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PREORDERS AND INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, we associate each intuitionistic fuzzy topology on a set X with two preorders on the set of all intutionistic fuzzy points $Pt(I^X)$ on X. Also we introduce the concept of two types of preorders on $Pt(I^X)$ namely C-preorders and K-preorders, and shows how these preorders related with the intuitionistic fuzzy topological properties on X.

Keywords: fuzzy topology; intuitionistic fuzzy topology; preorder; kernel topology; point closure topology.

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1. Introduction

Fuzzy sets and Intuitionistic fuzzy sets are very convenient postulations to elaborate the uncertainty in real world problems. Fuzzy sets have been accepted as a fruitful area of research in mathematics, as well as a tool for the assessment of different objects and processes in nature and society. As an augmentation of classical set theory, in 1965 Zadeh L. A. introduced fuzzy sets[8] and the first extension L-fuzzy sets, were bought by Goguen in 1967[5]. Another extension is the rough sets, which were defined by Z. Pawlak in 1981[12, 13] and Atanassov K initiated the notion of intuitionistic fuzzy sets(IFSs) in 1983[6, 7], and later in 1986, Chang C.

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L. launched the idea of fuzzy topology gratifying the similar concept of the axioms of topology. The theory of intuitionistic fuzzy topology(IFT) were introduced by D. Coker in 1997 [3, 4]. Now a days fuzzy sets and related theory widely uses in biological, social, linguistic, psychological and economic fields.

The correspondence between topologies and preorder have been described by several mathematicians like Ore. O. and Steiner A. K.[1]. In his works Steiner studied the anti-isomorphism between lattice of principal topologies and lattice of preorder relations on a set. Ore.O, associated closure operator and preorder on a set X through his work. Mathematicians Susan J Andima and Thron W. J. associated open sets and preorder on a set X and showed that a lot of topological properties like lower sepration axioms are order induced and also derived some new properties proposed by order properties [11]. As a generalisation of above study Madhavan Namboothiri N. M.[9] associated, to each fuzzy topology on a set X with two preoreder relations on $Pt(L^X)$ and investigated the natural correspondance between order properties with some fuzzy topological properties. Some studies connecting the correspondence between preorders on a set X and generalised fuzzy topologies on that set have done by Dhanya P. M.[10]. The above studies reveals the deep connection between order structures on a set X and the topological structures on the same set.

2. Preliminaries

In 1965, Zadeh introduced fuzzy set as a well defined collection of ordered pairs (x, μ), where μ corresponds to the membership value of x which always belongs to the interval $[0,1]$. The value of membership shows the strength of belongingness of each element. As a generalisation to this, in 1986 K Atanassov bought the notion of intuitionistic fuzzy set, which says that, each element in the set allocate a membership value and a non-membership value both belongs in the interval $[0,1]$ such that their sum will also be in the interval $[0,1]$. That means each element is connected to two functions, say membership value μ and non-membership value $\nu = 1 - \mu$, this shows there is no space for uncertainty. Chang C. L.[2] grafted the concept of fuzzy set into general topology. He extended the concept of general topology into fuzzy set by considering topology τ as a subset of I^X, for any set X with $0, 1 \in \tau$ and is closed under arbitrary unions and finite intersections.

Intuitionistic fuzzy topology is the generalisation of fuzzy topology. It is considered as the collection of intuitionistic fuzzy subsets of X in which $\underline{0}$ and $\underline{1}$ are members and is closed under arbitrary join and finite meets. The members of topology are called intuitionistic fuzzy open sets and complement of its members are called intuitionistic fuzzy closed sets. Using this concepts of intuitionistic open sets and intuitionistic closed sets we can construct the fuzzy interior of a set A as the join of all open sets contained in A and fuzzy closure of A as the meet of all closed sets containing A.

In this paper we deal with point closure topology and kernel topology on the set of intuitionistic fuzzy points. We can consider the intuitionistic fuzzy point as, for $x \in X,\ l,l' \in [0,1]$ and $l+l' \leq 1$

$$\mu_{x_{(l,l')}}(t) = \begin{cases} l & \text{if x = t} \\ 0 & \text{otherwise.} \end{cases}$$

$$\nu_{x((l,l')}(t) = \begin{cases} l' & \text{if x = t} \\ 1 & \text{otherwise.} \end{cases}$$

Note that $(\mu,\nu)/_{X-\{x\}} = \underline{0}/_{X-\{x\}}$. Through out this paper the set of all intuitionistic fuzzy points is denoted by $Pt([0,1]^X)$. Shortly $Pt(I^X)$. Let A be an intuitionistic fuzzy set, say $A = (\mu_A, \nu_A)$. Then $A(x) = (\mu_A(x), \nu_A(x)) = (l,l')$. If $(l,l') \neq (0,1)$ then $x_{(l,l')} \leq A$. Now consider $x_{(1,0)}$ and G be any intuitionistic fuzzy open set containing $x_{(1,0)}$, then $x_{(l,l')} \leq G$ and hence $G \wedge A \neq (0,1) = \underline{0}$. Thus $x_{(1,0)} \leq \bar{A}$. That is $\bar{A}(x) = (1,0)$. Thus if we select the neighbourhood structure as structure of open sets + membership relation between points and sets with membership relation as ordinary inclusion, then all closed sets must be crisp. Thus this membership relation does not evaluate the fuzzyness of the sets. Hence we need another type of membership relation and hence to define neighbourhood structure. So in this paper we consider the relation as for any point $x_{(l,l')} \in Pt(I^X)$ and $A \in I^X$ we say that $x_{(l,l')}$ quasi-coincidence with A or $x_{(l,l')}$ is quasi-coincident with A denoted by $x_{(l,l')}\ \hat{q}\ A$ if $\mu_A(x)+l > 1$ and $\nu_A(x)+l' < 1$. Thus $x_{(l,l')}\ \hat{q}\ A$ if and only if $l \not\leq \nu_A(x)$ or $l' \not\geq \mu_A(x)$. In general A is quasi-coincides with B if there exists $x \in X$ such that $\mu_A(x) \not\leq \nu_B(x)$ or $\nu_A(x) \not\geq \mu_B(x)$. More over a quasi - coincident neighbourhood of $x_{(l,l')}$ in an intuitionistic fuzzy topological space (I^X, δ) is an element U of δ such that $x_{(l,l')}\ \hat{q}\ U$.

