Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 4203-4215
https://doi.org/10.28919/jmcs/5572
ISSN: 1927-5307

ON NEW APPROACH OF EXISTENCE SOLUTIONS FOR ATANGANA-BALEANU FRACTIONAL NEUTRAL DIFFERENTIAL EQUATIONS WITH DEPENDENCE ON THE LIPSCHITZ FIRST DERIVATIVES

V. PANDIYAMMAL[1], U. KARTHIK RAJA[2,*]

[1]Department of Mathematics, Arulmigu Palaniandavar College of Arts and Culture, Palani - 624 601, India

[2]Research Centre & PG Department of Mathematics, The Madura College, Madurai - 625 011, India

Abstract. In this article, we establish the result on existence and uniqueness of a Atangana-Baleanu fractional neutral differential equations with dependence on the Lipschitz first derivative conditions in Banach space. These results are based on fixed point theorems. Moreover, an example is also provided to illustrate the main results.

Keywords: fractional calculus; neutral differential equations; AB-derivative; Lipschitz first derivatives; fixed point techniques.

2010 AMS Subject Classification: 34A08, 34K37, 34K40, 58C30.

1. Introduction

Fractional differential equations have appeared in various fields in the past few decades such as chemistry, physics, engineering, control theory, aerodynamics, electrodynamics of complex medium and control of dynamical systems and so on. In consequence, fractional differential equations is obtaining much significance and attention. For details, we refer readers to [12, 18, 19, 23] and references therein. The nonlocal characteristic of the fractional order operators

*Corresponding author
E-mail address: ukarthikraja@yahoo.co.in

Received February 19, 2021

is the main reasons for the popularity of the fractional calculus, which take into account the hereditary properties of several materials and processes.

Many researchers paid more attention to ABC-derivative with several conditions in various spaces in recent years. The AB fractional derivative is familiar to followings nonsingularity as known as nonlocality of the kernel, which acquires the generalized Mittag-Leffler function. Some of the latest studies on ABC-derivatives such as, Atangana and Koca find the chaos in a simple nonlinear system with AB-fractional derivatives [10]. Jarad et al. investigated a Ordinary Differential Equations in the form of AB-derivative [20]. Ravichandran et al. discussed in details the AB-fractional neutral integro-differntial equations [25].

More precisely Sene discussed Stokes' first problem for heated flat plate with AB-derivative [33]. Owolabi studied the modelling and simulation of a dynamical system with the Atangana-Baleanu fractional derivative [32]. A substantial deal of research work has been carry through on the application of fractional neutral derivative. Liu et al. discussed a coupled system of nonlinear neutral fractional differential equations [24]. Zhou et al. studied the fractional of neutral differntial equations with infinite delay [36].

In this paper, we are interested in the existence and uniqueness of solutions of the Atangana-Baleanu fractional neutral differential equation in the sense of Caputo to the following abstract form

$$({}_a^{ABC}D^{\alpha})(u(t)-g(t,u(t),u'(t,u(t)))) = f(t,u(t),u'(t,u(t))), \quad 1<\alpha\leq 2, \tag{1}$$

$$u(a)=u_0. \tag{2}$$

with $t\in C[a,b]$, where ${}_a^{ABC}D^{\alpha}$ is the left Caputo AB fractional derivative, $u(t),({}_a^{ABC}D^{\alpha})u,f\in C[a,b],f(a,u(a),u'(a,u(a)))=0$. Consider $\mathfrak{D}u(t)=u'(t,u(t))$. Then (1) becomes

$$({}_a^{ABC}D^{\alpha})(u(t)-g(t,u(t),\mathfrak{D}u(t))) = f(t,u(t),\mathfrak{D}u(t)), \quad 1<\alpha\leq 2, \tag{3}$$

$$u(a)=u_0 \tag{4}$$

The organization of the paper is as follows: In Section 2, we review some useful properties, definitions, propositions and lemmas of fractional calculus. The existence and uniqueness of solutions for AB-fractional neutral derivative results are proved in Section 3. In the last section is devoted to illustrate an example numerically solved.

2. PRELIMINARIES

In this section, we presents some definitions, lemmas and proposotions of fractonal calculus, which will be used throughout this paper.

