Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 2711-2726
https://doi.org/10.28919/jmcs/5545
ISSN: 1927-5307

EXTENSIONS OF FUZZY SOFT SETS AND RINGS

WAHEED AHMAD KHAN[1,*], ABDELGHANI TAOUTI[2]

[1]Department of Mathematics, University of Education Lahore, Attock campus, Pakistan
[2]ETS-Maths and NS Engineering Division, HCT, University City P.O. Box 7947, Sharjah, United Arab Emirates

Abstract. In this note, first we introduce the concept of extension of fuzzy soft subset (set) by utilizing its extension of mapping and re-define some operations of fuzzy soft sets based on the extension. Furthermore, we also discuss the extension of fuzzy soft set with respect to t-level soft sets of fuzzy soft sets. We explore the connection between both types of the extensions in example4.1. As an evidence we present the extension of fuzzy soft subrings and fuzzy soft ideals while considering the extension through t-level soft sets.

Keywords: extension of mapping; fuzzy soft sets; t-level soft set.
2010 AMS Subject Classification: 06E20, 08A72.

1. INTRODUCTION

Number of theories such as fuzzy sets [29], theory of vague sets [11], theory of interval mathematics [12], theory of rough sets [23] are famous and are using in analyzing data. But there are some imperfections in all these theories. In [21], Molodtsov introduced the concept of soft set and also he proved that soft set works more efficient than any other tool to remove uncertainties. Molodtsov successfully applied the soft theory in several directions, for instance smoothness of functions, game theory, operations research, Riemann integration, Perron integration, theory

*Corresponding author
E-mail address: sirwak2003@yahoo.com
Received February 13, 2021

of probability, theory of measurement, and so on. Numerous operations have been introduced and applied to soft sets (see [4], [26], [18] etc.,). Subsequently, the number of soft algebraic structures like soft groups [3], soft rings [1], soft near-semirings [14], fuzzy soft near-semirings [27] etc have been introduced in the literature.

Many researchers introduced the number of fuzzy algebraic structures in the literature, fuzzy rings were intrdouced by Liu [15]. Martinez [20] and Dixit et al. [8] further obtained certain ring theoretical analogues.

In [19], authors introduced the concept of the fuzzy soft sets by embedding the ideas of fuzzy sets. In a sequal, A. R. Roy and P. K. Maji gave some applications of fuzzy soft sets (see[25]). Further to this, Çağman et al. redefined the notion of fuzzy soft sets with new operations and applications [6]. Various fuzzy soft algebraic structures have been introduced such as fuzzy soft groups [5], fuzzy soft rings ([13]&[28]), fuzzy soft subnear-rings [22], fuzzy soft semirings [24], fuzzy soft subnear-semirings [27] etc. The concept of level soft sets of fuzzy soft set have been introduced and discussed in [5]. Furthermore, different kinds of level soft sets such as t-level soft sets, mid-level soft sets and top-level soft sets along with their applications have been discussed in [9].

In [16], extension of soft sets have been discussed. In this note, we introduce the notion of extension of fuzzy soft sets, we call it a fuzzy soft extension. We also re-define and simplify operations between fuzzy soft sets at the base of extension of fuzzy soft sets. We utilize the term "level soft set" and provide an alternate definition of extension of fuzzy soft sets. In a sequal, we provide the necessary and sufficient condition for the extension of fuzzy soft subsets. Finally, we introduce and discuss the extension of fuzzy soft subrings and fuzzy soft ideals of a ring.

2. Preliminaries

In this section, we recall few basic definitions and terminologies of soft set theory [21], fuzzy set theory [29] and fuzzy soft theory [19], [6], [25] that are useful for subsequent discussions.

Throughout, U refers to the initial universe, E for set of parameters, $\wp(U)$ is the power set of U.

Definition 2.1. [21]*A pair (F, E) is called a soft set over universal set U, where F is a mapping from a set of parameters (E) to power set $\wp(U)$ i.e., $F : E \to P(U)$.*

Furthermore, P. K. Maji et al. have introduced various operations on soft sets in [18], such as soft subset, intersection and bi-intersection of two soft sets, operations "*AND*", operation "*OR*" between two soft sets and so on.

Definition 2.2. [29] *A fuzzy set A in X is characterized by a membership characteristic function*

$\mu_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$, with the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A.

Definition 2.3. [8]*Let R be a ring and μ be a fuzzy subset of R. Then, μ is a fuzzy subring of R if (i) $\mu(x-y) \geq min(\mu(x), \mu(y))$, (ii) $\mu(xy) \geq min(\mu(x), \mu(y))$ for all $x, y \in R$.*

Fuzzy soft set has been introduced and discussed in [19], which is the fuzzy generalization of (crisp) soft sets. After that, N. Çağman, S. Enginoğlu and F. Çitak have further explored fuzzy soft sets and gave few applications [6].

