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A PROPOSAL FOR REVISITING SOME FIXED POINT RESULTS IN DISLOCATED QUASI-METRIC SPACES

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Abstract. In this paper, we prove some new fixed point theorems in a dislocated quasi-metric spaces for a self mapping, which unify and generalize some existing relevant fixed point theorems. Moreover, many examples are provided to illustrate our improvements.

Keywords: dislocated quasi-metric space; dq-converges; dq-limit; contraction mapping; fixed point.

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1. Introduction

In 1886, Pointcaré presented one of the most dynamic research subjects in nonlinear analysis, which is the notion of the fixed point. In 1922, Banach introduced a powerful tool in nonlinear analysis, which is the Banach contraction principle [1]. Since then, this contraction principle has been generalized in several directions and in different spaces see e.g., [2–5] end the references therein. In 2000, Hitzler et al introduced the concepts of dislocated metric spaces and established a fixed point theorem, which generalized the Banach contraction principle in such

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spaces [6]. Afterward, various generalizations of those spaces are introduced and many fixed point results were established see e.g., [7–10, 12] and references therein.

Here, we recall some relevant definitions which will be needed in our subsequent discussion.

Definition 1.1. *Let X be a non empty set and $d : X \times X \to \mathbb{R}^+$ be a function such that*

(1) $d(x,y) = d(y,x) = 0$ implies $x = y$,

(2) $d(x,y) \leq d(x,z) + d(z,y)$ for all x, y, $z \in X$.

Then, d is called dislocated quasi-metric (or simply dq-metric) on X.

Definition 1.2. *A sequence $\{x_n\}$ in a dq-metric space (X,d) is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that,*

$$d(x_m,x_n) \leq \varepsilon \ \text{ or } \ d(x_n,x_m) \leq \varepsilon, \ \text{ for all } \ m, \ n \geq N.$$

Definition 1.3. *A sequence $\{x_n\}$ is said to be dq-convergent to x in a dq-metric space X, if*

$$\lim_{n\infty} d(x_n,x) = \lim_{n\infty} d(x,x_n) = 0.$$

Here, x is called dq-limit of sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty$.

Definition 1.4. *Let (X,d_1) and (Y,d_2) be two dq-metric spaces, the function $f : X \to Y$ is said to be continuous if for each sequence $\{x_n\} \subset X$ which dq-converges to x in X, the sequence $\{f(x_n)\}$ is dq-converges to $f(x)$ in Y.*

Definition 1.5. *A dq-metric space (X,d) is called complete if every Cauchy sequence in X is dq-convergent.*

Recently, Anu [13] proved the following interesting generalization of Banach contraction principle for a continuous self mapping in dislocated quasi-metric space.

Theorem 1.1. *Let (X,d) be a complete dq-metric space and $T : X \to X$ a continuous mapping which satisfies*

$$\begin{aligned} d(Tx,Ty) \leq{}& \alpha d(x,y) + \beta \frac{d(x,Tx)\,d(y,Ty)}{d(x,y)+d(x,Tx)} + \gamma \frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(y,Ty)} \\ &+ \delta \frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(x,Tx)} + \mu(d(x,Tx)+d(y,Ty)) \end{aligned} \tag{1.1}$$

for all x, $y \in X$ with $d(x,y) \neq 0$, and where α, β, γ, δ, $\mu \in \mathbb{R}^+$ verifying

$$0 < \alpha + \frac{\beta}{2} + \gamma + 2\delta + 2\mu < 1.$$

Then, the self-mapping T has a unique fixed point.

The purpose of this paper is to extend some results concerning generalized contractions of Theorem 1.1. Indeed, a new contraction which generalized the one used in Theorem 1.1 is introduced, then a proof of Theorem 1.1, without any continuity requirement, is given and lastly some examples illustrating our results are provided.

2. Main Results

Here, we provide two new contraction conditions of fixed point theorems in dq-metric spaces.