The definition of Riemann-Liouville fractional integral and derivatives are given as follows:

- For $\alpha > 0$, the left R-L fractional integral of order α is given as [20]

$$({}_aI^{\alpha}u)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)ds. \tag{5}$$

- For $0 < \alpha < 1$, the left R-L fractional derivative of order α is given as [20]

$$({}_aD^{\alpha}u)(t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} u(s)ds\right) \tag{6}$$

- For $0 \leq \alpha \leq 1$, the Caputo fractional derivative of order α is given as [20]

$$({}_a^CD^{\alpha}u)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} u'(s)ds. \tag{7}$$

Definition 2.1. [7] *Let $u \in H^1(a,b)$, $a < b$ and α in [0,1]. The Caputo Atangana-Baleanu fractional derivative of u of order α is defined by*

$$({}_a^{ABC}D^{\alpha}u)(t) = \frac{B(\alpha)}{(1-\alpha)} \int_0^t u'(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right] ds. \tag{8}$$

where E_{α} is the Mittag-Leffler function defined by $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$ [27, 34] *and $B(\alpha) > 0$ is a normalizing function satisfying $B(0) = B(1) = 1$. The Riemann Atangana-Baleanu fractional derivative of u of order α is defined by*

$$({}_a^{ABR}D^{\alpha}u)(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t u(s)E_{\alpha}\left[-\alpha\frac{(t-s)^{\alpha}}{1-\alpha}\right] ds. \tag{9}$$

The associative fractional integral is defined by

$$({}_a^{AB}I^{\alpha}u)(t) = \frac{1-\alpha}{B(\alpha)}u(t) + \frac{\alpha}{B(\alpha)}({}_aI^{\alpha}u)(t) \tag{10}$$

where ${}_aI^{\alpha}$ is the left Riemann-Liouville fractional integral given in (5).

Lemma 2.2. [7] *Let $u \in H^1(a,b)$ and $\alpha \in [0,1]$. Then the following relation holds.*

$$({}_{a}^{ABC}D^{\alpha}u)(t) = ({}_{a}^{ABR}D^{\alpha}u)(t) - \frac{B(\alpha)}{1-\alpha}u(a)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-a)^{\alpha}\right). \tag{11}$$

Lemma 2.3. [20] *Suppose that $\alpha > 0$, $c(t)(1-\frac{1-\alpha}{B(\alpha)}d(t))^{-1}$ is a nonnegative, nondegreasing and locally integrable function on [a,b), $\frac{\alpha d(t)}{B(\alpha)}(1-\frac{1-\alpha}{B(\alpha)}d(t))^{-1}$ is non-negative and bounded on [a,b) and $u(t)$ is nonnegative and locally integrable [a,b) with*

$$u(t) \leq c(t) + d(t)({}_{a}^{AB}I^{\alpha}u)(t), \tag{12}$$

then

$$u(t) \leq \frac{c(t)B(\alpha)}{B(\alpha)-(1-\alpha)d(t)}E_{\alpha}\left(\frac{\alpha \ d(t)(t-a)^{\alpha}}{B(\alpha)-(1-\alpha)d(t)}\right). \tag{13}$$

Theorem 2.4. *(Ascoli-Arzela Theorem)(*[16]*) Let S be a compact metric spaces.Then $M \subset C(\Omega)$ is relatively compact iff M is uniformly bounded and uniformly equicontinuous.*

Theorem 2.5. *(Krasnoselskii Fixed Point Theorem)(*[16]*) Let S be a closed, bounded and convex subset of a real Banach space X and let T_1 and T_2 be operators on S satisfying the following conditions*

- *$T_1(s) + T_2(s) \subset S$*
- *T_1 is a strict contraction on S, i.e., there exist a $l \in [a,b)$ such that*
  $\|T_1(u) - T_1(v)\| \leq l\|u-v\| \ \ \forall\, u,v \in S$
- *T_2 is continuous on S and $T_2(s)$ is a relatively compact subset of X.*

Then, there exist a $u \in S$ such that $T_1u + T_2u = u$

Proposition 2.6. ([4]) For $0 \leq \alpha \leq 1$, we conclude that

$$\begin{aligned} ({}_{a}^{AB}I^{\alpha}({}_{a}^{ABC}D^{\alpha}u))(t) &= u(t) - u(a)E_{\alpha}(\lambda t^{\alpha}) - \frac{\alpha}{1-\alpha}u(a)E_{\alpha,\alpha+1}(\lambda t^{\alpha}) \\ &= u(t) - u(a). \end{aligned}$$

Proposition 2.7. ([22, 30]) $f'(u) \in D$ satisfy the Lipschitz condition.

i.e.,There exist a constant $l > 0$ such that

$$\|f'(u) - f'(v)\| \leq l\ (\|u-v\|), \ \ u,v \in D. \tag{14}$$

Definition 2.8. A continuous function $u : [a,b] \to \mathfrak{R}$ is called a mild solution of the following Atangana-Baleanu fractional derivative equation in the sense of Caputo

$$\begin{cases} ({}_a^{ABC}D^{\alpha})(u(t)-g(t)) = f(t), \quad 1<\alpha\leq 2, \\ u(a) = u_0 \end{cases}$$

for each $t \in C[a,b]$, $u(t)$ satisfies the following integral equation

$$u(t) = u_o - g(a) + g(t) + {}_a^{AB}I^{\alpha} f(t)$$

3. Existence and Uniqueness

In this section, we prove the existence and uniqueness solutions of (3) and (4) is studied with the following assumptions.