Definition 2.4. [19] *Let U be an initial universe set, E be a set of parameters and $A \subset E$. Let (U) denotes the fuzzy power set of U. A pair (F, A) is called a fuzzy soft set over U, where F is the mapping $F : A \to (U)$.*

Definition 2.5. [19] *A fuzzy soft set (F, A) over U is said to be a null fuzzy soft set denoted by $\varnothing$, if $\forall e \in A, F(e) = \varnothing$.*

Definition 2.6. [19] *A fuzzy soft set (F, A) over U is said to be an absolute fuzzy soft set, if $\forall$ $e \in A$, $F(e) = U$. It is denoted by $\widetilde{A}$. Clearly, $(\widetilde{A})^C = \varnothing$ and $\varnothing^C = \widetilde{A}$.*

Definition 2.7. [19] *The complement of a fuzzy soft set (F, A) is denoted by $(F, A)^C$ and is defined by $(F, A)^C = (F^C, \neg A)$, where $F^C : \neg A \to (U)$. The mapping is $F^C(\neg a) = U - F(a)$, for all $a \in A$.*

Definition 2.8. [19] *Let (F, A) and (G, B) be two fuzzy soft sets over the common universe U. Then,*

(1) (F, A) *is a fuzzy soft subset of* (G, B) *if* (i) $A \subset B$ *and* (ii) *For each* $a \in A$, $F(a)$ *is a fuzzy subset of* $G(a)$. *It is denoted by* $(F, A)\widetilde{\subseteq}(G, B)$.

(2) *Intersection of* (F, A) *and* (G, B) *is the fuzzy soft set* (H, C), *where* $C = A \cap B$ *and* $H(e) = F(e)$ *or* $G(e)$, *for all* $e \in C$. *It is denoted by* $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

(3) *Union of* (F, A) *and* (G, B) *is the fuzzy soft set* (H, C), *where* $C = A \cup B$ *and*

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B \\ G(e), & \text{if } e \in B - A \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

and is denoted by $(F, A)\widetilde{\cup}(G, B) = (H, C)$.

(4) (F, A) *AND* (G, B) *denoted by* $(F, A) \wedge (G, B)$ *and is defined by* $(F, A) \wedge (G, B) = (H, A\times B)$, *where* $H(a, b) = F(a) \cap G(b)$ *for all* $a, b \in (A\times B)$.

(2) (F, A) *OR* (G, B) *denoted by* $(F, A) \vee (G, B)$ *is defined by* $(F, A) \vee (G, B) = (H, A\times B)$, *where* $H(a, b) = F(a) \cup G(b)$ *for all* $a, b \in (A\times B)$.

It is easy to verify that every fuzzy soft set is a soft set. For proving this, let (F, E) be a fuzzy soft set, where $F : E \to I^X$ is a mapping given by $F(e) = F_e \in I^X$, for all $e \in E$. Here, F_e is a fuzzy set, i.e, $F_e : X \to I$. Now by using the fuzzy set F_e, we can define a function $\bar{F} : A \times I \to \wp(X)$ such that $\bar{F}(e, \alpha) = F(e)_\alpha$, for all $(e, \alpha) \in A \times I$, where $(F_e)_\alpha$ is α-level set of the fuzzy set F_e. Thus, (F, E, I) is a soft set.

Fuzzy soft rings and fuzzy soft ideals have been introduced and discussed in ([13]&[28]). However, both manuscripts containing the same essence, but we present definitions from both articles which are necessary for forthcoming dicussions and proving results.

Definition 2.9. [13] *Let* (F, A) *be a fuzzy soft set over R. Then,*

(1) (F, A) *is said to be a fuzzy soft ring over R, if* $F(x)$ *is a fuzzy subring of R for all* $x \in A$.

(2) *A fuzzy soft set* (G, I) *over R is called a fuzzy soft ideal of* (F, A), *if it satisfies:* (i) $I \subset A$, (ii) $G(x)$ *is a fuzzy ideal of* $F(x)$ *for all* $x \in I$.

(3) (F, A) *is said to be an identity fuzzy soft ring over R if* $F(x) = \{e\}$ *for all* $x \in A$, *where e is the identity element of R.*

(4) (F, A) *is said to be an absolute fuzzy soft ring over R, if* $F(x) = R$ *for all* $x \in A$.

Definition 2.10. [13] *Let (F, A) and (H, B) be two fuzzy soft rings over R. If $F(x) \subseteq H(x)$ for all $x \in A$, then $F(x)$ is a soft subring of $H(x)$.*

Definition 2.11. [28] *Let (f, A) be a non-null fuzzy soft set over R. Then,*

(1) (f, A) is a fuzzy soft ring over R iff $f(a) = f_a$ is a fuzzy subring of R for each $a \in A$, that is,

(i) $f_a(x-y) \geq f_a(x) \wedge f_a(y)$

(ii) $f_a(xy) \geq f_a(x) \wedge f_a(y)$, for all $x, y \in R$.

(2) A non-empty fuzzy soft set (g, B) over R is called a fuzzy soft ideal of (f, A), if it satisfies the followings.

(i) $g_a(x-y) \geq g_a(x) \wedge g_a(y)$,

(ii) $g_a(xy) \geq g_a(x) \wedge g_a(y)$, $x, y \in R$.

In [5], the authors introduced the notion of level soft sets of fuzzy soft sets while embedding the α-level set of the fuzzy set $(f_a)_\alpha$. Also, in [9] the authors utilized the term level soft sets while introducing the adjustable approach to fuzzy soft set based decision making problems.

Definition 2.12. [5] *Let (F, A) be a fuzzy soft set over X. The soft set $(F, A)_\alpha = \{(F_a)_\alpha : a \in A\}$, for each $\alpha \in (0, 1]$ is called an α-level soft set of the fuzzy soft set (F, A), where $(F_a)_\alpha$ is an α-level set of the fuzzy set F_a.*

Feng et al. (see[9]) have introduced the notion of t-level soft sets for fuzzy soft sets, which extends the classical level sets of fuzzy sets in a natural way.

