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LINE GRAPH ASSOCIATED TO THE INTERSECTION GRAPH OF IDEALS OF RINGS

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Abstract. Let R be a ring with unity and $I(R)^*$ are all non-trivial left ideals of R. The intersection graph of ideals of R is denoted by $G(R)$ is an undirected simple graph with vertex set $I(R)^*$ and two distinct vertices I and J are adjacent if and only if $I \cap J \neq 0$. In this article, we investigate some basic properties of the line graph associated to $G(R)$, denoted by $L(G(R))$. Moreover, we investigate completeness, unicyclicness, bipartiteness, planarity, outerplanarity, ring graph, diameter, girth and clique of $L(G(\mathbb{Z}_n))$. We also investigate some basic properties of $L(G(R))$ for left Artinian ring and finally, we determine the domination number and bondage number of $L(G(\mathbb{Z}_n))$.

Keywords: intersection graphs; line graphs; planarity; ring graph; diameter; clique; girth; domination parameter.

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1. Introduction

Recently, Chakrabarty et al. [3] introduced the intersection graphs of left ideals of a ring, where they took all the non-trivial left ideals of a ring as the vertex set and any two distinct

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vertices are adjacent if and only if their intersection is non-zero. They studied almost all fundamental concepts of the intersection graphs of ideals of rings. The central part of their interpretation depended upon the ring $\mathbb{Z}_n$. Akbari et al. [1] extended some characteristics between the graph-theoretic properties of intersection graph of ideals and some algebraic properties of rings.

In graph theory, one can associate to a given graph G to its line graph, denoted by $L(G)$, such that each vertex of $L(G)$ represents an edge of G, and any two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G share a common vertex. Whitney (1932) [12] introduced the one important theorem on the line graph, the structure of any connected graph can be recovered from its line graph i.e., there is a one-to-one correspondence between the class of connected graphs and the class of connected line graphs. Later, the term line graph comes from a paper from Harary and Norman (1960) [9]. With the class of intersection graphs at hand, it is natural to study the properties of their line graphs and seek any relation between them. There are some papers of line graphs associated with graphs of rings, see for instance [4], [7], [11], and [10] etc.

Let $G=(V,E)$ be a graph. The graph G is said to be connected if there exists a path between any two distinct vertices of G. On the other side, the graph G is called a null graph if G is a graph with no edges while the empty graph is a graph with no vertices. The graph G is called a complete graph if there exists an edge to every vertices of G. The distance between any two distinct vertices x and y are denoted by $d(x,y)$ and the diameter is defined as $diam(G) = sup\{d(x,y) \mid x,y \; are \; vertices \; of \; G\}$. The graph G is called unicyclic graph if G contains exactly one cycle and the length of a shortest cycle is called girth $gr(G)$ of G. The order of the maximal complete subgraph is called clique number, denoted by $\omega(G)$. A graph G is called embedded in the plane if it can be drawn on the plane so that no two edges intersect, such graph is called a planar graph. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A planar graph G is called outerplanar if it can be embedded in the plane such that all its vertices lie on the outer face. A chord is any edge of a graph G joining two nonadjacent vertices in a cycle. A cycle without chord is called primitive. A graph G has the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. Let $frank(G)$ be the number of primitive cycles of G. The number $frank(G)$ is called the free rank of G and the number $rank(G) = q - n + r$ is called the cycle rank of G, where $|V(G)| = n$, $|E(G)| = q$, and r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G and these two satisfy $rank(G) \leq frank(G)$. Gitler et al. [[6], Theorem 2.13], showed that:

Theorem 1.1. The following conditions are equivalent for a graph G with n vertices and q edges:

(1) G is a ring graph;
(2) $rank(G) = frank(G)$;
(3) G satisfies the primitive cycle property and G does not contain a subdivision of K_4 as a subgraph.

From the Theorem 1.1, it is clear that ring graphs are planar. Gitler et al. [6], stated that the blocks of graph G are important for calculating the numbers $frank(G)$ and $rank(G)$, and they also proved that outerplanar graphs are ring graphs. Note that the class of outerplanar graph is a proper subclass of ring graphs. A subset $D \subseteq V$ is called a dominating set of G if every vertex in V is either in D or adjacent to a vertex in D. The domination number $\gamma(G)$ is the minimum cardinality among the dominating sets of G. The bondage number $b(G)$ is the minimum number of edges whose removal increases the domination number of G. Any undefined terminology can be obtained in [2], [5], and [8].