A_1: Let $u \in C[a,b]$ and $g \in (C[a,b]\times\mathfrak{R}\times\mathfrak{R},\mathfrak{R})$ is continuous function and there exist a positive constants $\mathfrak{M}_1,\mathfrak{M}_2$ and $\mathfrak{M}$ such that

$$\|g(t,u_1,v_1)-g(t,u_2,v_2)\| \leq \mathfrak{M}_1(\|u_1-u_2\|+\|v_1-v_2\|)$$

for all u_1,v_1,u_2,v_2 in Y, $\mathfrak{M}_2 = \max_{t\in\mathfrak{R}}\|g(t,0,0)\|$ and $\mathfrak{M} = \max\{\mathfrak{M}_1,\mathfrak{M}_2\}$. Let $Y = C[\mathfrak{R},X]$ be the set continuous functions on $\mathfrak{R}$ with values in the Banach spaces X .

A_2: Let $u \in C[a,b]$ and $f \in (C[a,b]\times\mathfrak{R}\times\mathfrak{R},\mathfrak{R})$ is continuous function and there exist a positive constants $\mathfrak{N}_1,\mathfrak{N}_2$ and $\mathfrak{N}$ such that

$$\|f(t,u_1,v_1)-f(t,u_2,v_2)\| \leq \mathfrak{N}_1(\|u_1-u_2\|+\|v_1-v_2\|)$$

for all u_1,v_1,u_2,v_2 in Y, $\mathfrak{N}_2 = \max_{t\in\mathfrak{R}}\|f(t,0,0)\|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1,\mathfrak{N}_2\}$.

A_3: Let $u' \in C[a,b]$ satisfy the Lipschitz condition. i.e.,There exist a positive constants $\mathfrak{L}_1,\mathfrak{L}_2$ and $\mathfrak{L}$ such that

$$\|\mathfrak{D}(t,u)-\mathfrak{D}(t,v)\| \leq \mathfrak{L}_1(\|u-v\|),$$

for all u,v in Y. $\mathfrak{L}_2 = \max_{t\in D}\|\mathfrak{D}(t,0)\|$ and $\mathfrak{L} = \max\{\mathfrak{L}_1,\mathfrak{L}_2\}$.

A_4: For each $\lambda > 0$, Let $B_\lambda \in \{u \in Y : \|u\| \leq \lambda\} \subset Y$, them B_λ is clearly a bounded closed and convex set in $(C[a,b],\mathfrak{R})$ where $\lambda = ((1-2\mathfrak{C})^{-1}(\|u_0\|)+\mathfrak{C})$ and take $\mathfrak{C} = \max\{\mathfrak{M},\mathfrak{N}\}$ and $\mathfrak{C} < \frac{1}{2}$.

Lemma 3.1. *If* $\mathbf{A_3}$ *are satisfied, then the estimate*
$\|\mathfrak{D}u(t)\| \leq t(\mathfrak{L}_1\|u\| + \mathfrak{L}_2)$, $\|\mathfrak{D}u(t) - \mathfrak{D}v(t)\| \leq \mathfrak{L}t\|u-v\|$, *are satisfied for any* $t \in \mathfrak{R}$, *and* $u, v \in Y$.

Theorem 3.2. *Let* $u(t) \in C[a,b]$ *such that* $({}^{ABC}_{a}D^{\alpha}u)(t) \in C[a,b]$. *Suppose that* $f \in C([a,b] \times \mathfrak{R} \times \mathfrak{R}, \mathfrak{R})$ *satisfies* $\mathbf{A_1} - \mathbf{A_4}$. *Then, if* $g(a,u(a),\mathfrak{D}u(a)) = f(a,u(a),\mathfrak{D}u(a)) = 0$ *and* $\Big(\mathfrak{L}t + (1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big)\Big) \leq 1$ *the problem (3) and (4) has an unique solution.*

Proof. First, we show that u(t) satisfies the problem (3) and (4) iff u(t) satisfies the integral equation

$$u(t) = u_0 - g(a,u(a),\mathfrak{D}u(a)) + g(t,u(t),\mathfrak{D}u(t)) + {}^{AB}_{a}I^{\alpha} f(t,u(t),\mathfrak{D}u(t)) \tag{15}$$