Definition 2.13. [9] *Let $\Im = (F, A)$ be a fuzzy soft set over U, where $A \subseteq E$ and E is the parameter set. For $t \in [0, 1]$, the t-level soft set of the fuzzy soft set $\Im$ is a crisp soft set $L(\Im, t) = (F_t, A)$ defined as*

$$F_t(a) = L(F(a), t) = \{x \in U : F(a)(x) \geq t\}, \text{ for all } a \in A.$$

In the above definition, the level (or threshold) assigned to each parameter is always a constant value $t \in [0, 1]$. But in [9], the authors pointed out that sometimes it may be required that variable thresholds should be imposed among different parameters. In [10], the authors further studied and investigated the level soft sets based on variable thresholds given by a fuzzy set. If (U, E) be a soft universe and $A \subseteq E$. Let $\Im = (F, A)$ be a fuzzy soft set over U and let $\lambda : A \to [0, 1]$ be a fuzzy set in A called a threshold fuzzy set. The level soft set of $\Im = (F, A)$ with respect to λ is a soft set $L(\Im, \lambda) = (F_\lambda, A)$ defined by

$$F_\lambda(a) = L(F(a), \lambda) = \{x \in U : F(a)(x) \geq \lambda(a)\}, \text{ for all } a \in A \text{ [10]}.$$

In other words, a function instead of a constant number t, is being considered as the threshold on membership values.

Definition 2.14. [7] *Let (F, A) and (G, B) be the two fuzzy soft sets over the rings R_1 and R_2, respectively. Suppose $f : R_1 \to R_2$ and $g : A \to B$ be the two mappings. Then the pair (f, g) is called a fuzzy soft homomorphism if it satisfies the following conditions.*

(i) *f is a ring homomorphism from R_1 to R_2*

(ii) *g is a surjective mapping.*

(iii) *$f(F(x)) = G(g(x))$ for all $x \in A$.*

If f is an isomorphism from R_1 to R_2 and g is a bijective mapping then we call (f, g) a fuzzy soft isomorphism, we say that (F, A) is fuzzy soft isomorphic to (G, B) i.e., $(F, A) \simeq (G, B)$.

3. Fuzzy Soft Sets and Their Extensions

In this section, first we introduce and discuss the extension of fuzzy soft sets(subsets), and then we re-define some operations of fuzzy soft theory. Since, extension of fuzzy subsets, fuzzy subrings and fuzzy ideals have introduced and discuused in [17, Proposition 2.1]. By an extension of a fuzzy subset α of R to a fuzzy subset β of a set S containing R, we mean a fuzzy subset β of S such that $\beta = \alpha$ on R. We also introduce the extension of fuzzy soft sets by utilizing t-level soft sets, and also inter-relate both types of extensions (see in example4.2). In due course, we apply these extensions to fuzzy soft subrings and fuzzy soft ideals.

Definition 3.1. *Let R be a nonempty subset of a set S and let A be a fuzzy subset of R. If B is an extension of A to a fuzzy subset of S, then $A_t \cap B_s = A_s$ for all s, t such that* $0 \leq t \leq s \leq 1$[17, Proposition 2.1].

Definition 3.2. *Let (F,A) and (G,B) be the two fuzzy soft sets over a common universe U. We say that (G, B) is a fuzzy soft extension of (F, A) if for each $e \in A \subset B \subset E$, $G(e)$ is the fuzzy extension of $F(e)$.*

Example 3.1. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of parameters. Let $A = \{e_1, e_2, e_3\} \subset E$. Then, the tabular representation of fuzzy soft set $\Im = (F, A)$ is given in table1.*

Table 1

U	e_1	e_2	e_3
h_1	0.4	1	0.5
h_2	0.6	0.5	0.6
h_3	0.5	0.5	0.8
h_4	0.9	0.5	0.2
h_5	0.3	0.7	0.9

We can view the fuzzy soft set $\Im = (F, A)$ as the collection of the following fuzzy approximations.

$F(e_1) = \{(h_1, 0.4)\ (h_2, 0.6)\ (h_3, 0.5)\ (h_4, 0.9)\ (h_5, 0.3)\}$; $F(e_2) = \{(h_1, 1)\ (h_2, 0.5)\ (h_3, 0.5)\ (h_4, 0.5)\ (h_5, 0.7)\}$

$F(e_3) = \{(h_1, 0.5)\ (h_2, 0.6)\ (h_3, 0.8)\ (h_4, 0.2)\ (h_5, 0.9)\}$.

Again, by considering $U = \{h_1, h_2, h_3, h_4, h_5\}$ with set of parameters $B = \{e_1, e_2, e_3, e_4\}$ and table2 is the tabular representation of $\Im^* = (\alpha, B)$.

*Table*2

U	e_1	e_2	e_3	e_4
h_1	0.4	1	0.5	0
h_2	0.6	0.5	0.6	0
h_3	0.5	0.5	0.8	0
h_4	0.9	0.5	0.2	0
h_5	0.3	0.7	0.9	0

We can view the fuzzy soft set $\Im^* = (G, B)$ as the collection of the following fuzzy approximations:

$\alpha(e_1) = \{(h_1, 0.4)\ (h_2, 0.6)\ (h_3, 0.5)\ (h_4, 0.9)\ (h_5, 0.3)\}$, $\alpha(e_2) = \{(h_1, 1)\ (h_2, 0.5)\ (h_3, 0.5)\ (h_4, 0.5)\ (h_5, 0.7)\}$, $\alpha(e_3) = \{(h_1, 0.5)\ (h_2, 0.6)\ (h_3, 0.8)\ (h_4, 0.2)\ (h_5, 0.9)\}$, $\alpha(e_4) = \varnothing_{E-A}$.