The organization of our article is as follow: In Sec. 2, we investigate some basic properties of $L(G(\mathbb{Z}_n))$ such as completeness, bipartiteness, unicyclicness, planarity, outerplanarity and ring graphs. In Sec. 3, we determined the diameter, girth and clique of $L(G(\mathbb{Z}_n))$ and $L(G(R))$ and finally in Sec. 4, we determined some domination parameters of $L(G(\mathbb{Z}_n))$.

2. On the Structures and Properties of $L(G(R))$

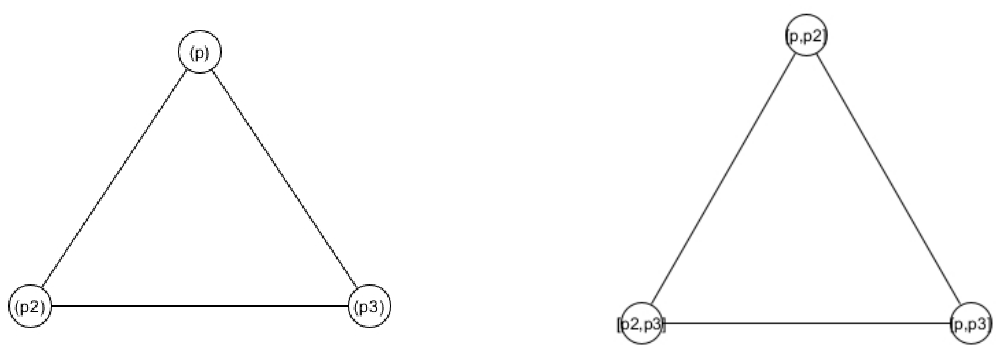
In this section, we study some basic properties of the line graph associated with intersection graph of ideals of ring. We denote the line graph of $G(R)$ by $L(G(R))$. Vertex of the line graph of $G(R)$ are all the edges of $G(R)$ and any two distinct vertices in $L(G(R))$ are adjacent if and only if their corresponding edges share a common vertex in $G(R)$.

Remark 2.1. Chakrabarty et al. [[3], Corollary 2.5] prove that for any graph $G(R)$ of a ring R, whenever $G(R)$ is not connected, it is a null graph (i.e. it has no edge). Therefore, whenever $G(R)$ is not connected, its line graph $L(G(R))$ is always empty graph. Thus, here we investigate the line graph of all the connected graph of $G(R)$.

Example 2.2. Let $n = p_1^{m_1} p_2^{m_2} ... p_i^{m_i}$, where p_i are all distinct primes and $i = 1, 2, ..., n$. In the following we observed the intersection graph of $G(\mathbb{Z}_n)$ and its line graph $L(G(\mathbb{Z}_n))$ for certain values of n.

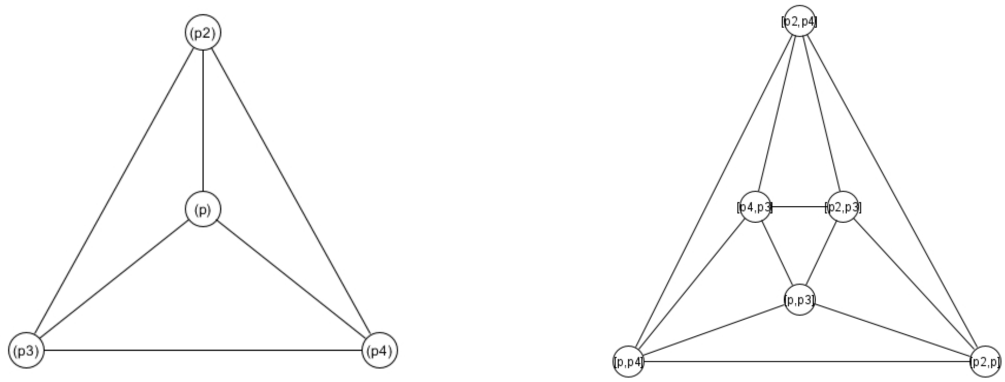
(a) $G(\mathbb{Z}_{p^3})$ (b) $L(G(\mathbb{Z}_{p^3}))$