Let $u(t)$ satisfy (3). To apply the AB fractional integral to both sides of (3), we get

$$({}^{AB}_{a}I^{\alpha}({}^{ABC}_{a}D^{\alpha})(u(t) - g(t,u(t),\mathfrak{D}u(t)))) = {}^{AB}_{a}I^{\alpha} f(t,u(t),\mathfrak{D}u(t)) \tag{16}$$

Now, constructing use of Proposition 2.4, we get

$$u(t) - g(t,u(t),\mathfrak{D}u(t)) - (u(a) - g(a,u(a),\mathfrak{D}u(a))) = {}^{AB}_{a}I^{\alpha} f(t,u(t),\mathfrak{D}u(t)) \tag{17}$$

Since $u(a) = u_0$ from (4) and $f(a,u(a),\mathfrak{D}u(a)) = 0$, (15) is satisfied. If $u(t)$ satifies (15), then by using that $f(a,u(a),\mathfrak{D}u(a)) = 0$ it is obvious that $u(a) = u_0$.
To apply the Riemann-Liouville AB fractional derivative to both sides of (15) and utilize that $({}^{AB}_{a}D^{\alpha}({}^{AB}_{a}I^{\alpha}u))(t) = u(t)$. We get

$$({}^{ABR}_{a}D^{\alpha}u)(t) = u_0({}^{ABR}_{a}D^{\alpha}1)(t) + ({}^{ABR}_{a}D^{\alpha})g(t,u(t),\mathfrak{D}u(t)) + ({}^{ABR}_{a}D^{\alpha}({}^{AB}_{a}I^{\alpha}))(t)f(t,u(t),\mathfrak{D}u(t)) \tag{18}$$

Thus, we have

$$({}^{ABR}_{a}D^{\alpha})(u(t) - g(t,u(t),\mathfrak{D}u(t))) = (u_0 - g(a,u(a),\mathfrak{D}u(a)))E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-a)^{\alpha}\right) + f(t,u(t),\mathfrak{D}u(t)) \tag{19}$$

Then, the result is acquired by getting from theorem(1) in [7]. Now, we can consider the operator T defined by

$$Tu(t) = u_0 - g(a,u(a),\mathfrak{D}u(a)) + g(t,u(t),\mathfrak{D}u(t))) + {}^{AB}_{a}I^{\alpha} f(t,u(t),\mathfrak{D}u(t)).$$

Then, by A_3, $\|u\| \leq \lambda$ we get

$$\begin{aligned}
\|Tu(t)\| &\leq \|u_0\| + \mathfrak{M}(\|u\| + t(\mathfrak{L}_1\|u\| + \mathfrak{L}_2)) + \frac{1-\alpha}{B(\alpha)}(\mathfrak{N}_1(\|u\| + \mathfrak{L}_1 t\|u\|) \\
&\quad + \frac{1-\alpha}{B(\alpha)}\mathfrak{N}_2 + \frac{\alpha}{B(\alpha)}(\mathfrak{N}_1\|u\| + \mathfrak{L}t\|u\|)(({}_a^{AB}I^{\alpha})(t) \\
&\leq \|u_0\| + \mathfrak{C}\|u\| + \mathfrak{C}\Big(\mathfrak{L}t + \frac{1-\alpha}{B(\alpha)}(1+\mathfrak{L}t) + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}(1+\mathfrak{L}t)\Big)\|u\| \\
&\quad + \mathfrak{C}\Big(\mathfrak{L}t + \frac{1-\alpha}{B(\alpha)}(1+\mathfrak{L}t) + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}(1+\mathfrak{L}t)\Big) \\
&\leq \lambda(1-2\mathfrak{C}) + 2\mathfrak{C}\lambda \\
&= \lambda
\end{aligned}$$

i.e., $\|Tu(t)\| \leq \lambda$. Now to prove uniqueness

$$\begin{aligned}
\|T(u) - T(v)\| &\leq \|g(t,u_1(t),\mathfrak{D}u(t)) + g(a,u(a),\mathfrak{D}u(a)) + {}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t)) \\
&\quad - [g(t,v(t),\mathfrak{D}v(t)) + g(a,v(a),\mathfrak{D}v(a)) + {}_a^{AB}I^{\alpha} f(t,v(t),\mathfrak{D}v(t))]\| \\
&\leq \mathfrak{M}(1+\mathfrak{L}t)\|u-v\| + \frac{1-\alpha}{B(\alpha)}(\mathfrak{N}(1+\mathfrak{L}t))\|u-v\|] \\
&\quad + \frac{\alpha}{B(\alpha)}(\mathfrak{M}(1+\mathfrak{L}t))\|u-v\|(({}_a^{AB}I^{\alpha})(t) \\
&\leq \mathfrak{C}\|u-v\| + \mathfrak{C}\Big(\mathfrak{L}t + (1+\mathfrak{L}t)\Big(\frac{1-\alpha}{B(\alpha)} + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big)\Big)\|u-v\| \\
&\leq 2\mathfrak{C}\|u-v\| \\
&\leq \|u-v\|
\end{aligned}$$