We note that $\alpha_{|A} = F$, $\alpha_{|(B-A)}(B-A) = \varnothing$.*Hence,* $\alpha = \alpha_{(F, A)}$.

We observed that $(F, A) \subseteq (\alpha, B)$ but $(\alpha, B) \nsubseteq (F, A)$. Also, if (F, A) is a fuzzy soft subset of (G, B) then it doesn't imply that (G, B) is a fuzzy soft extension of (F, A). Moreover, it is worth noting that the extension of fuzzy soft set is again a fuzzy soft set having a common domain. Thus, by extension of a fuzzy soft set (F, A) i.e., $\alpha_{(F, A)}$, we mean $F(e_i) = \alpha_{(F, A)}(e_i)$ for all $e_i \in A \subset B$. We may re-write this as follows.

4. Fuzzy Soft Sets and Their Extensions

In this section, first we introduce and discuss the extension of fuzzy soft sets(subsets), and then we re-define some operations of fuzzy soft theory. By an extension of a fuzzy subset α of R to a fuzzy subset β of a set S containing R, we mean a fuzzy subset β of S such that $\beta = \alpha$ on R. We also introduce the extension of fuzzy soft sets by utilizing t-level soft sets, and also inter-relate both types of extensions (see in example4.2). In due course, we apply these extensions to fuzzy soft subrings and fuzzy soft ideals.

Definition 4.1. *Let* (F, A) *be a fuzzy soft set over* U *, where* F *is a mapping given by* $F : A \to (U)$. *Then, the extension of mapping* F *on* E *is the mapping* $\alpha : E \to (U)$ *such that* $\alpha_{|A} = F$, $\alpha_{|(E-A)}(E-A) = \varnothing_{E-A}$ *(where* $\varnothing_{E-A} = \alpha(e)$ *for all* $e \in E-A$ *), and is denoted by* $(\alpha, E)_{(F, A)}$ *(in short,* $\alpha_{(F, A)}$*).*

In the sense of fuzzy subsets, we may define the extension of fuzzy soft subsets of U as follows.

Definition 4.2. *Let* (F, A) *and* (α, E) *be two fuzzy soft sets over the common universe* U. *Then,* (α, E) *is the fuzzy soft extension of* (F, A) *if,* (i) $A \subseteq E$ *and* (ii) *For each* $a \in A$, *the fuzzy sets* $\alpha(a)$ *and* $F(a)$ *are equal i.e.,* $\alpha(a) = F(a)$.

Example 4.1. *Let* $U = \{h_1, h_2, h_3, h_4, h_5\}$ *and* $E = \{e_1, e_2, e_3, e_4, e_5\}$ *be the set of parameters. Let* $A = \{e_1, e_2, e_3\} \subset E$. *Then, the tabular representation of fuzzy soft set* $\Im = (F, A)$ *is given in table*1.

*Table*1

U	e_1	e_2	e_3
h_1	0.4	1	0.5
h_2	0.6	0.5	0.6
h_3	0.5	0.5	0.8
h_4	0.9	0.5	0.2
h_5	0.3	0.7	0.9

We can view the fuzzy soft set $\Im = (F, A)$ *as the collection of the following fuzzy approximations.*

$F(e_1) = \{(h_1, 0.4)\ (h_2, 0.6)\ (h_3, 0.5)\ (h_4, 0.9)\ (h_5, 0.3)\}$; $F(e_2) = \{(h_1, 1)\ (h_2, 0.5)\ (h_3, 0.5)\ (h_4, 0.5)\ (h_5, 0.7)\}$

$F(e_3) = \{(h_1, 0.5)\ (h_2, 0.6)\ (h_3, 0.8)\ (h_4, 0.2)\ (h_5, 0.9)\}$.

Again, by considering $U = \{h_1, h_2, h_3, h_4, h_5\}$ *with set of parameters* $B = \{e_1, e_2, e_3, e_4\}$, *table*2 *is the tabular representation of* $\Im^* = (\alpha, B)$.

*Table*2

U	e_1	e_2	e_3	e_4
h_1	0.4	1	0.5	0
h_2	0.6	0.5	0.6	0
h_3	0.5	0.5	0.8	0
h_4	0.9	0.5	0.2	0
h_5	0.3	0.7	0.9	0

We can view the fuzzy soft set $\Im^* = (G, B)$ *as the collection of the following fuzzy approximations:*

$\alpha(e_1) = \{(h_1, 0.4)\ (h_2, 0.6)\ (h_3, 0.5)\ (h_4, 0.9)\ (h_5, 0.3)\}$, $\alpha(e_2) = \{(h_1, 1)\ (h_2, 0.5)\ (h_3, 0.5)\ (h_4, 0.5)\ (h_5, 0.7)\}$, $\alpha(e_3) = \{(h_1, 0.5)\ (h_2, 0.6)\ (h_3, 0.8)\ (h_4, 0.2)\ (h_5, 0.9)\}$, $\alpha(e_4) = \varnothing_{E-A}$.

We note that $\alpha_{|A} = F$, $\alpha_{|(B-A)}(B-A) = \varnothing$.*Hence,* $\alpha = \alpha_{(F, A)}$.