Through out the paper, preorder relations has a prominent role. A reflexive and transitive binary relation is called a preorder relation and if a preorder which is antisymmetric is called as a partial order.

3. Topology and Corresponding Preorders

Let (I^X, δ) be an intuitionistic fuzzy topological space. Define two relations R_δ and S_δ on $Pt(I^X)$ by $x_{(l,l')}\ R_\delta\ y_{(m,m')}$ if and only if every quasi - neighbourhood of $y_{(m,m')}$ quasi - coincides with $x_{(l,l')}$ and $x_{(l,l')}\ S_\delta\ y_{(m,m')}$ if and only if every open set containing $y_{(m,m')}$ contains $x_{(l,l')}$. It is interesting to see that R_δ and S_δ are preorders on $Pt(I^X)$ and we denote these preorders respectively by $\rho_1(\delta)$ and $\rho_2(\delta)$. Now we can view ρ_1 and ρ_2 as functions from the set of all intuitionistic fuzzy topologies on X to the set of all preorders on $Pt(I^X)$ and it can easly verified that these functions are order reversing in the sense that, if δ_1 and δ_2 are two intuitionistic fuzzy topologies on X with $\delta_1 \subseteq \delta_2$, then $\rho_1(\delta_2) \subseteq \rho_1(\delta_1)$ and $\rho_2(\delta_2) \subseteq \rho_2(\delta_1)$. Also note that in an intuitionistic fuzzy topological space (I^X, δ), the closure of $x_{(l,l')}$ for any $x_{(l,l')} \in Pt(I^X)$ is $\{\overline{x_{(l,l')}}\} = \vee\{y_{(m,m')} \in Pt(I^X) : (x_{(l,l')}, y_{(m,m')}) \in \rho_1(\delta)\}$ and the kernel of $x_{(l,l')}$ for any $x_{(l,l')} \in Pt(I^X)$ is $\{\widehat{x_{(l,l')}}\} = \vee\{y_{(m,m')} : (y_{(m,m')}, x_{(l,l')}) \in \rho_2(\delta)\}$. More over we can observe that if two intuitionistic fuzzy topologies on X have the same image under ρ_1, then they have same point closures and if two intuitionistic fuzzy topologies on X have the same image under ρ_2, then they have same kernels. Thus for a given preorder R on $Pt(I^X)$, it make sense to write $\{\overline{x_{(l,l')}}\}$ and $\{\widehat{x_{(l,l')}}\}$ even though no topology is specified.

Definition 3.1. For any preorder R on $Pt(I^X)$ and for any $x_{(l,l')} \in Pt(I^X)$, the closure of $x_{(l,l')}$ with respect to R is denoted by $\{\overline{x_{(l,l')}}\}^R$ and is defined as $\{\overline{x_{(l,l')}}\}^R = \vee\{y_{(m,m')} : (x_{(l,l')}, y_{(m,m')}) \in R\}$. Similarly the kernel of $x_{(l,l')}$ with respect to R is denoted by $\{\widehat{x_{(l,l')}}\}^R$ and is defined as $\{\widehat{x_{(l,l')}}\}^R = \vee\{y_{(m,m')} : (y_{(m,m')}, x_{(l,l')}) \in R\}$. The smallest intuitionistic fuzzy topology on X such that for all $x_{(l,l')} \in Pt(I^X)$, $\{\overline{x_{(l,l')}}\}^R$ are closed is called the point closure intuitionistic fuzzy topology of R and is denoted by $\mu(R)$ and the smallest intuitionistic fuzzy topology on X such that for all $x_{(l,l')} \in Pt(I^X)$, $\{\widehat{x_{(l,l')}}\}$ are open is called the kernel intuitionistic fuzzy topology of R and is denoted by $\nu(R)$.

From above definition it is clear that, for any preorder R in $Pt(I^X)$, $\{(\{\overline{x_{(l,l')}}\}^R)', x_{(l,l')} \in Pt(I^X)\} \cup \{\underline{1}\}$is a subbase for the point closure topology $\mu(R)$ and $\{\{\widehat{x_{(l,l')}}\}^R, x_{(l,l')} \in pt(I^X)\}$ is a base for the kernel topology $\nu(R)$.

Remark 3.1. Let (I^X, τ) be an intuitionistic fuzzy topological space.
Then for any $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$, if $y_{(m,m')} \leq \{\overline{x_{(l,l')}}\}$, then $(x_{(l,l')}, y_{(m,m')}) \in \rho_1(\tau)$.