Hence, the operator $Tu(t), t \in B_{\lambda}$ proved the existence and uniqueness conditions and has a fixed point by Banach contraction principle in Banach spaces X.

Next, we investigate the problem (3) and (4) has a fixed point by utilizing Krasnoselskii's fixed point theroem.

Theorem 3.3. *If* $\mathbf{A_1} - \mathbf{A_4}$ *are satisfied and* $q(t_2 - t_1) = [\mathfrak{N}(\|u(t_2) - u(t_1)\| + \mathfrak{L}t\|u(t_2) - u(t_1)\|)]$, *then the problem (3) and (4) has a solution.*

Proof. Now, for any $\lambda_0 > 0$ and $u \in B_{\lambda_0}$, we define two operator T_1 and T_2 on B_{λ_0} as follows

$$(T_1u)(t) = u_0 - g(0,u(0),0) + g(t,u(t),\mathfrak{D}u(t)) \tag{20}$$

$$(T_2u)(t) = {}_a^{AB}I^{\alpha} f(t,u(t),\mathfrak{D}u(t)). \tag{21}$$

Obviously, u is a solution of (3) and (4) iff the operator $T_1u + T_2u = u$ has a solution $u \in B_{\lambda_0}$

This proof will be given in three steps.

Step 1. $\|T_1u + T_2u\| \leq \lambda_0$ whenever $u \in B_{\lambda_0}$.

For every $u \in B_{\lambda_0}$, we have

$$\begin{aligned}
\|(T_1u)(t) + (T_2u)(t)\| &\leq \|u_0\| + \mathfrak{M}(\|u\| + t(\mathfrak{L}_1\|u\| + \mathfrak{L}_2)) + \frac{1-\alpha}{B(\alpha)}(\mathfrak{N}_1(\|u\| + \mathfrak{L}_1 t\|u\|) \\
&\quad + \frac{1-\alpha}{B(\alpha)}\mathfrak{N}_2 + \frac{\alpha}{B(\alpha)}(\mathfrak{N}_1\|u\| + \mathfrak{L}t\|u\|)({}_a^{AB}I^{\alpha})(t) + \frac{\alpha}{B(\alpha)}\mathfrak{N}_2({}_a^{AB}I^{\alpha})(t) \\
&\leq \|u_0\| + \mathfrak{C}\|u\| + \mathfrak{C}\Big(\mathfrak{L}t + \frac{1-\alpha}{B(\alpha)}(1+\mathfrak{L}t) + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}(1+\mathfrak{L}t)\Big)\|u\| \\
&\quad + \mathfrak{C}\Big(\mathfrak{L}t + \frac{1-\alpha}{B(\alpha)}(1+\mathfrak{L}t) + \frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}(1+\mathfrak{L}t)\Big) \\
&\leq \lambda(1-2\mathfrak{C}) + 2\mathfrak{C}\lambda \\
&\leq \lambda_0
\end{aligned}$$

Hence, $\|T_1u + T_2u\| \leq \lambda_0$ for every $u \in B_{\lambda_0}$.

Step 2. T_1 is a contraction on B_{λ_0} for every $u,v \in B_{\lambda_0}$, according to $\mathbf{A_4}$ and (20), we have

$$\begin{aligned}
\|(T_1u)(t) - (T_1v)(t\| &\leq \|u_0 - v_0\| + l\|u_0 - v_0\| + \mathfrak{M}\|u-v\| + \mathfrak{M}\mathfrak{L}t\|u-v\| \\
&\leq \|u_0 - v_0\|[1 + l + \mathfrak{M}\|u-v\| + \mathfrak{M}\mathfrak{L}t\|u-v\|] \\
&\leq R\|u_0 - v_0\|
\end{aligned}$$

which implies that $\|T_1u - T_1v\| \leq R\|u_0 - v_0\|$, since $R = 1$, where $R = 1 + l + \mathfrak{M}\|u-v\| + \mathfrak{M}\mathfrak{L}t\|u-v\|$. This shows that T_1 is a contraction.