We observed that $(F, A) \subseteq (\alpha, B)$ but $(\alpha, B) \nsubseteq (F, A)$. Also, if (F, A) is a fuzzy soft subset of (G, B) then it doesn't imply that (G, B) is a fuzzy soft extension of (F, A). Moreover, it is worth noting that the extension of fuzzy soft set is again a fuzzy soft set having a common domain. Thus, by extension of a fuzzy soft set (F, A) i.e., $\alpha_{(F, A)}$, we mean $F(e_i) = \alpha_{(F, A)}(e_i)$ for all $e_i \in A \subset B$. We may re-write this as follows.

Rematk 4.1. *The extension $\alpha_{(F, A)}$ of a fuzzy soft set (F, A) is*

$$\alpha_{(F, A)}(e) = \begin{cases} F(e), & \text{if } e \in A \\ \varnothing_{E-A}, & \text{if } e \in E - A \end{cases}$$

Definition 4.3. *If $\alpha_{(F, A)}$ and $\beta_{(G, B)}$ are fuzzy soft extensions of (F, A) and (G, B) over the common universe R, respectively. Then, (F, A) is said to be a fuzzy soft subset of (G, B) if and only if $\alpha_{(F, A)}(e) \subseteq \beta_{(G, B)}(e)$ for all $e \in E$. It is denoted by $(F, A)\tilde{\subseteq}(G, B)$.*

We call (F, A) is fuzzy soft equal to (G, B) if and only if (F, A) is a fuzzy soft subset of (G, B) and (G, B) is a fuzzy soft subset of (F, A), *i.e.*, $\alpha_{(F, A)} = \alpha_{(G, B)}$. We will mention it by $(F, A) \equiv (G, B)$.

First, Maji et al. defined the intersection, union, complement of fuzzy soft sets (see definition 2.8(2)), then these operation were revised and improved by B. Ahmad et al. by imposing some restrictions (see[2]). Here, we re-define these operations which are based on the extension of mappings and are also free from any restrictions.

Definition 4.4. *If $\alpha_{(F, A)}$ and $\beta_{(G, B)}$ are fuzzy soft extensions of (F, A) and (G, B) over the common universe R. Then, the fuzzy soft intersection of (F, A) and (G, B) i.e., $(F, A) \tilde{\cap} (G, B)$ exists if and only if there exist a fuzzy soft set (H, C) satisfying $\forall e \in E$, $\alpha_{(F, A)}(e) \cap \beta_{(G, A)}(e) = \gamma_{(H, C)}(e)$. It is denoted by $(F, A) \tilde{\cap} (G, B) \equiv (H, C)$.*

Definition 4.5. *If $\alpha_{(F, A)}$ and $\beta_{(G, B)}$ are fuzzy soft extensions of (F, A) and (G, B) over the common universe R. Then the fuzzy soft union of (F, A) and (G, B) i.e., $(F, A) \tilde{\cup} (G, B)$ exists if and only if there exist a fuzzy soft set (H, C) satisfying $\forall e \in E$, $\alpha_{(F, A)}(e) \cup \beta_{(G, B)}(e) = \gamma_{(H, C)}(e)$. It is noted as $(F, A) \tilde{\cup} (G, B) \equiv (H, C)$.*

Definition 4.6. *Let $\alpha_{(F, A)}$ and $\beta_{(G, B)}$ are fuzzy soft extensions of (F, A) and (G, B) over a common universe R. Then, $(F, A) \tilde{\neg} (G, B)$ is said to be a fuzzy soft difference of (F, A) and (G, B) if and only if there exist a fuzzy soft set (H, C) satisfying, $\alpha_{(F, A)} - \beta_{(F, A)} = \gamma_{(H, C)}$, for all $e \in E$, and is denoted by $(F, A) \tilde{\neg} (G, B) \equiv (H, C)$.*

If (F, A) is an absolute fuzzy soft set then $(F, A) \tilde{\neg} (F, B)$ is the fuzzy soft complement of the set (F, B), and is denoted by $(F, B)^{\tilde{C}}$.

Here, we must concern with the algebraic properties of fuzzy soft set operations based on extension of mappings. We present few results for the sake of investigations.

Theorem 4.1. *Let $\alpha_{(F, A)}, \beta_{(G, B)}$ and $\gamma_{(H, C)}$ are fuzzy soft extensions of (F, A), (G, B) and (H, C) over the common universe R. Then*

(1) $(F, A) \tilde{\neg} ((G, B) \tilde{\cap} (H, C)) \equiv ((F, A) \tilde{\neg} (G, B)) \tilde{\cup} ((F, A) \tilde{\neg} (H, C))$

(2) $(F, A) \tilde{\neg} ((G, B) \tilde{\cup} (H, C)) \equiv ((F, A) \tilde{\neg} (G, B)) \tilde{\cap} ((F, A) \tilde{\neg} (H, C))$

Proof. (1) For all $e \in E$,

$$\begin{aligned}(F, A) \tilde{\neg} ((G, B) \tilde{\cap} (H, C))(e) &= \alpha_{(F, A)}(e) - (\beta_{(G, B)} \cap \gamma_{(H, C)})(e) \\ &= (\alpha_{(F, A)}(e) - \beta_{(G, B)}(e)) \cup (\alpha_{(F, A)}(e) - \gamma_{(H, C)}(e)) \\ &= ((F, A) \tilde{\neg} (G, B))(e) \cup ((F, A) \tilde{\neg} (H, C))(e) \\ &\equiv ((F, A) \tilde{\neg} (G, B) \tilde{\cup} (F, A) \tilde{\neg} (H, C))(e).\end{aligned}$$

(2) Similar to that of the proof (1). □

One can easily deduce the fuzzy soft De Morgan laws.