Proof. Let $y_{(m,m')} \leq \{\overline{x_{(l,l')}}\}$ then there is a collection $\{y_{(k_i,k_i')} \in Pt(I^X), i \in \Lambda\}$ with $(x_{(l,l')}, y_{(k_i,k_i')}) \in \rho_1(\tau)$ for each $i \in \Lambda$ and $m \leq \vee_{i\in\Lambda} k_i$ and $m' \geq \wedge_{i\in\Lambda} k_i'$. Now let U be a quasi-neighbourhood of $y_{(m,m')}$ implies $m \nleq \nu_U(y)$ or $m' \ngeq \mu_U(y)$. If $k_i \leq \nu_U(y), \forall i \in \Lambda$ then $\vee_{i\in\Lambda} k_i \leq \nu_U(y)$ and hence $m \leq \nu_U(y)$. If $k_i' \geq \mu_U(y) \forall i \in \Lambda$ then $\wedge_{i\in\Lambda} k_i' \geq \mu_U(y)$ and hence $m' \geq \mu_U(y)$. That is, since $m \nleq \nu_U(y)$ or $m' \ngeq \mu_U(y)$, there is an $i \in \Lambda$ such that $k_i \nleq \nu_U(y)$ or there is an $i \in \Lambda$ such that $k_i' \ngeq \mu_U(y)$. That is $U \in Q(y_{(k_i,k_i')})$. Therefore $U \hat{q}\, x_{(l,l')}$. That is for every quasi-neighbourhood of $y_{(m,m')}$ quasi-coincides with $x_{(l,l')}$. Therefore $(x_{(l,l')}, y_{(m,m')}) \in \rho_1(\tau)$. □

Remark 3.2. Let (I^X, τ) be an intuitionistic fuzzy topological space.
Then for any $x_{(l,l')}$, $y_{(m,m')} \in Pt(I^X)$, $y_{(m,m')} \leq \{\widehat{x_{(l,l')}}\}$ implies that $(y_{(m,m')}, x_{(l,l')}) \in \rho_2(\tau)$.

Proof. Let $y_{(m,m')} \leq \{\widehat{x_{(l,l')}}\}$, then there is a collection $\{y_{(k_i,k_i')} \in Pt(I^X), i \in \Lambda\}$ with $(y_{(k_i,k_i')}, x_{(l,l')}) \in \rho_2(\tau)$ $\forall i \in \Lambda$ and $m \leq \vee_{i\in\Lambda} k_i$ and $m' \geq \wedge_{i\in\Lambda} k_i'$. Then every τ -open set U containing $x_{(l,l')}$ contains $y_{(k_i,k_i')}, \forall i \in \Lambda$. Hence U contains $y_{(k_i,k_i')}, \forall i \in \Lambda$. Hence U contains $\vee_{i\in\Lambda} y(k_i, k_i')$. Thus every τ -open set containing $x_{(l,l')}$ contains $y_{(m,m')}$ and hence$(y_{(m,m')}, x_{(l,l')}) \in \rho_2(\tau)$. □

Theorem 3.2. *Let (I^X, τ) be an intuitionistic fuzzy topological space with $\rho_1(\tau) = R$, then $\mu(R) \subseteq \tau$ and $\rho_1(\mu(R)) = R$.*

Proof. Given $\rho_1(\tau) = R$, then clearly for any $x_{(l,l')} \in Pt(I^X)$, $\{\overline{x_{(l,l')}}\}^\tau = \{\overline{x_{(l,l')}}\}^R$, then for any $x_{(l,l')} \in Pt(I^X)$, $\{\overline{x_{(l,l')}}\} = \vee\{y_{(m,m')} : (x_{(l,l')}, y_{(m,m')}) \in R\}$, is closed with respect to τ. By the definition, $\mu(R)$ is the smallest topology in which $\{\overline{x_{(l,l')}}\}^R$ are closed. Hence $\mu(R) \subseteq \tau$. Since ρ_1 is order reversing we have

$$\rho_1(\mu(R)) \supseteq \rho_1(\tau) = R$$

Also, $(x_{(l,l')}, y_{(m,m')}) \in \rho_1(\mu(R))$ implies that $y_{(m,m')} \leq \{\overline{x_{(l,l')}}\}^{\mu(R)}$ and hence $y_{(m,m')} \leq \{\overline{x_{(l,l')}}\}^{R}$. Therefore $y_{(m,m')} \leq \{\overline{x_{(l,l')}}\}^{\tau}$ implies that $(x_{(l,l')}, y_{(m,m')}) \in \rho_1(\tau) = R$. Thus $\rho_1(\mu(R)) \subseteq R$ and hence $\rho_1(\mu(R)) = R$. □

Theorem 3.3. *Let (I^X, τ) be an intuitionistic fuzzy topological space with $\rho_2(\tau)$ = R, then $\tau \subseteq \nu(R)$ and $\rho_2(\nu(R))$ = R.*

Proof. For any $U \in \tau$, we have $U = \vee \{\{x_{(l,l')}\}^{\tau} : x_{(l,l')} \leq \mathrm{U}\} = \vee \{\{x_{(l,l')}\}^{R} : x_{(l,l')} \leq \mathrm{U}\}$. Thus $U \in \nu(R)$ and hence $\tau \subseteq \nu(R)$.

Now $\tau \subseteq \nu(R)$, implies that $\rho_2(\nu(R)) \subseteq \rho_2(\tau) = R$.