Step 3. T_2 is completely continuous operator.

First we have to prove that T_2 is continuous on B_{λ_0}. For every $u_n, u \subset B_{\lambda_0}$, $n = 1,2,3....$ with $\lim_{n\to u}\|u_n - u\| = 0$, we get $\lim_{n\to u} u_n(t) = u(t)$, for $t \in [a,b]$.

Thus by $\mathbf{A_2}$, we know $\lim_{n\to\infty} f(t,u_n(t),\mathfrak{D}u_n(t)) = f(t,u(t),\mathfrak{D}u(t))$ for every $t \in [a,b]$.

We can conclude that

$$\sup_{s\in[a,b]} \|f(t,u_n(t),\mathfrak{D}u_n(t)) - f(t,u(t),\mathfrak{D}u(t))\| \to 0 \text{ as } n\to\infty$$

On other hand, for $t\in[a,b]$ we can obtain that

$$\begin{aligned}
\|(T_2u_n)(t)-(T_2u)(t)\| &\leq \|{}_a^{AB}I^\alpha f(t,u_n(t),\mathfrak{D}u_n(t)) - {}_a^{AB}I^\alpha f(t,u(t),\mathfrak{D}u(t))\| \\
&\leq \frac{1-\alpha}{B(\alpha)}\|{}_a^{AB}I^\alpha f(t,u_n(t),\mathfrak{D}u_n(t)) - {}_a^{AB}I^\alpha f(t,u(t),\mathfrak{D}u(t))\| \\
&\quad + \frac{\alpha}{B(\alpha)}\|{}_a^{AB}I^\alpha f(t,u_n(t),\mathfrak{D}u_n(t)) - {}_a^{AB}I^\alpha f(t,u(t),\mathfrak{D}u(t))\|{}_aI^\alpha(t) \\
&\leq \frac{1-\alpha}{B(\alpha)}\sup_{s\in[a,b]}\|f(t,u_n(t),\mathfrak{D}u_n(t)) - f(t,u(t),\mathfrak{D}u(t))\| \\
&\quad + \frac{(b-a)^\alpha}{B(\alpha)\Gamma(\alpha)}\sup_{s\in[a,b]}\|f(t,u_n(t),\mathfrak{D}u_n(t)) - f(t,u(t),\mathfrak{D}u(t))\| \\
&\leq \left(\frac{1-\alpha}{B(\alpha)} - \frac{(b-a)^\alpha}{B(\alpha)\Gamma(\alpha)}\right)\sup_{s\in[a,b]}\|f(t,u_n(t),\mathfrak{D}u_n(t)) - f(t,u(t),\mathfrak{D}u(t))\|
\end{aligned}$$

Hence $\|(T_2u_n)(t)-(T_2u)(t)\|\to 0$ as $n\to\infty$. Therefore T_2 is continuous on B_{λ_0}.

Now, we have to show that $T_2u, u\in B_{\lambda_0}$ is relatively compact which is enough to prove that the function $T_2u, u\in B_{\lambda_0}$ uniformly bounded and equicontinuous, and $\forall\, t\in[a,b]$

$\|T_2u\|\leq\lambda_0$, for any $u\in B_{\lambda_0}$, therefore $(T_2u)(t), u\in B_{\lambda_0}$ is bounded uniformly.

Now, we show that $(T_2u)(t), u\in B_{\lambda_0}$ is a equicontinuous.

For any $u\in B_{\lambda_0}$ and $a\leq t_1\leq t_2\leq t$, we get

$$\begin{aligned}
\|(T_2u)(t_2)-(T_2u)(t_1)\| &\leq \|{}_a^{AB}I^\alpha f(t_2,u(t_2),\mathfrak{D}u(t_2)) - {}_a^{AB}I^\alpha f(t_1,u(t_1),\mathfrak{D}u(t_1))\| \\
&\leq \frac{1-\alpha}{B(\alpha)}\|f(t_2,u(t_2),\mathfrak{D}u(t_2)) - f(t_1,u(t_1),\mathfrak{D}u(t_1))\| \\
&\quad + \frac{\alpha}{B(\alpha)}{}_aI^\alpha\|f(t_2,u(t_2),\mathfrak{D}u(t_2)) - f(t_1,u(t_1),\mathfrak{D}u(t_1))\| \\
&\leq \frac{1-\alpha}{B(\alpha)}(\mathfrak{N}(\|u(t_2)-u(t_1)\| + \mathfrak{L}t\|u(t_2)-u(t_1)\|)) \\
&\quad + \frac{\alpha}{B(\alpha)}(\mathfrak{N}(\|u(t_2)-u(t_1)\| + \mathfrak{L}t\|u(t_2)-u(t_1)\|))({}_aI^\alpha)(t_2-t_1)
\end{aligned}$$

$$\leq \frac{1-\alpha}{B(\alpha)} q(t_2-t_1) + \frac{\alpha}{B(\alpha)} q(t_2-t_1) \frac{(t_2-t_1)^{\alpha}}{\alpha\Gamma(\alpha)}$$
$$\leq q\Big(\frac{1-\alpha}{B(\alpha)} - \frac{(t_2-t_1)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\Big)(t_2-t_1)$$