Theorem 2.1. *Let (X,d) be a complete dq-metric space and $T : X \to X$ a mapping which satisfies*

$$d(Tx,Ty) \leq \lambda \max \left\{ \begin{array}{l} 2d(x,y),\ 4\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)+d(x,Tx)},\ 2\frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(y,Ty)}, \\ \frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(x,Tx)},\ d(x,Tx)+d(Ty,y) \end{array} \right\}, \tag{2.1}$$

for all x, $y \in X$ with $d(x,y) \neq 0$ and where $\lambda \in [0, \frac{1}{2})$. Then, T has a unique fixed point.

Proof. Let T be a self mapping of X such that the condition (2.1) holds. We consider

$$M(x,y) = \max \left\{ \begin{array}{l} 2d(x,y), 4\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)+d(x,Tx)}, 2\frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(y,Ty)}, \\ \frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(x,Tx)},\ d(x,Tx)+d(x,y) \end{array} \right\},\ \forall x,\, y \in X.$$

Next, we will distinguish the following two cases :

- If $M(x,y) = 4\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)+d(x,Tx)}$ and in this case, we consider $\eta \in X$. First, if $T\eta = \eta$, the mapping T has a fixed point. Next, we assume that $T\eta \neq \eta$. Thus, by taking $x = \eta$ and $y = T\eta$ in inequality (2.1), we obtain

$$d(T\eta, T^2\eta) \leq 4\lambda\, \frac{d(\eta,T\eta)\,d(T\eta,T^2\eta)}{d(\eta,T\eta)+d(\eta,T\eta)} \leq\ 2\lambda\ d(T\eta,T^2\eta). \tag{2.2}$$

Remembering $\lambda \in [0, \frac{1}{2})$, we find $d(T\eta, T^2\eta) = 0$. Moreover, from (2.1), we have

$$d(T^2\eta, T\eta) \leq \ 4\lambda \ \frac{d(T\eta, T^2\eta)\, d(\eta, T\eta)}{d(T\eta, \eta) + d(T\eta, T^2\eta)} = 0 \tag{2.3}$$

Then, $d(T\eta, T^2\eta) = d(T^2\eta, T\eta) = 0$ and hence $T\eta$ is a fixed point of T.

- If $M(x,y) = \max\left\{2d(x,y), \frac{2\,d(x,Tx)d(x,Ty)}{d(x,y)+d(y,Ty)}, \frac{d(x,Tx)d(x,Ty)}{d(x,y)+d(x,Tx)}, d(x,Tx)+d(x,y)\right\}$, let $x_0 \in X$ and consider a sequence $\{x_n\}$ in X defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $N \in \mathbb{N}$ such that $x_{N+1} = x_N$, then x_N is a fixed point of T. Next, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}^*$, we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda\, M(x_{n-1}, x_n), \tag{2.4}$$

where the quantity $M(x_{n-1}, x_n)$ is given by

$$\begin{aligned} M(x_{n-1},x_n) &= \max\left\{\begin{matrix} 2d(x_{n-1},x_n), \frac{2\,d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)}{d(x_{n-1},x_n)+d(x_n,Tx_n)}, \\ \frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)}{d(x_{n-1},x_n)+d(x_{n-1},Tx_{n-1})}, d(x_{n-1},Tx_{n-1})+d(x_{n-1},x_n) \end{matrix}\right\} \\ &= \max\left\{\begin{matrix} 2d(x_{n-1},x_n), \frac{2\,d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})}, \\ \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{d(x_{n-1},x_n)+d(x_{n-1},x_n)}, d(x_{n-1},x_n)+d(x_{n-1},x_n) \end{matrix}\right\} \\ &= \max\left\{2d(x_{n-1},x_n), \frac{2\,d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})}, \frac{d(x_{n-1},x_{n+1})}{2}\right\}. \end{aligned} \tag{2.5}$$