Rematk 4.2. *Let $\alpha_{(F, A)}$ and $\beta_{(G, B)}$ are fuzzy soft extensions of (F, A) and (G, B) over the common universe R. Then*

(1) $((F, A) \tilde{\cap} (G,B))^{\tilde{C}} \equiv (F, A)^{\tilde{C}} \tilde{\cup} (G, B)^{\tilde{C}}$

(2) $((F, A) \tilde{\cup} (G, B))^{\tilde{C}} \equiv (F, A)^{\tilde{C}} \tilde{\cap} (G, B)^{\tilde{C}}$

Theorem 4.2. *Let $\varnothing$ and $\widetilde{U}$ are null and absolute fuzzy soft sets over U and $\alpha_{(\widetilde{U}, E)}, \beta_{(\varnothing, E)}$ are the fuzzy soft extensions of $(\widetilde{U}, E)$ and $(\varnothing, E)$, respectively. Then*

(1) $\widetilde{U}^{\tilde{C}} \equiv \varnothing$

(2) $\varnothing^{\tilde{C}} \equiv \widetilde{U}$

Proof. (1) For all $e \in E$, $\widetilde{U}^{\tilde{C}}(e) = \alpha^{C}_{(\widetilde{U}, E)}(e) \equiv \varnothing$

(2) For all $e \in E$, $\varnothing^{\tilde{C}}(e) = \alpha^{C}_{(\varnothing, E)}(e) = \widetilde{U} - \alpha_{(\varnothing, E)}(e) \equiv \widetilde{U}$ □

Extensions of fuzzy soft sets based on t-level soft sets

In this part we introduce the extension of fuzzy soft sets (subsets) at the base of t-level soft sets of fuzzy soft sets. We also relate both the definitions[4.1&4.7] of extension of fuzzy soft subsets (sets) in example4.2. Throughout this part, for the sake of clarity F_R is a fuzzy soft extension of F_S.

Definition 4.7. *Let A be a nonempty subset of E and (F, A) (i.e., F_A) be a fuzzy soft set over U. If (α, E) is a fuzzy soft extension of F over U, then $L(F(a), t)\tilde{\cap} L(\alpha(a), s) = L(F(a), s)$, for all s, t such that $0 < t \leq s \leq 1$.*

Example 4.2. *Since α is an extension of a fuzzy soft set F, as we have observed in example4.1. We continue with example4.1 to show that $L(F(a), t)\tilde{\cap} L(\alpha(a), s) = L(F(a), s)$, for all s, t such that $0 < t \leq s \leq 1$. Suppose we take $t = 0.3$, $s = 0.6$ with having $A = \{e_1, e_2, e_3\} \subset B = \{e_1, e_2, e_3, e_4\}$. Then, 0.3-level sets of fuzzy sets $F(e_1)$, $F(e_2)$, $F(e_2)$ are as follows: $L(F(e_1), 0.3) = U$, $L(F(e_2), 0.3) = U$, $L(F(e_3), 0.3) = \{h_1, h_2, h_3, h_5\}$. Then, the 0.3-level soft set of the fuzzy soft set $\Im = (F, A)$ is a soft set $L(\Im, 0.3) = (F_{0.3}, A)$, where $F_{0.3}$ is the set-valued map from $A \to \wp(U)$ is defined by $F_{0.3}(e_i) = L(F(e_i), 0.3)$. Since, we know that the approximate value sets of the level soft set $L(\Im, 0.3)$ are the 0.3-level set of the fuzzy set $F(e_i)$ for every parameter $e_i \in A$. Hence, the tabular representation of level soft $L(\Im, 0.3)$ is as under:*

U	e_1	e_2	e_3
h_1	1	1	1
h_2	1	1	1
h_3	1	1	1
h_4	1	1	0
h_5	1	1	1

Similarly, 0.6-level sets of fuzzy sets of $\alpha(e_1)$, $\alpha(e_2)$, $\alpha(e_3)$ are: $L(\alpha(e_1), 0.6) = \{h_2, h_4\}$, $L(\alpha(e_2), 0.6) = \{h_1, h_5\}$, $L(\alpha(e_3), 0.6) = \{h_2, h_3, h_5\}$, $L(\alpha(e_4), 0.6) = \varnothing$. the tabular representation of 0.6-level soft set of a fuzzy soft set $L(\Im^*, 0.6)$ is as under:

U	e_1	e_2	e_3	e_4
h_1	0	1	0	0
h_2	1	0	1	0
h_3	0	0	1	0
h_4	1	0	0	0
h_5	0	1	1	0

Finally, $L(F(e_1), 0.3)\tilde{\cap}\, L(\alpha(e_1), 0.6) = U \cap \{h_2, h_4\} = \{h_2, h_4\} = L(F(e_1), 0.6)$*,* $L(F(e_2), 0.3)\tilde{\cap}\, L(\alpha(e_2), 0.6) = U \cap \{h_1, h_5\} = L(F(e_2), 0.6)$ *and* $L(F(e_3), 0.3)\tilde{\cap}\, L(\alpha(e_3), 0.6) = \{h_1, h_2, h_3, h_5\} \cap \{h_2, h_3, h_5\} = \{h_2, h_3, h_5\} = L(F(e_3), 0.6)$. *Similarly, one can prove for any values of* s, t *such that* $0 < t \leq s \leq 1$.