$\|(T_2u)(t_2)-(T_2u)(t_1)\| \to 0$ as $t_2 \to t_1$. Therefore, the operator T_2 is a equicontinuous on B_{λ_0}. Hence, which implies T_2 is relatively compact on B_{λ_0}.

Therefore T_2 is satisfies the condition of theorem 2.4 and theorem 2,5, we can conclude that T_2 has a fixed point. Therefore the problem (3) and (4), the operator T has a fixed point u.

4. EXAMPLE

In this section an example is presented for the existence results to the following problem.

$$({}_0^{ABC}D^{\frac{3}{2}})(u(t)-\frac{t}{3\sqrt(\pi)}sin(u(t)+u'(t)))) = \frac{t}{3\sqrt(\pi)}cos(u(t)+u'(t)), \tag{22}$$

$$u(0)=1,\ \ t\in[1,2],\ B(\alpha)=1 \tag{23}$$

Notice that $g(0,u(0),\mathfrak{D}u(0)) = f(0,u(0),\mathfrak{D}u(0)) = 0$ and $u'(t) \in C[1,2]$ satisfy the Lipschitz conditions.

Let $g(t,u,v) = \frac{t}{3\sqrt(\pi)}sin(u+v),\ \ f(t,u,v) = \frac{t}{3\sqrt(\pi)}cos(u+v),\ t\in[1,2]$.
It is easy to see that

$$({}_0^{ABC}D^{\frac{3}{2}})(u(t)-g(t,u,v)) = f(t,u,v), \tag{24}$$

$$u(0)=1,\ \ t\in[1,2],\ B(\alpha)=1 \tag{25}$$

Therefore, by Banach contraction principle theorem (24) and (25) has an unique solution, this can be written as
$u(t) = \lim_{n\to\infty} u_n(t)$, where

$$u_n(t) \ \ = \ \ 1+\frac{1}{3\sqrt{\pi}}g_{n-1}(t)+\frac{1-\alpha}{3\sqrt{\pi}}f_{n-1}(t)+\frac{\alpha}{3\sqrt{\pi}\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, f_{n-1}(s)ds,$$

where $n = 1,2,3,...$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] T. Abdeljawad, A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel, J. Inequal. Appl. 2017 (2017), 130.

[2] T. Abdeljawad, Q.M. Al-Mdallal, Discrete Mittag-Leffler kernel type fractional difference initial value problems and Gronwall?s inequality, J. Comput. Appl. Math. 339 (2018), 218-230.

[3] T. Abdeljawad, D. Baleanu, On Fractional Derivatives with Exponential Kernel and their Discrete Versions, Rep. Math. Phys. 80 (2017), 11-27.

[4] T. Abdeljawad, D. Baleanu, Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels, Adv. Differ. Equ. 2016 (2016), 232.

[5] T. Abdeljawad, D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, J Nonlinear Sci. Appl. 10 (2017), 1098-1107.

[6] Y. Adjabi, F. Jarad, D. Baleanu , T. Abdeljawad, On cauchy problems with Caputo Hadamard fractional derivatives, J. Comput. Anal. Appl. 21 (2016), 661-681.

[7] A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model. Therm. Sci. 20 (2016), 763-769.

[8] A. Atangana, J.F. Gomez-Aguilar, Numerical approximation of Riemann-Liouville definition of fractional derivative: From Riemann-Liouville to Atangana-Baleanu: Atangana and Gomez-Aguilar, Numer. Meth. Part. Differ. Equ. 34 (2018), 1502-1523.

[9] A. Atangana, J.F. Gomez-Aguilar, Hyperchaotic behaviour obtained via a nonlocal operator with exponential decay and Mittag-Leffler laws, Chaos Solitons Fractals. 102 (2017), 285-294.

[10] A. Atangana, I. Koca, Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order, Chaos Soliton Fractals. 89 (2016), 447-454.

[11] A. Khan, H. Khan, J.F. Gomez-Aguilar, T. Abdeljawad, Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel, Chaos Solitons Fractals. 127 (2019), 422-427.