In addition, we use the triangular inequality to get that, for all $n \in \mathbb{N}^*$, we have

$$\frac{d(x_{n-1},x_n)\, d(x_{n-1},x_{n+1})}{d(x_{n-1},x_n) + d(x_n,x_{n+1})} \leq \frac{d(x_{n-1},x_n)\, d(x_{n-1},x_{n+1})}{d(x_{n-1},x_{n+1})} = d(x_{n-1},x_n). \tag{2.6}$$

Combining inequalities (2.5) and (2.6), we deduce

$$M(x_{n-1},x_n) \leq \max\left\{2\,d(x_{n-1},x_n), \frac{1}{2}\,d(x_{n-1},x_{n+1})\right\}, \ \forall n \in \mathbb{N}^*. \tag{2.7}$$

Next, it follows from (2.4) and (2.7) that for $n \in \mathbb{N}^*$, we have

$$d(x_n,x_{n+1}) \leq 2\lambda\, d(x_{n-1},x_n) \ \text{ or } \ d(x_n,x_{n+1}) \leq \frac{\lambda}{2}\, d(x_{n-1},x_{n+1}), \tag{2.8}$$

which implies that for $n \in \mathbb{N}^*$, we find

$$d(x_n,x_{n+1}) \leq 2\lambda\, d(x_{n-1},x_n) \ \text{ or } \ d(x_n,x_{n+1}) \leq \frac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}}\, d(x_{n-1},x_n), \tag{2.9}$$

Then, since $2\lambda \in [0,1)$ and $\frac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}} \in [0,1)$, we conclude that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \ \forall n \in \mathbb{N}^*. \tag{2.10}$$

Then, $\{x_n\}$ is a Cauchy sequence in a complete space X and there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_{n+1}, u) = \lim_{n\to\infty} (Tx_n, u) = \lim_{n\to\infty} (x_n, u) = 0, \tag{2.11}$$

$$\lim_{n\to\infty} d(u, x_{n+1}) = \lim_{n\to\infty} (u, Tx_n) = \lim_{n\to\infty} (u, x_n) = 0. \tag{2.12}$$

Let us now consider $n \in \mathbb{N}^*$, we have

$$\begin{aligned} d(Tu, Tx_n) &\leq \lambda M(u, x_n) \\ &\leq \lambda \max \left\{ \begin{array}{l} 2d(u,x_n),\ 2\frac{d(u,Tu)d(u,Tx_n)}{d(u,x_n)+d(x_n,Tx_n)}, \\ \frac{d(u,Tx_n)d(u,Tx_n)}{d(u,x_n)+d(u,Tu)},\ d(u,Tu)+d(Tx_n,x_n) \end{array} \right\} \\ &\leq \lambda \max \left\{ \begin{array}{l} 2d(u,x_n),\ 2d(u,Tu), \\ \frac{d(u,Tx_n)d(u,Tx_n)}{d(u,x_n)+d(u,Tu)},\ d(u,Tu)+d(Tx_n,x_n) \end{array} \right\}. \end{aligned} \tag{2.13}$$

Letting $n \to \infty$ in the previous inequality, we obtain

$$d(Tu, u) \leq \lambda \max \{2d(u,Tu),\ d(u,Tu)\} = 2\lambda\, d(u,Tu). \tag{2.14}$$

In addition, we have

$$\begin{aligned} d(Tx_n, Tu) &\leq \lambda M(x_n, u) \\ &\leq \lambda \max \left\{ \begin{array}{l} 2d(x_n,u),\ 2\frac{d(x_n,Tx_n)\,d(x_n,Tu)}{d(x_n,u)+d(u,Tu)}, \\ \frac{d(x_n,Tx_n)\,d(x_n,Tu)}{d(x_n,u)+d(x_n,Tx_n)},\ d(x_n,Tx_n)+d(Tu,u) \end{array} \right\}. \end{aligned} \tag{2.15}$$

