\mathfrak{K}_-$ respectively such that $T^+t_n{}^+ = x_n{}^+$ and $T^-z_n{}^- = y_n{}^-$ for each $n \in \mathbb{N}$. Consider the operator $T : \mathfrak{K} \longrightarrow \mathfrak{K}$ defined by $Tk = T^+k^+ + T^-k^-$ for each $k \in \mathfrak{K}$ which is linear, continuous and invertible and let's see that $\left\{\frac{\sqrt{2}}{2}(t_n{}^+ + z_n{}^-)\right\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ is base J-ortonormal on $\mathfrak{K}$. Indeed,

$$\begin{aligned}
\left[\frac{\sqrt{2}}{2}(t_n{}^+ + z_n{}^-), \frac{\sqrt{2}}{2}(t_m{}^+ + z_m{}^-)\right]_J &= \left[\frac{\sqrt{2}}{2}(t_n{}^+ - z_n{}^-), \frac{\sqrt{2}}{2}(t_m{}^+ + z_m{}^-)\right] \\
&= \left[\frac{\sqrt{2}}{2}t_n{}^+, \frac{\sqrt{2}}{2}t_m{}^+\right] - \left[\frac{\sqrt{2}}{2}z_n{}^-, \frac{\sqrt{2}}{2}z_m{}^-\right] \\
&= \frac{1}{2}[t_n^+, t_m^+] - \frac{1}{2}{}^{-}_{n}, z_m^-] \\
&= \frac{1}{2}\left([t_n^+, t_m^+] - [z_n^-, z_m^-]\right) = \delta_{nm}
\end{aligned}$$

Then, $\left\{\frac{\sqrt{2}}{2}(t_n{}^+ + z_n{}^-)\right\}_{n\in\mathbb{N}}$ is base J-ortonormal in $\mathfrak{K}$. Also like,

$$\begin{aligned}
T\left(\frac{\sqrt{2}}{2}(t_n{}^+ + z_n{}^-)\right) &= \frac{\sqrt{2}}{2}T(t_n{}^+ + z_n{}^-) \\
&= \frac{\sqrt{2}}{2}T^+(t_n{}^+) + \frac{\sqrt{2}}{2}T^-(z_n{}^-) \\
&= \frac{\sqrt{2}}{2}(x_n{}^+ + y_n{}^-)
\end{aligned}$$

Therefore, $\left\{\frac{\sqrt{2}}{2}(x_n{}^+ + y_n{}^-)\right\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ is a Riesz base for $(\mathfrak{K}, [\cdot,\cdot]_J)$. $\square$

Proposition 16. *Let $\mathfrak{K}_1$, $\mathfrak{K}_2$ Krein spaces, and J_1, J_2 be the fundamental symmetries associated with the decompositions $\mathfrak{K}_1 = \mathfrak{K}_1^+ \oplus \mathfrak{K}_1^-$ and $\mathfrak{K}_2 = \mathfrak{K}_2^+ \oplus \mathfrak{K}_2^-$ respectively. If $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_1$ y $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_2$ are Riesz bases, then $\left\{\frac{\sqrt{2}}{2}(x_n \oplus y_n)\right\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_1 \oplus \mathfrak{K}_2$ is a Riesz base for $\mathfrak{K}_1 \oplus \mathfrak{K}_2$.*

Proof. Let $T_1 : \mathfrak{K}_1 \longrightarrow \mathfrak{K}_1$, $T_2 : \mathfrak{K}_2 \longrightarrow \mathfrak{K}_2$ and $\{e_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_1$, $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_2$ linear, continuous operators and invertible and the orthonormal bases that make $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_1$ and $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_2$ Riesz bases in $\mathfrak{K}_1$ and $\mathfrak{K}_2$ respectively. Notice that the following product

$$[\cdot,\cdot] : \mathfrak{K}_1 \oplus \mathfrak{K}_2 \times \mathfrak{K}_1 \oplus \mathfrak{K}_2 \longrightarrow \mathbb{C}$$

defined by

$$[\cdot,\cdot] = [\cdot,\cdot]_1 + [\cdot,\cdot]_2$$

turns out to be an inner product. Consider the operator $T : \mathfrak{K}_1 \oplus \mathfrak{K}_2 \longrightarrow \mathfrak{K}_1 \oplus \mathfrak{K}_2$ given by $T = T_1 \oplus T_2$. This operator is linear, continuous and invertible, since T_1 and T_2 are. It remains to show that $\left\{\frac{\sqrt{2}}{2}(e_n \oplus t_n)\right\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}_1 \oplus \mathfrak{K}_2$ is the orthonormal basis of $\mathfrak{K}_1 \oplus \mathfrak{K}_2$. In effect, note that:

$$\begin{aligned}\left[\frac{\sqrt{2}}{2}(e_k \oplus t_k), \frac{\sqrt{2}}{2}(e_n \oplus t_n)\right] &= \left[\frac{\sqrt{2}}{2}e_k, \frac{\sqrt{2}}{2}e_n\right]_1 + \left[\frac{\sqrt{2}}{2}t_k, \frac{\sqrt{2}}{2}t_n\right]_2 \\ &= \frac{1}{2}[e_k, e_n]_1 + \frac{1}{2}[t_k, t_n]_2 = \pm\frac{1}{2}\delta_{kn} + \pm\frac{1}{2}\delta_{kn} = \pm\delta_{kn}\end{aligned}$$