Let $(x_{(l,l')}, y_{(m,m')}) \in R$ and U be an open set with respect to $\nu(R)$ containing $y_{(m,m')}$. Since $\{\{\widehat{x_{(l,l')}}\}^{R}: x_{(l,l')} \in pt(I^X)\}$ is a base for the kernel topology $\nu(R)$, there is an $z_{(n,n')} \in pt(I^X)$ such that $y_{(m,m')} \leq \widehat{\{z_{(n,n')}\}}^{R} \leq U$. Here $y_{(m,m')} \leq \widehat{\{z_{(n,n')}\}}^{R}$ implies that $(y_{(m,m')}, z_{(n,n')}) \in R$. Thus $(x_{(l,l')}, z_{(n,n')}) \in R$ and hence $x_{(l,l')} \leq \widehat{z_{(n,n')}}\}^{R} \leq U$. Thus every open set containing $y_{(m,m')}$ with respect to $\nu(R)$ contains $x_{(l,l')}$. Hence we get $(x_{(l,l')}, y_{(m,m')}) \in \rho_2(\mu(R))$. Thus $R \subseteq \rho_2(\nu(R))$. Hence $\rho_2(\nu(R)) = R$. □

Theorem 3.4. *Let $(I^X, \mathscr{T})$ be an intuitionistic fuzzy topological space. Then there exists an ordinary topology τ on X such that $\mathscr{T} = \{(\chi_A, 1-\chi_A), A \in \tau\}$ if and only if $\rho_1(\mathscr{T}) = \rho_2(\mathscr{T})$.*

Proof. Suppose that there is an ordinary topology τ on X such that $\mathscr{T} = \{(\chi_A, 1-\chi_A), A \in \tau\}$. Note that if $U \in \mathscr{T}$ then $U = (\chi_A, 1-\chi_A)$, $A \in \tau$. Thus $U \in Q(x_{(l,l')})$ if and only if $x_{(l,l')} \leq U$. Also $U \nleq (y_{(m,m')})', U \in \mathscr{T}$ if and only if $y_{(m,m')} \leq U$.

Thus $(x_{(l,l')}, y_{(m,m')}) \in \rho_1(\mathscr{T}) \iff$ every quasi-neighbourhood of $y_{(m,m')}$ is quasi-coincides with $x_{(l,l')} \iff$ every quasi-neighbourhood U of $y_{(m,m')}$ has the property that $\mu_U(y) \nleq l'$ or $\nu_U(y) \ngeq l \iff$ every quasi-neighbourhood U of $y_{(m,m')}$ has the property that $x_{(l,l')} \leq U \iff$ every $\mathscr{T}$- open set containing $y_{(m,m')}$ contains $x_{(l,l')} \iff (x_{(l,l')}, y_{(m,m')}) \in \rho_2(\mathscr{T})$. Hence $\rho_1(\mathscr{T}) = \rho_2(\mathscr{T})$.

Conversely suppose that $\rho_1(\mathscr{T}) = \rho_2(\mathscr{T})$. Let $U \in \mathscr{T}$ and $x_{(l,l')} \in pt(I^X)$. Then $x_{(l,l')} \leq U$ implies that $(x_{(l,l')}, x_{(l,l')}) \in \rho_2(\mathscr{T}) = \rho_1(\mathscr{T})$. Hence it is clear that for all if $l \neq 0$, $x_{(m,m')} \leq U$, for $m \geq l$ and $m' \leq l'$. Thus for all $x \in X$, $U(x) = \underline{1}$ or $U(x) = \underline{0}$. Thus U =($\chi_A, 1-\chi_A), A \subseteq$

X. Let $\tau = \{A \subseteq X : (\chi_A, 1-\chi_A) \in \mathscr{T}\}$. Then τ is an ordinary topology on X, and $\mathscr{T} = \{(\chi_A, 1-\chi_A), A \in \tau\}$ □

Remark 3.3. Above theorem reveals that the pair of preorders associated to an intuitionistic fuzzy topology we defined in this section is the generalisation of preorder associated to an ordinary topology given by Andima S. J. [11].

Example 3.1. *Let* $X = \{a,\ b\}$ *and* Δ *be the diagonal relation on* $Pt(I^X)$ *and* R *be the preorder on* $Pt(I^X)$ *given by* $R = \Delta \cup \{(a_{(0.4,0.6)}, b_{(0.4,0.6)})\}$. *Then for* $a_{(i,j)} \in Pt(I^X)$, $(i,\ j) \neq (0.4,\ 0.6)$, $\{\overline{a_{(i,j)}}\}^R = a_{(i,j)}$ *and* $(\{\overline{a_{(i,j)}}\}^R)' = a_{(j,i)} \vee b_{(1,0)}$. *Also* $\{\overline{a_{(0.4,0.6)}}\}^R = a_{(0.4,0.6)} \vee b_{(0.4,0.6)}$ *and hence* $(\{\overline{a_{(0.4,0.6)}}\}^R)' = a_{(0.6,0.4)} \vee b_{(0.6,0.4)}$. *Similarly for* $b_{(i,j)} \in Pt(I^X)$, $\{\overline{b_{(i,j)}}\}^R = b_{(i,j)}$ *and* $(\{\overline{b_{(i,j)}}\}^R)' = b_{(j,i)} \vee a_{(1,0)}$. *Since* $\mu(R)$ *is the smallest intuitionistic fuzzy topology on X such that* $a_{(i,j)}$ *with* $(i,\ j) \neq (0.4,\ 0.6)$ *and* $0 \leq i+j \leq 1$, $a_{(0.6,0.4)} \vee b_{(0.6,0.4)}$, $b_{(i,j)}$ *with* $i,\ j \in I$ *and* $0 \leq i+j \leq 1$ *are open it is clear that* $\mu(R) = I^X$.