Keeping in mind the following inequality

$$\frac{d(x_n,Tx_n)\,d(x_n,Tu)}{d(x_n,u)+d(x_n,Tx_n)} \leq d(x_n,Tu), \tag{2.16}$$

the inequality (2.15) leads to

$$(2.17) \qquad d(Tx_n, Tu) \leq \lambda \max \left\{ \begin{array}{l} 2\,d(x_n,u)\,,\; 2\,\frac{d(x_n,Tx_n)\,d(x_n,Tu)}{d(x_n,u)+d(u,Tu)}, \\ \\ d(x_n,Tu)\,,\; d(x_n,Tx_n)+d(Tu,u) \end{array} \right\}.$$

Passing to limit in (2.17) as $n \to \infty$ and using (2.14), we get

$$(2.18) \qquad d(u,Tu) \leq \lambda \max\{d(u,Tu)\,,\; d(Tu,u)\} \leq \max(\lambda, 2\lambda^2)\,d(u,Tu).$$

Thus, since $\lambda,\, 2\lambda^2 \in [0, \frac{1}{2})$, it follows from (2.18) and (2.14) that

$$(2.19) \qquad d(u,Tu) = d(Tu,u) = 0.$$

Hence, $Tu = u$, and then T has at least one fixed point in X, which finishes the existence part.

For the uniqueness, let u, $v \in X$ two fixed points of T such that $u \neq v$. From (2.1), we have

$$(2.20) \qquad \begin{aligned} d(u,u) &= d(Tu,Tu) \\ &\leq \lambda \max \left\{ \begin{array}{l} 2\,d(u,u)\,,\; 4\,\frac{d(u,Tu)\,d(u,Tu)}{d(u,u)+d(u,Tu)}\,,\; 2\,\frac{d(u,Tu)\,d(u,Tu)}{d(u,u)+d(u,Tu)}, \\ \\ \frac{d(u,Tu)\,d(u,Tu)}{d(u,u)+d(u,Tu)}\,,\; [d(u,Tu)+d(Tu,u)] \end{array} \right\} \\ &\leq \lambda \max\left\{ 2\,d(u,u)\,,\; \frac{d(u,u)}{2} \right\} \\ &\leq 2\lambda\, d(u,u). \end{aligned}$$

Since $\lambda \in [0, \frac{1}{2})$, the above inequality implies that $d(u,u) = 0$, and similarly, we have

$$d(v,v) = 0.$$

On the other hand, we use (2.1) to conclude that

$$
\begin{aligned}
d(u,v) = d(Tu,Tv) &\leq \lambda \max \left\{ \begin{array}{l} 2\,d(u,v),\; 4\,\frac{d(u,Tu)\,d(v,Tv)}{d(u,v)+d(u,Tu)},\; 2\,\frac{d(u,Tu)\,d(u,Tv)}{d(u,v)+d(v,Tv)}, \\ \frac{d(u,Tu)\,d(u,Tv)}{d(u,v)+d(u,Tu)},\; [d(u,Tu)+d(Tv,v)] \end{array} \right\} \\
&\leq 2\lambda\, d(u,v).
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
d(v,u) = d(Tv,Tu) &\leq \lambda \max \left\{ \begin{array}{l} 2\,d(v,u),\; 4\,\frac{d(v,Tv)\,d(u,Tu)}{d(v,u)+d(v,Tv)},\; 2\,\frac{d(v,Tv)\,d(v,Tu)}{d(v,u)+d(u,Tu)}, \\ \frac{d(v,Tv)\,d(v,Tu)}{d(v,u)+d(v,Tu)},\; [d(v,Tv)+d(Tu,u)] \end{array} \right\} \\
&\leq 2\lambda\, d(v,u).
\end{aligned} \tag{2.22}
$$

Lastly, since $2\lambda \in [0,1)$, the inequalities (2.21) and (2.22) imply that $d(u,v) = d(v,u) = 0$, which is a contradiction. Then we have one and only one fixed point. $\square$

Example 2.2. *Consider the set $X = \left\{0, \frac{1}{9}, 100\right\}$ endowed with the dq-metric d given by*

$$d(x,y) = x + 2y,\ \forall x, y \in X.$$

We construct a mapping $T : X \to X$ by $T0 = 0$, $T100 = \frac{1}{9}$ and $T\frac{1}{9} = 0$. For $\lambda = \frac{1}{3}$, it is clear that all the assumptions of Theorem 2.1 hold, and then, 0 is the unique fixed point of T .