Also,

$$\begin{aligned}T(J_1 \oplus J_2)\left(\frac{\sqrt{2}}{2}(e_n \oplus t_n)\right) &= T_1J_1\left(\frac{\sqrt{2}}{2}e_n\right) \oplus T_2J_2\left(\frac{\sqrt{2}}{2}t_n\right) \\ &= \frac{\sqrt{2}}{2}(T_1J_1e_n \oplus T_2J_2t_n) = \frac{\sqrt{2}}{2}(J_1x_n \oplus J_2y_n) \\ &= (J_1 \oplus J_2)\left(\frac{\sqrt{2}}{2}(x_n \oplus y_n)\right)\end{aligned}$$

$\therefore \{\frac{\sqrt{2}}{2}(x_n \oplus y_n)\}$ is a Riesz base for $\mathfrak{K}_1 \oplus \mathfrak{K}_2$. □

Theorem 17. *Let $(\mathfrak{K}, [\cdot,\cdot])$ be a Krein space with fundamental symmetry J associated with the decomposition into $\mathfrak{K} = \mathfrak{K}^+ \oplus \mathfrak{K}^-$ and $\{x_n\}_{n\in\mathbb{N}}$ a Riesz basis for $\mathfrak{K}$. If P is an orthogonal projector of $\mathfrak{K}$ commuting with J, then $\{Px_n\}_{n\in\mathbb{N}}$ is a Riesz basis for Krein's space $P\mathfrak{K}$.*

Proof. Let T be the linear, continuous and inverse operator that makes the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{K}$ a Riesz basis for $\mathfrak{K}$. Now, if P is an orthogonal projector that commutes with J, it is concluded by the Lemma 6 that $P\mathfrak{K}$ is a Kerin space with the fundamental symmetry PJ associated with the decomposition $P\mathfrak{K} = P\mathfrak{K}^+ \oplus P\mathfrak{K}^-$. On the other hand, if $\Gamma : \mathfrak{K} \to \mathfrak{K}$ is the operator defined by

$$\Gamma := PTP$$

then Γ is also linear, continuous and invertible and by hypothesis there is $\{e_n\}_{n\in\mathbb{N}}$ orthonormal basis of the Kerin space $\mathfrak{K}$ that complies with $TJe_n = Jx_n$ for each $n \in \mathbb{N}$, and consequently $Je_n = T^{-1}Jx_n$. Then, taking into account that P commutes with J and that $P^2 = P$ we obtain

$$\begin{aligned}\Gamma(PJ)e_n &= (PTP)(PJ)e_n = (PT)P^2Je_n = (PT)P(T^{-1}J)x_n \\ &= (PTP)(T^{-1}J)x_n = (PTP)PP^{-1}(T^{-1}J)x_n = (PTP^2)(P^{-1}T^{-1})Jx_n \\ &= P(TP)(TP)^{-1}Jx_n = PJx_n = PP(Jx_n) = (PJ)(Px_n)\end{aligned}$$

Therefore, it is concluded that Γ makes $\{Px_n\}_{n\in\mathbb{N}}$ a Riesz basis for $P\mathfrak{K}$. $\square$

2.2. Example. Consider the vector space $\mathbb{C}^2$ over $\mathbb{C}$, with the usual sum and product and the function $[\cdot,\cdot]:\mathbb{C}^2\times\mathbb{C}^2\longrightarrow\mathbb{C}$ given by

$$[(x_1,y_1),(x_2,y_2)] = x_1\overline{x_2} - y_1\overline{y_2}. \tag{2.2}$$

Well, it turns out that the space with inner product $(\mathbb{C}^2,[\cdot,\cdot])$ is a Krein space with fundamental decomposition $\mathbb{C}^2 = \mathscr{K}^+[\dotplus]\mathscr{K}^-$, where $\mathscr{K}^+ = \{(x,0):\ x\in\mathbb{C}\}$ and $\mathscr{K}^- = \{(0,y):\ y\in\mathbb{C}\}$. Then, the fundamental symmetry

$$J((x,y)) = P^+(x,y) - P^-(x,y) = (x,-y)$$

determines the J -norm $\|\cdot\|_J$ such that

$$\|(x,y)\|_J = [J(x,y),(x,y)]^{1/2} = (x\cdot\bar{x} - (-y)\cdot\bar{y})^{1/2} = \sqrt{|x|^2+|y|^2}$$

On the other hand, considering the orthonormal base $\{(1,0),(0,1)\}$ and the operator $T=J$ linear, continuous and invertible, the Riesz base $\{(1,0),(0,-1)\}$ since $TJ=I$ and therefore

$$TJ(1,0) = (1,0) = J(1,0) \quad \textit{and} \quad TJ(0,1) = (0,1) = J(0,-1)$$

3. Conclusion

It is possible to conclude, by virtue of the theorem 11, that a Riesz basis does not depend on the fundamental decomposition of the Krein space. Therefore, we are able to extend the Riesz basis definition, from Hilbert spaces to indefinite metric spaces, known as Krein spaces.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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