On the other hand, if we change the value of $\alpha(e_2) = \{(h_1, 1)\ (h_2, 0.5)\ (h_3, 0.5)\ (h_4, 0.5)\ (h_5, 0.7)\}$ by $\alpha(e_2) = \{(h_1, 0.5)\ (h_2, 0.5)\ (h_3, 0.5)\ (h_4, 0.5)\ (h_5, 0.7)\}$ in example4.1. Then, α is no more an extension of F and we can see that $L(F(e_2), 0.3)\ \tilde{\cap}\, L(\alpha(e_2), 0.6) = U \cap \{h_5\} = \{h_5\} \neq L(F(e_2), 0.6) = \{h_1, h_5\}$. It is also notable that there is an important role of t-level sets of a fuzzy subsets (sets) of U involved in whole of the procedure.

We are utilizing t-level soft set of a fuzzy soft set, where $t \in [0, 1]$ for the extension of fuzzy soft sets. Now we try to provide a necessary and sufficient condition for the existence of extension of fuzzy soft sets at the base of t-level sets.

Theorem 4.3. *Let A be a nonempty subset of E and (f, A) and (α, E) are the fuzzy soft sets over U. If*

$$\Re = \{L(\alpha(a), t) : f_a(x) \geq t\} \text{ for all } a \in A \text{ and } x \in U$$

is the collection of subsets of R, such that $(1) \cup L(\alpha(a), t) = R$, (2) *for all s, t such that $s > t$ and $f_a(x) \geq s, t$, if and only if $L(\alpha(a), t) \subset L(f_a, s)$, and* (3) *for all s, t such that $s > t$ and $f_a(x) \geq s, t$, $(f_a, t) \cap (\alpha(a), s) = (f_a, s)$. Then, a fuzzy soft set f_a has an extension to a fuzzy soft set f_a^e of R such that $L(f_a^e, t) \supseteq L(\alpha(a), t)$ for all $f_a(x) \geq t$.*

Proof. We define a fuzzy soft set f_a^e of U such that for all $y \in U$,

$$f_a^e(y) = sup\{t : y \in L(\alpha(a), t)$$

if $y \in L(\alpha(a), t)$ then $f_a^e(x) \geq t$, so $y \in L(f_a^e, t)$. Hence, $L(f_a^e, t) \subseteq f_a^e(y)$. Now we show that f_a^e is the extension of f_a, let $b \in E$ such that $\alpha(b)(x) = t$ for all $x \in U$, then $b \in L(f_a^e, t)$. If $b \in L(f_a^e, s)$ for some $s > t$ then $\alpha(b)(x) \geq s > t$, a contradiction. Hence, $b \notin L(f_a^e, s)$ for all $s > t$. Since, $b \in L(f_a^e, t) \subset L(\alpha(a), t)$, and $b \notin L(\alpha(a), s)$ for all $s > t$ by condition (3), $\alpha(b)(x) = t$. □

Let us consider (F, A) is a a fuzzy soft set over R. Then, (F, A) has the supremum property if and only if every nonempty subset of $F(a)$ has a maximal element. On the other hand, F has the supremum property if and only if $F(a)$ is well ordered undered relations "$>$". If we impose this supremum property on $F(a)$ in theorem4.3, then we have the following result.

Theorem 4.4. *Let A be a nonempty subset of a set B and let (F, A) be a fuzzy soft set over U such that (F, A) has the supremum (sup) property. If*

$$\Re = \{L(\alpha(a), t) : f_a(x) \geq t\} \textit{ for all } a \in A \textit{ and } x \in U$$

is the collection of subsets of U which satisfies (1), (2), and (3) of theorem 4.3, then f_a has a unique extension to a fuzzy soft set f_a^e of U such that $L(f_a^e(x), t) = f_a(x)$.

Proof. We define f_a^e as in theorem 4.3. Then, for all $y \in U$,

$$f_a^e(y) = max\{n : y \in L(\alpha_a, n)\}.$$

Thus, $x \in L(f_a, t)$ if and only if $f_a^e(x) \geq t \Leftrightarrow x \in L(f_a, t)$. Since, f_a has the supremum property, $(\alpha_a(x)) \subset f_a(x)$. Hence, $(f_a^e)(x) = \alpha_a(x)$. For uniqueness, consider that g_a^e be an extension of f_a such that $L(g_a^e(a), t) = L(\alpha_a, t)$ and $g_a^e(x) = f_a(x)$. Then, $L(g_1^e, t) = L(\alpha(a), t)$ for all $f_a(x) \geq t$ for all $a \in A$ and $x \in U$. Hence, it follows that $g_a^e = f_a^e$. □

Theorem 4.5. *Let $\aleph$ be a non-empty subset of $[0, 1]$, R be a ring and $\{I_t : t \in \aleph\}$ be the collection of ideals of R such that $R = \cup I_t$ for all $s, t \in \aleph$, $s > t$ if and only if $I_s \subset I_t$. If the fuzzy soft subset (α, I) of (over) R defined by $\alpha_a(x) = sup\{t : x \in I_t\}$, for all $x \in R$, then α is a fuzzy soft ideal of (over) R.*