FIGURE 1. $G(\mathbb{Z}_{p^3})$ & $L(G(\mathbb{Z}_{p^3}))$



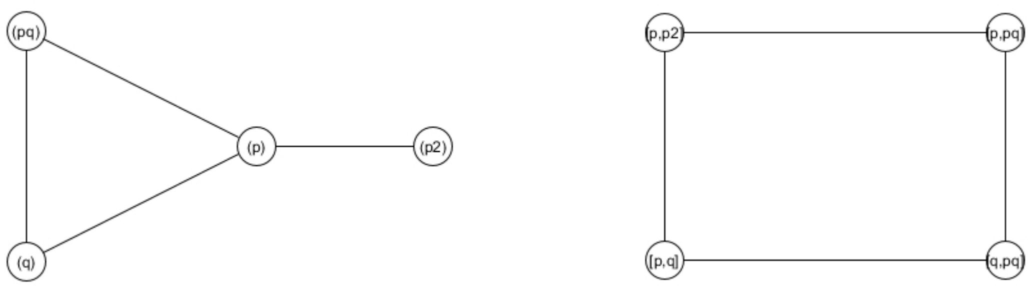
(a) $G(\mathbb{Z}_{p^4})$ (b) $L(G(\mathbb{Z}_{p^4}))$

FIGURE 2. $G(\mathbb{Z}_{p^4})$ & $L(G(\mathbb{Z}_{p^4}))$



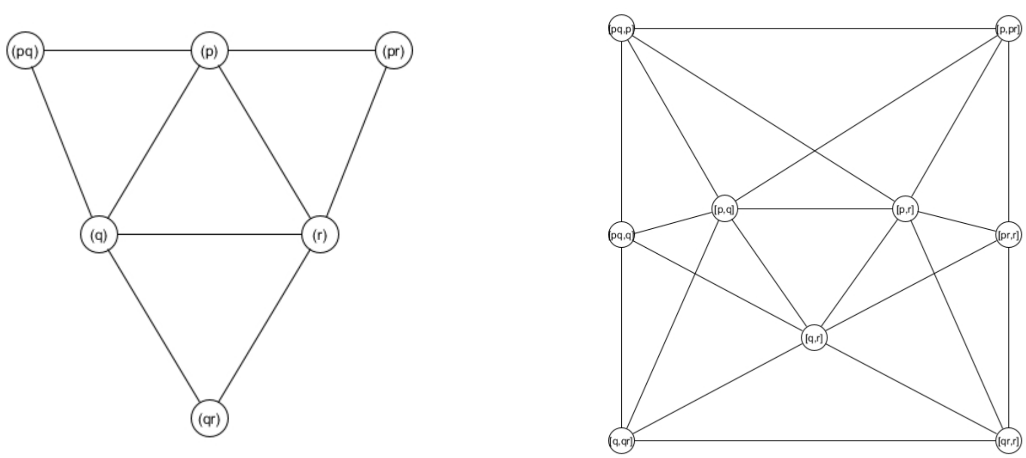
(a) $G(\mathbb{Z}_{p^5})$ (b) $L(G(\mathbb{Z}_{p^5}))$

FIGURE 3. $G(\mathbb{Z}_{p^5})$ & $L(G(\mathbb{Z}_{p^5}))$



(a) $G(\mathbb{Z}_{p^2q})$ (b) $L(G(\mathbb{Z}_{p^2q}))$

FIGURE 4. $G(\mathbb{Z}_{p^2q})$ & $L(G(\mathbb{Z}_{p^2q}))$



(a) $G(\mathbb{Z}_{pqr})$ (b) $L(G(\mathbb{Z}_{pqr}))$

FIGURE 5. $G(\mathbb{Z}_{pqr})$ & $L(G(\mathbb{Z}_{pqr}))$

Theorem 2.3. Let n, m are any two positive integers. Then $L(G(\mathbb{Z}_n))$ is complete if and only if $n = p^m$, where p is a prime number and $m = 3$ or 4.

Proof. Let $m = 3$ or 4, then it is easy to see that the line graph is either K_1 or K_3 by figures 1 (b), 2 (b).