We note here that from Theorem 2.1, we can deduce immediately Theorem 1.1. Now, we give a new result similar to Theorem 1.1, in which we omit the continuity assumption of T.

Theorem 2.3. *Let (X,d) be a complete dq-metric space and $T : X \to X$ a self-mapping. If*

$$
\begin{aligned}
d(Tx,Ty) \leq\ & \alpha d(x,y) + \beta \frac{d(x,Tx)\,d(y,Ty)}{d(x,y)+d(x,Tx)} + \gamma \frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(y,Ty)} \\
& + \delta \frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(x,Tx)} + \mu\,[d(x,Tx)+d(y,Ty)]
\end{aligned} \tag{2.23}
$$

for all x, $y \in X$ with $d(x,y) \neq 0$ and where α, β, γ, δ, $\mu \in \mathbb{R}^+$ such that

$$0 < \alpha + \frac{\beta}{2} + \gamma + 2\delta + 2\mu < 1.$$

Then, the self-mapping T has a unique fixed point.

Proof. Let T a self mapping of X which verifies assumptions of Theorem 2.3 and consider

$$M(x,y)=\max\left\{\begin{array}{l}2\,d(x,y),4\,\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)+d(x,Tx)},2\,\frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(y,Ty)},\\ \frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(x,Tx)},[d(x,Tx)+d(x,y)]\end{array}\right\}. \tag{2.24}$$

Then, by using (2.23) and (2.24), we find

$$d(Tx,Ty)\leq\frac{\alpha}{2}M(x,y)+\frac{\beta}{4}M(x,y)+\frac{\gamma}{2}M(x,y)+\delta M(x,y)+\mu M(x,y) \tag{2.25}$$

Taking $\lambda=\frac{\alpha}{2}+\frac{\beta}{4}+\frac{\gamma}{2}+\delta+\mu\in[0,\frac{1}{2})$, the inequality (2.25) can be written as follows.

$$d(Tx,Ty)\leq\lambda\,M(x,y). \tag{2.26}$$

Hence, T satisfies the conditions of Theorem 2.1 and then T has a unique fixed point in X. □

We now state another result, which generalized Theorem 2.1.

Theorem 2.4. *Let (X,d) be a complete dq-metric space and $T:X\to X$ a self mapping. If*

$$d(Tx,Ty)\leq\lambda\,\max\left\{\begin{array}{l}a\,d(x,y),2a\,\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)+d(x,Tx)},a\,\frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(y,Ty)},\\ \frac{d(x,Tx)\,d(x,Ty)}{d(x,y)+d(x,Tx)},[d(x,Tx)+d(x,y)]\end{array}\right\}, \tag{2.27}$$

for all x, $y\in X$ with $d(x,y)\neq 0$, $\lambda\in[0,\frac{1}{a})$ and $a\geq 2$. Then, T has a unique fixed point in X.

Proof. It can be obtained in a similar way to that used in the proof of Theorem 2.1. □

Finally, we give the following example illustrating the main result Theorem 2.4.

Example 2.5. *Let $X=\left\{0,\frac{1}{13},11\right\}$ equipped with the following dq-metric*

$$d(x,y)=x+2y,\ \forall x,y\in X.$$

We consider the mapping $T:X\to X$ defined by

$$T0=0,\ T11=\frac{1}{13},\ T\frac{1}{13}=0.$$

For $\lambda=\frac{1}{3}$, $a=3$, all assumptions of Theorem 2.4 hold. Then, 0 is the unique fixed point of T.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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