Proof. Let $x, y \in R$ and $f_a(x - y) = n$. Then, $x - y \in I_t$ for all $n \geq t$ and $x - y \notin I_s$ for all $s > n$. Thus, either (1) $x, y \in I_t$, for all $t \leq n$ and either $x \in I_s$ for all $s > n$ or $y \in I_s$ for all $s > n$ or (2) there exists $m \in [0, 1]$ such that for all $t \in \aleph$, $m \leq t \leq n$, $x \in I_t$ and $y \in I_t$. Suppose case(1) hold true, then either $f_a(x) = n$ or $f_a(y) = n$. Again, if case(2) holds true, then $f_a(x) \leq m$ and $f_a(y) \leq m$. In either case, $f_a(x-y) \geq \min(f_a(x), f_a(y)) \Rightarrow f_a(x-y) \geq f_a(x) \wedge f_a(y))$. Suppose that $f_a(xy) = n$ then, $xy \in I_t$ for all $t \leq n$, and $xy \notin I_s$ for all $s > n$. Hence, $f_a(x) = n \geq max(f_a(x), f_a(y)) = f_a(x) \wedge f_a(y)$. Thus, f_a is a fuzzy soft ideal of (over) R. □

Rematk 4.3. *Let $\aleph$ be a non-empty subset of $[0, 1]$. Let R be a ring and let $\{R_t : t \in \aleph\}$ be the collection of subrings of R such that $R = \cup R_t$ and for all $s, t \in \aleph$, $s > t$ if and only if $R_s \subset R_t$. If the fuzzy soft subset (α, R_t) of (over) R defined by $\alpha(x) = sup\{t : x \in R_t\}$, for all $x \in R$, then α is a fuzzy soft subring of R.*

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] U. Acar, F. Koyuncu, B. Tanay, Soft sets and soft rings, Computers Math. Appl. 59 (2010), 3458–3463.

[2] B. Ahmad, A. Kharal, On Fuzzy Soft Sets, Adv. Fuzzy Syst. 2009 (2009), 586507.

[3] H. Aktas, Naim Cagman, Soft sets and soft groups, Inform. Sci. 177 (2007), 2726-2735.

[4] M.I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir, On some new operations in soft set theory, Computers Math. Appl. 57 (2009), 1547–1553.

[5] A. Aygünoglu, H. Aygün, Introduction to fuzzy soft groups, Computers Math. Appl. 58 (2009), 1279-1286.

[6] N. Çağman, S. Enginoğlu, F. Çitak, Fuzzy soft set theory and its applications, Iran. J. Fuzzy Syst. 8(3) (2011), 137-147.

[7] Y. Çelik, C. Ekiz, S. Yamak, Applications of fuzzy soft sets in ring theory, Ann. Fuzzy Math. Inform. 5(3) (2013), 451-462.

[8] V.N. Dixit, R. Kumar, N. Ajmal, On fuzzy rings, Fuzzy Sets Syst. 49 (1992), 205-213.

[9] F. Feng, Y.B. Jun, X. Liu, L. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234 (2010), 10-20.

[10] F. Feng, Y.M. Li, H.X. Lei, Level soft sets based on fuzzy sets, in Quantitative Logic and Soft Computing, G.J. Wang, B. Zhao, Y. Li, Eds., vol. 5 of World Scientific Proceedings Series on Computer Engineering and Information Science, pp. 480–487, World Scientific Publishing, 2012.

[11] W.-L. Gau, D.J. Buehrer, Vague sets, IEEE Trans. Syst., Man, Cybern. 23 (1993), 610–614.

[12] M.B. Gorzałczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets Syst. 21 (1987), 1–17.

[13] E. İnan, M.A. Öztürk, Fuzzy soft rings and fuzzy soft ideals, Neural Comput. Appl. 21 (2012), 1-8.

[14] W.A. Khan, A. Rehman, A. Taouti, Soft near-semirings, AIMS Math. 5(6) (2020), 6464-6478.

[15] W. Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets Syst. 8 (1982), 133-139.

[16] Z. Ma, W. Yang, B. Hu, Soft set theory based on its extension, Fuzzy Inform. Eng. 2 (2010), 423–432.

[17] D.S. Malik, J.N. Mordeson, Extensions of fuzzy subrings and fuzzy ideals, Fuzzy Sets Syst. 45 (1992), 245-251.

[18] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003), 555-562.

[19] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9(3) (2001), 589-602.

[20] L. Martinez, Prime and primary L-fuzzy ideals of L-fuzzy rings, Fuzzy Sets Syst. 101 (1999), 489-494.

[21] D. Molodtsov, Soft set theory-First results, Computers Math. Appl. 37(4/5) (1999), 19-31.

[22] M. A. Öztürk, E. İnan, Fuzzy soft subnear-rings and $(\in, \in \vee q)$-fuzzy soft subnear-rings, Computers Math. Appl. 63 (2012), 617-628.

[23] Z. Pawlak, Rough sets, Int. J. Inform. Computer Sci. 11 (1982), 341-356.

[24] M.M.K. Rao, B. Venkateswarlu, Fuzzy soft k-ideals over semirings and fuzzy soft semiring homomorphism, J. Hyperstruct. 4(2) (2015), 93-116.

[25] A.R. Roy, P.K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2007), 412-418.

[26] A. Sezgin, A.O. Atagün, On operations of soft sets, Computers Math. Appl. 61 (2011), 1457-1467.

[27] A. Taouti, W.A. Khan, Fuzzy subnear-semirings and fuzzy soft subnear-semirings, AIMS Math. 6(3) (2021), 2268-2286.

[28] B.P. Varol, A. Aygunöglu, H. Aygün, On fuzzy soft rings, J. Hyperstruct. 1(2) (2012), 1-15.

[29] L.A. Zadeh, Fuzzy sets, Inform. Control. 8 (1965), 338-353.