Conversely, let $L(G(\mathbb{Z}_n))$ is a complete graph. Therefore, $G(\mathbb{Z}_n)$ is either a star graph or a complete graph of K_3; Otherwise $L(G(\mathbb{Z}_n))$ is not a complete graph. Therefore, $n = 3$ or 4.

Theorem 2.4. Let R be a left Artinian ring. Then $L(G(R))$ is complete graph if and only if $R \cong \mathbb{Z}_{p^i}$, where $i = 3$ or 4.

Proof. The sufficiency is obvious. For necessity, assume that $L(G(R)) = K_n$, $(n \neq 3)$. Then $G(R)$ is a star graph. Therefore, $R \cong \mathbb{Z}_{p^3}$. Again, let $L(G(R)) = K_3$, then either $G(R) = K_{1,3}$ or K_3. Since, $G(\mathbb{Z}_n) \neq K_{1,3}$ Therefore we conclude that $G(R) = K_3$ by [[3], Theorem 2.9]. Therefore, $R \cong \mathbb{Z}_{p^4}$.

Theorem 2.5. Let $n = p^4$ or p^2q, then $L(G(\mathbb{Z}_n))$ is a unicyclic graph.

Proof. Let $n = p^4$, then $L(G(\mathbb{Z}_n))$ is K_3 by Theorem 2.3 and if $n = p^2q$, then from the figure 4 (b), $L(G(\mathbb{Z}_n))$ is C_4. Which yields $L(G(\mathbb{Z}_n))$ is a unicyclic graph.

The above Theorem 2.5 has an important consequence. We know that a graph G is a bipartite graph if and only if G has no odd cycle [[5], K*ö*nig (1936)]. Since, it also satisfied in the case of line graph. So, we immediately have the following Theorem:

Theorem 2.6. Let n be any positive integer. Then $L(G(\mathbb{Z}_n))$ is a complete bipartite graph if and only if $n = p^2q$.

Theorem 2.7. Let n is any positive integer. Then $L(G(\mathbb{Z}_n))$ has a cycle (of length 3 or 4) if and only if $n = mt$, where $m = p^4, p^2q$ or pqr, p, q, r are distinct primes and $t \in \mathbb{N}$.

Proof. Let $n = mt$, where $m = p^4, p^2q$ or pqr, p, q, r are distinct primes and $t \in \mathbb{N}$. Then $L(G(\mathbb{Z}_n))$ will contain a subgraph isomorphic to $L(G(\mathbb{Z}_m))$. Now from the figures: 2 (b), 3 (b), 4 (b) and 5 (b), it is clear that $L(G(\mathbb{Z}_n))$ contains a cycle (of length 3 or 4).

Conversely, let $L(G(\mathbb{Z}_n))$ contains a cycle (of length 3 or 4). Let $n = p_1.p_2...p_k$ $(k > 1)$, where p_i's are prime numbers but may not be all distinct. If $k < 3$, then $L(G(\mathbb{Z}_n))$ is empty graph. Therefore, $n = p^3, p^2q, pqr$, where p, q, r are distinct primes. If $n = p^3$, then $L(G(\mathbb{Z}_n))$ is K_1 (cf. Fig. 1 (b)). In other two cases $L(G(\mathbb{Z}_n))$ contain cycles and satisfy the required condition.

If $k \geq 4$, then n is always a multiple of m, where $m = p^4, p^2q$ or pqr for some distinct primes p, q, r. Which complete the proof.

In the following, we investigate the planarity and non-planarity properties of $L(G(\mathbb{Z}_n))$.

Theorem 2.8. Let n be any positive integer. If $n = p^m (m \leq 5)$ or p^2q, where p and q are distinct primes, then the line graph $L(G(\mathbb{Z}_n))$ of $G(\mathbb{Z}_n)$ is planar.

Proof. Follows directly from the figues: 2 (b), 3 (b), and 4 (b).

Theorem 2.9. Let n be any positive integer. If $n = p^m (m \geq 6), p^3q, p^2q^2, p^2qr, pqr$ or $pqrs$, where $p, q, r,$ and s are distinct primes, then the line graph $L(G(\mathbb{Z}_n))$ of $G(\mathbb{Z}_n)$ is non-planar.