Now $\{\widehat{a_{(i,j)}}\}^R = a_{(i,j)}$, *for* $i,j \in I$, $0 \leq i+j \leq 1$, $\{\widehat{b_{(i,j)}}\}^R = b_{(i,j)}$ *for* $i,j \in I$, $0 \leq i+j \leq 1$ *with* $(i,\ j) \neq (0.4,\ 0.6)$ *and* $\{\widehat{b_{(0.4,0.6)}}\}^R = a_{(0.4,0.6)} \vee b_{(0.4,0.6)}$. *Thus it is trivial that* $\nu(R) = I^X$. *Also note that*

$\rho_1(\mu(R)) = \{(a_{(i,i')}, a_{(j,j')}) : j \leq i \text{ and } j' \geq i',\ \ i,i',j,j' \in I, 0 \leq i+i' \leq 1,\ 0 \leq j+j' \leq 1\} \cup \{(b_{(k,k')}, b_{(l,l')}) : l \leq k \text{ and } l' \geq k',\ \ k,k',l,l' \in I,\ 0 \leq k+k' \leq 1,\ 0 \leq l+l' \leq 1\}$ *and*

$\rho_2(\nu(R)) = \{(a_{(i,i')}, a_{(j,j')}) : i \leq j \text{ and } i' \geq j',\ \ i,i',j,j' \in I,\ 0 \leq i+i' \leq 1,\ 0 \leq j+j' \leq 1\} \cup \{(b_{(k,k')}, b_{(l,l')}) : k \leq l \text{ and } k' \geq l'\ \ k,k',l,l' \in I,\ 0 \leq k+k' \leq 1,\ 0 \leq l+l' \leq 1\}$

Remark 3.4. From Example3.1, it is clear that for an arbitrary preorder R on $Pt(I^X)$, R need not be comparable with $\rho_1(\mu(R))$ or $\rho_2(\mu(R))$. Also it reveals that if R is an arbitrary preorder on $Pt(I^X)$ and T is an intuitionistic fuzzy topology on X the condition $\mu(R) \subseteq T \subseteq \nu(R)$ not implies $\rho_1(T) = R$ or $\rho_2(T) = R$.

Definition 3.5. Let X be any set and a preorder R on $Pt(I^X)$ is said to be a K-preorder if for each $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$, $(x_{(l,l')}, y_{(m,m')}) \in R$ implies $(x_{(p,p')}, y_{(m,m')}) \in R$ for all $p \leq l$, $p' \geq l', p \neq 0$.

Theorem 3.6. *Let R be a preorder on $Pt(I^X)$. Then there is an intuitionistic fuzzy topology τ on X such that $\rho_2(\tau) = R$ if and only if R is a K-preorder.*

Proof. Let R be a preorder on $Pt(I^X)$ such that there is an intuitionistic fuzzy topology τ on X with $\rho_2(\tau) = R$. Then for $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$, $(x_{(l,l')}, y_{(m,m')}) \in R \Rightarrow$ every τ-open set containing $y_{(m,m')}$ contains $x_{(l,l')}$ and hence every τ-open set containing $y_{(m,m')}$ contains $x_{(p,p')}$, for all $p \leq l, p' \geq l', p \neq 0$. That is $(x_{(p,p')}, y_{(m,m')}) \in R$ for all $p \leq l, p' \geq l', p \neq 0$. Thus R is a K-preorder on $Pt(I^X)$.

Conversely suppose that R is a K-preorder on $Pt(I^X)$. Then for $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$, $(x_{(l,l')}, y_{(m,m')}) \in \rho_2(\nu(R)) \Rightarrow x_{(l,l')} \leq \{\widehat{y_{(m,m')}}\}^R$, $(x_{(p,p')}, y_{(m,m')}) \in R$ for some $p \geq l, p' \leq l', p \neq 0$. Since R is a K-preorder $(x_{(l,l')}, y_{(m,m')}) \in R$. Thus we have $\rho_2(\nu(R)) \subseteq R$. Now let $(x_{(l,l')}, y_{(m,m')}) \in R$. Let G be a $\nu(R)$-open set containing $y_{(m,m')}$. Then we can write $G = \vee\{\{\widehat{z_{(q,q')}}\}^R : z_{(q,q')} \leq G\}$. Since $(x_{(l,l')}, y_{(m,m')}) \in R$, it is clear that $x_{(l,l')} \leq \{\widehat{y_{(m,m')}}\} \leq G$. Thus every $\nu(R)$-open set containing $y_{(m,m')}$ contains $x_{(l,l')}$. Hence $(x_{(l,l')}, y_{(m,m')}) \in \rho_2(\nu(R))$. Thus $R \subseteq \rho_2(\nu(R))$. Hence $\rho_2(\nu(R)) = R$. □

Definition 3.7. Let X be any set. A preorder R on $Pt(I^X)$ is said to be C-preorder if for each $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$, $(x_{(l,l')}, y_{(p,p')}) \in R$ for all $p \leq m,\ p' \geq m',\ p \neq 0$ whenever $(x_{(l,l')}, y_{(m,m')}) \in R$.

Theorem 3.8. *Let R be a preoder on $Pt(I^X)$. Then there is an intuitionistic fuzzy topolgy τ on X such that $\rho_1(\tau) = R$ if and only if R is a C-preorder.*

Proof. Let R be a preorder on $Pt(I^X)$ such that there is an intuitionistic fuzzy topology τ on X with $\rho_1(\tau) = R$. Then for all $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$, $(x_{(l,l')}, y_{(m,m')}) \in R \Rightarrow$ every quasi-neighbourhood of $y_{(m,m')}$ with respect to τ quasi-coincides with $x_{(l,l')}$. As every quasi-neighbourhood of $y_{(p,\ p')}$, $p \leq m,\ p' \geq m', p \neq 0$ is a quasi-neighbourhood of $y_{(m,m')}$, every quasi-neighbourhood of $y_{(p,p')}$ with respect to τ quasi-coincides with $x_{(l,l')}$.

Thus $(x_{(l,l')}, y_{(p,\ p')}) \in R$ for all $p \leq m,\ p' \geq m', p \neq 0$. Thus R is a C-preorder .