[12] Z. Bai, H. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005), 495-505.

[13] D. Baleanu, M. Inc, A. Yusuf, A.I. Aliyu, Space-time fractional Rosenou-Haynam equation: Lie symmetry analysis, explicit solutions and conservation laws, Adv. Differ. Equ. 2018 (2018), 46.

[14] D. Baleanu, M. Inc, A. Yusuf, A.I. Aliyu, Time fractional third-order evolution equation: symmetry analysis, explicit solutions, and conservation laws, J. Comput. Nonlinear Dyn. 13 (2018), 021011.

[15] J.D. Djida, A. Atangana, I. Area, Numerical computation of a fractional derivative with non-local and non-singular kernel, Math. Model. Nat. Phenom. 12 (2017), 4-13.

[16] U. Cakan, I. Ozdemir, An application of Krasnoselskii fixed point theorem to some nonlinear functional integral equations, Nevsehir Bilim ve Teknoloji Dergisi, 3 (2014), 66-73.

[17] Y.Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, Adv. Differ. Equ. 2014 (2014), 10.

[18] L. Gaul, P. Klein, S. Kemple, Damping description involving fractional operators, Mech. Syst. Signal Process. 5 (1991), 81-88.

[19] M. Donatelli, M. Mazza, S. Serra-Capizzano, Spectral analysis and structure preserving preconditioners for fractional diffusion equations, J. Comput. Phys. 307 (2016), 262-279.

[20] F. Jarad, T. Abdeljawad, Z. Hammouch, On a class of ordinary differential equations in the frame of Atagana-Baleanu derivative, Chaos Solitons Fractals, 117 (2018), 16-20.

[21] F. Jarad, T. Abdeljawad, D Baleanu, Caputo-type modification of the Hadamard fractional derivative. Adv. Differ. Equ. 2012 (2012), 142.

[22] D. Lera, Y.D. Sergeyev, Acceleration of Univariate Global Optimization Algorithms Working with Lipschitz Functions and Lipschitz First Derivatives, SIAM J. Optim. 23 (2013), 508-529.

[23] H.-L. Li, Y.-L. Jiang, Z. Wang, L. Zhang, Z. Teng, Global Mittag-Leffler stability of coupled system of fractional-order differential equations on network, Appl. Math. Comput. 270 (2015), 269-277.

[24] S. Liu, G. Wang, L. Zhang, Existence results for a coupled system of nonlinear neutral fractional differential equations, Appl. Math. Lett. 26 (2013), 1120-1124.

[25] K. Logeswari, C. Ravichandran, A new exploration on existence of fractional neutral integro- differential equations in the concept of Atangana-Baleanu derivative, Physica A: Stat. Mech. Appl. 544 (2020), 123454.

[26] I. Koca, Analysis of rubella disease model with non-local and non-singular fractional derivatives, Int. J. Optim. Control, Theor. Appl. 8 (2017), 17-25.

[27] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam, 2006.

[28] I. Koca, A. Atangana, Solutions of Cattaneo-Hristov model of elastic heat diffusion with Caputo-Frbrizio and Atagana-Baleanu fractional derivatives, Therm. Sci. 21(2017), 2299-2305.

[29] I. Koca, Modelling the spread of Ebola virus with Atangana-Baleanu fractional operators, Eur. Phys. J. Plus. 133 (2018), 100.

[30] D.E. Kvasov, Y.D. Sergeyev, A univariate global search working with a set of Lipschitz constants for the first derivative, Optim. Lett. 3 (2009), 303-318.

[31] M. Toufik, A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models, Eur. Phys. J. Plus. 132 (2017), 444.

[32] K.M. Owolabi, Modelling and simulation of a dynamical system with the Atangana-Baleanu fractional derivative, Eur. Phys. J. Plus. 133 (2018), 15.

[33] N. Sene, Stokes? first problem for heated flat plate with Atangana-Baleanu fractional derivative, Chaos Solitons Fractals. 117 (2018), 68-75.

[34] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives,: Theory and Applications, Gordon and Breach, Amsterdam, 1993.

[35] A. Yusuf, M. Inc, A. Isa Aliyu, D. Baleanu, Efficiency of the new fractional derivative with nonsingular Mittag-Leffler kernel to some nonlinear partial differential equations, Chaos Solitons Fractals. 116 (2018), 220-226.

[36] Y. Zhou, F. Jiao, J. Li, Existence and uniqueness for fractional neutral differential equations with infinite delay, Nonlinear Anal., Theory Meth. Appl. 71 (2009), 3249-3256.