Proof. Let $n = p^m (m \geq 6), p^3q, p^2q^2, p^2qr, pqr$ or $pqrs$, where $p, q, r,$ and s are distinct primes. Then $L(G(\mathbb{Z}_n))$ contains a subdivision of K_5 and hence they are non-planar.

In the following, we investigate the outerplanar properties of $L(G(\mathbb{Z}_n))$.

Theorem 2.10. Let n be any positive integer. If $n = p^m (m \leq 4)$ or p^2q, where p and q are distinct primes, then $L(G(\mathbb{Z}_n))$ is outerplanar.

Proof. For the proof of the theorem, we have the following two cases:

Case 1. Let $n = 2, 3, 4$, then $L(G(\mathbb{Z}_n))$ is an empty graph, K_1, and K_3, which shows that $L(G(\mathbb{Z}_n))$ is outerplanar graph.

Case 2. Let $n = p^2q$, then $L(G(\mathbb{Z}_n))$ is a unicyclic graph by Theorem 2.5. Which shows that $L(G(\mathbb{Z}_{p^2q}))$ is outerplanar graph.

Theorem 2.11. Let n be any positive integer. If $n = p^m$, where $m \geq 5$, then the line graph of $G(\mathbb{Z}_n)$ is not outerplanar.

Proof. Let $n = p^m$, where $m \geq 5$. Then it is clear that $L(G(\mathbb{Z}_n))$ contains a subdivision of K_4 (cf. fig. 4 for $m = 5$). Which yields $L(G(\mathbb{Z}_n))$ is not outerplanar graph.

Since, every outerplanar graph is a ring graph. Therefore, we have the following two results:

Theorem 2.12. Let $m \leq 4$ be any positive integer and $m > 1$. If $n = p^m$, then $L(G(\mathbb{Z}_n))$ is a ring graph.

Theorem 2.13. Let $m \geq 5$ be any integer. If $n = p^m$, then $L(G(\mathbb{Z}_n))$ is not a ring graph.

3. DIAMETER, GIRTH & CLIQUE

In this section we determined the diameter, girth and clique number of $L(G(\mathbb{Z}_n))$ and $L(G(R))$. We begin with the following Theorem.

Theorem 3.1. Let n is any positive integer such that $L(G(\mathbb{Z}_n))$ is not empty graph. Then, $diam(L(G(\mathbb{Z}_n))) \in \{0,1,2\}$.

Proof. Let $n = p_1^{m_1} p_2^{m_2} ... p_i^{m_i}$, where p_i are distinct primes and $i = 1,2,...,n$. If $i = 1$, then $m_1 > 2$, then $G(\mathbb{Z}_{p^{m_1}})$ is a complete graph. Since, $d - 1 \leq diam(L(G(\mathbb{Z}_{p^{m_1}}))) \leq d + 1$, where d is the diameter of $G(\mathbb{Z}_{p^{m_1}})$. Therefore, $diam(L(G(\mathbb{Z}_n))) \in \{0,1,2\}$. Now, for other values of n, the proof is straight forward from the figure 5.

Theorem 3.2. If $L(G(\mathbb{Z}_n))$ contains a cycle, then $gr(L(G(\mathbb{Z}_n))) \in \{3,4\}$.

Proof. Let $n = p_1^{m_1} p_2^{m_2} ... p_i^{m_i}$, where p_i are distinct primes and $i = 1,2,...,n$. In the observation of factorization, suppose n has exactly one prime. Then $m_1 > 3$. In this case, $[p, p^2]$, $[p, p^3]$ and $[p^2, p^3]$ form the shortest length of cycle. If $i, = 2$ such that $m_1 = 2$ and $m_2 = 1$, then $[p, p^2]$, $[p, pq]$, $[q, pq]$ and $[p,q]$ form a cycle of length four and the graph is a cycle graph by Theorem 2.5. Similarly, for the other values of n, we will get smallest length of three cycle. Therefore, $gr(L(G(\mathbb{Z}_n))) \in \{3,4\}$.

Theorem 3.3. Let R be a left Artinian ring. Then $gr(L(G(R))) \in \{3,\infty\}$.