Conversely suppose that R is a C-preorder on $Pt(I^X)$. Let $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$ with $(x_{(l,l')}, y_{(m,m')}) \in \rho_1(\mu(R))$. Then every quasi-neighbourhood of $y_{(m,m')}$ with respect to $\mu(R)$ quasi-coincides with $x_{(l,l')}$. Let $G = \{\overline{x_{(l,l')}}\}^R$ then $G' \in \mu(R)$. Thus if $y_{(m,m')} \not\leq G$ then $m \not\leq \mu_G(y)$ or

$m' \ngeq \nu_G(y)$. That is G' is a quasi-neighbourhood of $y_{(m,m')}$ with respect to $\mu(R)$. Thus G' quasi-coincides with $x_{(l,l')}$. That is $\nu_G(x) \nleq l'$ or $\mu_(x) \ngeq l$. That is $G' \nleq (x_{(l,l')})'$. As $G = \{\overline{x_{(l,l')}}\}^R$, $x_{(l,l')} \leq G$ and hence $G' \leq (x_{(l,l')})'$, which is a contradiction. Thus $y_{(m,m')} \leq G = \{\overline{x_{(l,l')}}\}^R$. Hence $(x_{(l,l')}, y_{(p,p')}) \in R$ for some $p \geq m, p' \leq m', p \neq 0$. Since R is a C-preorder $(x_{(l,l')}, y_{(m,m')}) \in R$. Thus $\rho_1(\mu(R)) \subseteq R$. Now for $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$ assume that $(x_{(l,l')}, y_{(m,m')}) \in R$. Then $y_{(m,m')} \leq \{\overline{x_{(l,l')}}\}^R$. let U be a quasi-neighbourhood of $y_{(m,m')}$ with respect to $\mu(R)$. If $x_{(l,l')} \leq U'$, then $\{\overline{x_{(l,l')}}\}^R \leq U'$ and hence $y_{(m,m')} \leq U'$, which is a contradiction. Thus $x_{(l,l')} \nleq U'$. That is $l \nleq \nu_U$ or $l' \ngeq \mu_U$. That is, every quasi-neighbourhood of $y_{(m,m')}$ with respect to $\mu(R)$ quasi-coincides with $x_{(l,l')}$. Hence $(x_{(l,l')}, y_{(m,m')}) \in \rho_1(\mu(R))$. Thus $R \subseteq \rho_1(\mu(R))$. Hence $\rho_1(\mu(R)) = R$. □

Remark 3.5. If R is a K-preorder on $Pt(I^X)$, then for any $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$ $(x_{(l,l')},\ y_{(p,p')}) \in R$ for all $p \geq m, p' \leq m', p \neq 0$, whenever $(x_{(l,l')}, y_{(m,m')}) \in R$. Similarly if R is a C-preorder on $Pt(I^X)$ then for any $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$, $(x_{(p,p')}, y_{(m,m')}) \in R$ for all $p \geq l, p' \leq l', p \neq 0$, whenever $(x_{(l,l')}, y_{(m,m')}) \in R$.

Remark 3.6. If R is both a C-preorder and a K-preorder on $Pt(I^X)$, then it is equivalent to a preorder on X, as R has the property that for any $x_{(l,l')}, y_{(m,m')} \in Pt(I^X)$, $(x_{(l,l')}, y_{(m,m')}) \in R$ implies that $(x_{(p,p')}, y_{(q,q')}) \in R$ for all $x_{(p,p')}, y_{(q,q')} \in Pt(I^X)$.

Proposition 3.1. Let H be the collection of all K-preorders on $Pt(I^X)$, then ν is order reversing on H.

Proof. Let R_1 and R_2 belong to H with $R_1 \subseteq R_2$ and $G \in \nu(R_2)$. If $y_{(m,m')} \leq \{\widehat{x_{(l,l')}}\}^{R_1}$ then $(y_{(m,m')}, x_{(l,l')}) \in R_1$ and hence $(y_{(m,m')}, x_{(l,l')}) \in R_2$, since $R_1 \subseteq R_2$ and consequently $y_{(m,m')} \leq \{\widehat{x_{(l,l')}}\}^{R_2}$. Thus $\{\widehat{x_{(l,l')}}\}^{R_1} \leq \{\widehat{x_{(l,l')}}\}^{R_2}$ for all $x_{(l,l')} \in Pt(I^X)$ and hence $G = \vee\{\{\widehat{x_{(l,l')}}\}^{R_2} : x_{(l,l')} \leq G\} \geq \vee\{\{\widehat{x_{(l,l')}}\}^{R_1} : x_{(l,l')} \leq G\} \geq G$. Therefore $G \in \nu(R_1)$ and hence $\nu(R_2) \subseteq \mu(R_1)$ □

Remark 3.7. Let M be the collection of all C-preorders on $Pt(I^X)$ then μ is neither order preserving nor order reversing.

Following example illustrates this.

Example 3.2. *Let* $X = (-\omega)\cup\{a,b\}$ *where* $-\omega = \{0,-1,-2,-3,........\}$ *and a and b are two points not in* $(-\omega)$. *Let us define an ordering S on* $(-\omega)\times\,[1,0]$ *by,* $S = \{(x_{(l,l')}, y_{(m,m',)}) : x,y \in -\omega,\ x \leq y,\ 0 < l+l' \leq 1, 0 < m+m' \leq 1, 0 < l \leq 1,\ 0 < l' \leq 1\}$, *where* $\leq$ *is the usual ordering on* $(-\omega)$. *Define* $R = S\cup\{(a_{(p,p')}, a_{(q,q')}) : 0 < p+p' \leq 1, 0 < q+q' \leq 1,\ 0 < p \leq 1,\ 0 < q \leq 1\}\cup\{(a_{(p,p')}, b_{(q,q')}) : 0 < p+p' \leq 1, 0 < q+q' \leq 1,\ 0 < p \leq 1,\ 0 < q \leq 1\}\cup\{(b_{(p,p')}, b_{(q,q')}) : 0 < p+p' \leq 1, 0 < q+q' \leq 1,\ 0 < p \leq 1,\ 0 < q \leq 1\}\cup\Delta$ *and* $R^* = R\cup\{(a_{(p,p')}, (-n)_{(r,r')}) : n \in \omega, 0 \leq p+p' \leq 1, 0 \leq r+r' \leq 1,\ 0 < p \leq 1,\ 0 < r \leq 1\}\cup\{(b_{(q,q')}, (-n)_{(r,r')}) : n \in \omega, 0 < q+q' \leq 1, 0 < r+r' \leq 1,\ 0 < q \leq 1,\ 0 < r \leq 1\}\cup\{((-n)_{(p,p')}, (-n)_{(r,r')}) : n \in \omega, 0 \leq p+p' \leq 1, 0 \leq r+r' \leq 1,\ 0 < p \leq 1,\ \ 0 < r \leq 1\}$, *where* Δ *is the diagonal relation on* $Pt(I^X)$. *Note that R and* R^* *are C-preorders on* $Pt(I^X)$. *Then there exists two intuitionistic fuzzy topologies* τ_1 *and* τ_2 *on X such that* $\rho_1(\tau_1) = R$ *and* $\rho_1(\tau_2) = R^*$. *Now* $\{\overline{b_{(l,l')}}\}^{\mu(R)} = (\chi_{\{b\}}, 1-\chi_{\{b\}})$ *and* $\{\overline{b_{(l,l')}}\}^{\mu(R^*)} = (\chi_{\{b\}\cup(-\omega)}, 1-\chi_{\{b\}\cup(-\omega)})$. *Thus* $b_{(1,0)}$ *is closed with respect to* $\mu(R)$ *but not closed with respect to* $\mu(R^*)$. *Let* $A = \{\overline{b_{(l,l')}}\}^{\mu(R^*)} = (\chi_{\{b\}\cup(-\omega)}, 1-\chi_{\{b\}\cup(-\omega)})$. *Then A is closed in* $\mu(R^*)$. *Note that* $A' = a_{(1,0)}$. *Now for any* $n \in \omega$, $\{\overline{(-n)_{(r,r')}}\}^R = B_m$, *where,* $B_m = \vee\{(-m)_{(s,s')} : -m \in (-\omega),\ (-n) \leq (-m), 0 < s+s' \leq 1,\ 0 < s \leq 1\} = (\chi_{A_n},\ 1-\chi_{A_n})$, *where* $A_n = \{-m \in (-\omega) : -n \leq -m\}$. *Therefore* $(\{\overline{(-n)_{(r,r')}}\}^R)' = C\vee a_{(1,0)}\vee b_{(1,0)}$, *where* $C = (\chi_{D_n},\ 1-\chi_{D_n})$, $.D_n = \{-m \in (-\omega) : -m < -n\}$. *Also since* $\{\overline{a_{(1,0)}}\}^R = a_{(1,0)}\vee b_{(1,0)}$ *implies that* $(\{\overline{a_{(p,p')}}\}^R)' = (\chi_{(-\omega)},\ 1-\chi_{(-\omega)})$ *and* $(\{\overline{b_{(q,q')}}\}^R)' = (\chi_{(-\omega)\cup\{a\}},\ 1-\chi_{(-\omega)\cup\{a\}})$. *Thus it is clear that* A' *cannot be expressed as a join of finite meets of the family* $\{(\{\overline{x_{l,l')}}\}^R)' : x \in X, 0 < l+l' \leq 1\}\cup\{\underline{1}\}$. *Therefore A is not closed with respect to* $\mu(R)$ *and hence* $\mu(R)$ *and* $\mu(R^*)$ *are not comparable.*

Theorem 3.9. *Let R be a preorder on* $Pt(I^X)$ *such that for* $x,y \in X$ *if,* $(x_{(l,l')}, y_{(m,m')}) \in R$ *then* $(x_{(p,p')}, y_{(q,q')}) \in R$ *for all* $p,q \in I-\{0\}$ *[that is, R is both C-preoder and K-preorder]. Then* $\mu(R) = \nu(R)$ *if and only if for all* $x_{(l,l')} \in Pt(I^X)$, $\{\widehat{x_{(l,l')}}\}'$ *is the join of finite number of point closures.*