Proof. Let R be a left Artinian ring and I_i $(i = 1,2,...,n)$ are non-trivial left ideals of R. If $i = 2$ then $L(G(R))$ is K_1. Again let $i \geq 3$, the it is easy to see that $[I_1, I_2], [I_2, I_3], [I_3, I_1]$ form a shortest length of cycle in $L(G(R))$. Therefore, $gr(L(G(R))) \in \{3,\infty\}$.

For $m = 3$, $\omega(L(G(\mathbb{Z}_{p^3}))) = 1$, since $L(G(\mathbb{Z}_{p^3}))$ is K_1. In the following we give the boundness of clique number for $m \geq 4$.

Theorem 3.4. Let $n = p^m$, where $m \geq 4$ is any integer. Then $\omega(L(G(\mathbb{Z}_n))) \geq 3$.

Proof. We assume that $m = 4$, then $\mathbb{Z}_{p^4}$ have $(p),(p^2),(p^3)$ ideals. Now, we can easily see that $[p, p^2], [p^2, p^3], [p^3, p]$ form a complete graph of K_3. Also for $m = 5$, it is clear that $\omega(L(G(\mathbb{Z}_n))) = 3$ from the figure 3 (b). Therefore, $\omega(L(G(\mathbb{Z}_n))) \geq 3$.

Theorem 3.5. Let R be a left Artinian ring. Then $\omega(L(G(R))) = 1$ or $\omega(L(G(R))) \geq 3$.

Proof. Let R be a left Artinian ring. Suppose I_i $(i = 1,2,...,n)$ are non-trivial left ideals of R. Then oviously all the ideals are adjacent to I_1 in $G(R)$. If $i = 2$, then $L(G(R))$ is K_1. Let $i \geq 3$, then it is easy to check that $\omega(L(G(R))) \geq 3$. Thus the result hold.

4. Some Domination Parameters

In this section, we determined the domination number and the bondage number of the line graph associated with the intersection graph of ideals of ring. We begin with the following Theorem.

Theorem 4.1. Let n,m are any two positive integers such that $n = p^m$, where p is a prime number and $m = 3,4$. Then $\gamma(L(G(\mathbb{Z}_n))) \in \{0,1\}$.

Proof. Let $m = 3,4$. Then $L(G(\mathbb{Z}_n))$ is complete graph of K_1 and K_2, which complete the proof.

Theorem 4.2. Let n,m are any two positive integers. Then the following holds:

$$\gamma(L(G(\mathbb{Z}_n))) = \begin{cases} \geq 2 \; if \; n = p^m, \; where \; m \geq 5 \\ 2 \; if \; n = p^2q \; or \; pqr, \; where \; p, \; q, \; r \; are \; distinct \; primes \end{cases}$$

Proof. Let $n = p^m$ and $m \geq 5$. If $m = 5$, then $(p), \ (p^2), \ (p^3), \ (p^4)$ are non-trivial ideals and $G(\mathbb{Z}_{p^5})$ is K_4. Now if we draw the line graph of $G(\mathbb{Z}_{p^5})$, then $[p, p^2]$ and $[p^3, p^4]$ dominate all the vertices in $L(G(\mathbb{Z}_{p^5}))$ (cf. fig. 3). Therefore, $\gamma(L(G(\mathbb{Z}_{p^5}))) \geq 2$.

Let $n = p^2q$, then $L(G(\mathbb{Z}_n))$ is a cyclic graph of length four by the Theorem 2.5. Therefore, $\gamma(L(G(\mathbb{Z}_n))) = 2$.

Let $n = pqr$, where $p, \ q, \ r$ are distinct primes. Then $[p,q]$ and $[qr, r]$ dominate all the vertices in $L(G(\mathbb{Z}_n))$ (cf. fig. 5), which yields $\gamma(L(G(\mathbb{Z}_n))) = 2$.

Theorem 4.3. Let n,m are any two positive integers. Then the following holds:

$$b(L(G(\mathbb{Z}_n))) = \begin{cases} \geq 2 \; if \; n = p^m, \; where \; m \geq 4 \\ 3 \; if \; n = p^2q, \; where \; p, \; q \; are \; distinct \; primes \\ 4 \; if \; n = pqr, \; where \; p, \; q, \; r \; are \; distinct \; primes \end{cases}$$

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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