Proof. Assume that $\mu(R) = \nu(R)$. For $a_{(l,l')} \in Pt(I^X)$, if $\{\widehat{a_{(l,l')}}\}' = \underline{0}$, then it is the join of an empty collection of point closures. Suppose that $a_{(l,l')} \in Pt(I^X)$ and $\{\widehat{a_{(l,l')}}\}' \neq \underline{0}$. Since $\{\widehat{a_{(l,l')}}\}'$ is closed in $\mu(R) = \nu(R)$, $\{\widehat{a_{(l,l')}}\}'$ is the meet of finite join of point closures. That is $\{\widehat{a_{(l,l')}}\}' = \wedge\{F_\alpha : \alpha \in \mathscr{A}\}$, where $\mathscr{A}$ is an index set and for each $\alpha \in \mathscr{A}$, F_α is a finite join of point closures. From the property of R it is clear that $a_{(1,0)} \leq \{\widehat{a_{(l,l')}}\}$ and hence $a_{(k,k')} \not\leq \{\widehat{a_{(l,l')}}\}'$ for all $k \in I - \{0\}$. Now $a_{(k,k')} \not\leq \{\widehat{a_{(l,l')}}\}'$ implies that there is a $\beta_k \in \mathscr{A}$ such that $a_{(k,k')} \not\leq F_{\beta_k}$. Let $F_{\beta_k} = \vee_{i=1}^{n}\{\overline{x_{i(l_i,l'_i)}}\}$. Then $b_{(m,m')} \leq F_{\beta_k}$ implies that there is a subcollection $\{i_1, i_2, i_3, ...\}$ of $\{1,2,3,.......\}$ such that $b_{(m,m')} \leq b_{(l_{i_1},l'_{i_1})} \vee b_{(l_{i_2},l'_{i_2})} \vee ..... \vee b_{(l_{i_m},l'_{i_m})}$ where $b_{(l_{i_j},l'_{i_j})} \leq \{\overline{x_{i_j(l_{i_j},l'_{i_j})}}\}$ for $j = 1,2,.....m$. Now $b_{(l_{i_j},l'_{i_j})} \leq \{\overline{x_{i_j(l_{i_j},l'_{i_j})}}\}$ implies that $b_{(l_{i_j},l'_{i_j})} \leq \vee_k b_{(l_{i_{j_k}},l'_{i_{j_k}})}$ where $(x_{i_j(l_{i_j},l'_{i_j})}, b_{(l_{i_{j_k}},l'_{i_{j_k}})}) \in R$ for all k. Then by the property of R $(x_{i_j(l_{i_j},l'_{i_j})}, \vee_k b_{(l_{i_{j_k}},l'_{i_{j_k}})}) \in R$ and hence $(x_{i_j(l_{i_j},l'_{i_j})}, b_{(l_{i_j},l'_{i_j})}) \in R$. Thus $(x_{i_j(l_{i_j},l'_{i_j})}, b_{(m,m')}) \in R$ for $j = 1,2,3,.......m$. Take $q = i_j$for some $j \in \{1,2,3,.......m\}$. Then $b_{(m,m')} \leq \{\overline{x_{q_{(l_q,l'_q)}}}\} \leq F_{\beta_k}$ and $(x_{q_{(l_q,l'_q)}}, b_{(m,m')}) \in R$. If $(b_{(m,m')}, a_{(k,k')}) \in R$, then $(x_{q_{(l_q,l'_q)}}, a_{(k,k')}) \in R$ and hence $a_{(k,k')} \leq \{\overline{x_{q_{(l_q,l'_q)}}}\} \leq F_{\beta_k}$ a contradiction. Thus $b_{(m,m')} \leq b_{(1,0)} \leq \{\widehat{a_{(l,l')}}\}'$. Since $b_{(m,m')}$ is arbitrary we have $F_{\beta_k} \leq \{\widehat{a_{(l,l')}}\}'$. Also it is clear that $\{\widehat{a_{(l,l')}}\}' \leq F_{\beta_k}$. Thus $F_{\beta_k} = \{\widehat{a_{(l,l')}}\}'$.

Conversely suppose that for all $x_{(l,l')} \in Pt(I^X)$, $\{\widehat{x_{(l,l')}}\}'$ is the join of finite number of point closures. Then $\{\widehat{x_{(l,l')}}\}'$ is closed in $\mu(R)$ and hence $\{\widehat{x_{(l,l')}}\}$ is open in $\mu(R)$. Thus $\nu(R) \subseteq \mu(R)$. Now $R^* = \{(x,y) : (x_{(1,0)}, y_{(1,0)}) \in R)\}$ is a preorder in X. Then there is an ordinary topology T on X such that $\rho(T) = R^*$ [11]. Let $\tau = \{(\chi_A, 1-\chi_A) : A \in T\}$. Then by theorem 3.4 $\rho_1(\tau) = \rho_2(\tau)$. Hence from Theorem 3.2 and Theorem 3.3 $\rho_1(\mu(R)) = R = \rho_2(\nu(R))$ and $\mu(R) \subseteq \nu(R)$. Thus $\nu(R) = \mu(R)$. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A.K. Steiner, The lattice of topologies: structure and complementation, Trans. Amer. Math. Soc. 122 (1966), 379-398.

[2] C.L. Chang, Fuzzy topological space, J. Math. Anal. Appl. 24 (1968), 182-190,.

[3] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets Syst. 88 (1997), 81–89.

[4] D. Coker and A. H. Es: On fuzzy compactness in intuitionistic fuzzy topological spaces. J. Fuzzy Math. 3 (1995), 899–909.

[5] J. Goguen, L-fuzzy sets. J. Math. Anal. Appl. 18 (1967), 145–174.

[6] K.T. Atanassov, Intuitionistic Fuzzy Sets, in: Intuitionistic Fuzzy Sets, Physica-Verlag HD, Heidelberg, 1999: pp. 1–137.

[7] K.T. Atanassov, Intuitionistic fuzzy sets. In Proceedings of the VII ITKR's Session, Sofia, Bulgaria, 7–9 June 1983 (Deposed in Central Sci.- Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgaria); reprinted in Int. J. Bioautom. 20 (2016), S1–S6.

[8] L.A. Zadeh, Fuzzy sets, Inform. Control. 8 (1965), 338-353.

[9] N.M.M. Namboothiri, Order induced L-fuzzy topological properties, Adv. Fuzzy Sets Syst. 9 (2011), 167-197.

[10] P.M. Dhanya, Some problems on generalised topologies and fuzzy generalised topologies, Ph.D Thesis, Department of Mathematics, University of Calicut, (2016).

[11] S. Andima, W.J. Thron, Order-induced topological properties, Pac. J. Math. 75 (1978), 297–318.

[12] Z. Pawlak, Rough Sets Research Report PAS 431, Institute of Computer Science, Polish Academy of Sciences, Warsaw, Poland, 1981.

[13] Z. Pawlak, Rough sets, Int. J. Computer Inform. Sci. 11 (1982), 341–356.