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# ON MAXIMUM MODULUS OF POLAR DERIVATIVE OF A POLYNOMIAL

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**Abstract.** Let $p(z)$ be a polynomial of degree $n$ and for any real or complex number $\alpha$, $D_\alpha p(z)$ denotes the polar derivative of $p(z)$ with respect to $\alpha$, then

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z)\,.$$

In this paper, we consider the more general class of polynomials $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, not vanishing in $|z| < k$, $k > 0$, to estimate $\max_{|z|=\rho} |D_\alpha p(z)|$ in terms of $\max_{|z|=r} |p(z)|$ by involving some coefficients of $p(z)$, where $0 < r \le \rho \le k$. Interestingly, the results improve and extend other well known inequalities to polar derivative. Moreover, our results give several interesting results as special cases.

**Keywords:** Bernstein's inequality; maximum modulus; polynomial; polar derivative.

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## 1. Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree $n$ and let

$$M(p,t) = \max_{|z|=t} |p(z)|, \quad \text{and} \quad m(p,t) = \min_{|z|=t} |p(z)|.$$

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Concerning the estimate of $\max|p'(z)|$ in terms of $\max|p(z)|$ on the unit circle $|z| = 1$, we have

$$M(p',1) \leq nM(p,1)\,. \tag{1.1}$$

Inequality (1.1) is the famous result known as Bernstein's inequality [17] and equality holds if and only if $p(z)$ has all its zeros at the origin.

If we restrict ourselves to the class of polynomials not vanishing in $|z| < 1$, then Erdös conjectured and Lax [13] proved

$$M(p',1) \leq \frac{n}{2}M(p,1). \tag{1.2}$$

Inequality (1.2) is best possible for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

As a generalization of inequality (1.2), Malik [14] proved that if $p(z)$ is a polynomial of degree $n$ not vanishing in $|z| < k$, $k \geq 1$, then

$$M(p',1) \leq \frac{n}{1+k}M(p,1). \tag{1.3}$$

The result is best possible and equality holds for $p(z) = (z+k)^n$.

Further, in seeking generalization of (1.3) for the same class of polynomials, Bidkham and Dewan [6] proved for $1 \leq \rho \leq k$,

$$M(p',\rho) \leq \frac{n(\rho+k)^{n-1}}{(1+k)^n}M(p,1). \tag{1.4}$$

Dewan and Mir [8] improved as well as generalized inequality (1.4) by proving

*Theorem* A. If $p(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n$ having no zero in $|z| < k$, $k \geq 1$, then for $0 < r \leq \rho \leq k$,

$$\begin{aligned} M(p',\rho) \leq \frac{n(k+\rho)^{n-1}}{(k+r)^n} & \left\{1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2+\rho^2)n|a_0| + 2k^2\rho|a_1|}\right. \\ & \left. \times \left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\}M(p,r)\,. \end{aligned} \tag{1.5}$$

Further, Mir et al. [15] recently generalized and improved Theorem A by establishing

*Theorem* B. If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\le\mu\le n$, is a polynomial of degree $n$ having no zero in $|z|<k$, $k>0$, then for $0<r\le\rho\le k$,

$$M(p',\rho)\le\frac{n\rho^{\mu-1}(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+r^\mu)^{\frac{n}{\mu}}}\left[\left\{1-\frac{k^\mu(k-\rho)nR}{\mu S}\left(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\right)\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}-1}\right\}\times\right.$$
$$\left.M(p,r)-\frac{(k^\mu+r^\mu)^{\frac{n}{\mu}+1}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}}}\left\{\frac{T}{S}\left(\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1-\frac{n}{\mu}\left(\frac{\rho^\mu-r^\mu}{k^\mu+r^\mu}\right)\right)m(p,k)\right\}\right],\tag{1.6}$$

where here and throughout this paper, $R=(n|a_0|-k^\mu\mu|a_\mu|)$, $S=\{n|a_0|(\rho^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)\}$ and $T=(n|a_0|\rho+\mu|a_\mu|k^{\mu+1})$.

Let $D_\alpha p(z)$ denotes the polar derivative of a polynomial $p(z)$ of degree $n$ with respect to a real or complex number $\alpha$, then

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z).$$

The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p'(z)\,.$$

Aziz [1] extended Theorem A to the polar derivative of $p(z)$ by proving

*Theorem* C. If $p(z)=\sum\limits_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree $n$ having no zero in $|z|<k$, $k\ge1$, then for every real or complex number $\alpha$, with $|\alpha|\ge1$,

$$M(D_\alpha p,1)\le n\left(\frac{k+|\alpha|}{1+k}\right)M(p,1)\,.\tag{1.7}$$

Inequality (1.7) is best possible for $p(z)=(z+k)^n$ with a real number $\alpha\ge1$ and $k\ge1$.

In this paper, we consider the more general class of polynomials $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\le\mu\le n$, not vanishing in $|z|<k$, $k>0$, with $0<r\le\rho\le k$, and find estimates of the maximum modulus of the polar derivative of $p(z)$ on the circle $|z|=\rho$ in terms of $M(p,r)$ by involving some of the coefficients of $p(z)$. In fact, we first prove

**Theorem 1.1.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, then for every real or complex number $\alpha$ with $|\alpha| \ge \rho$ and $0 < r \le \rho \le k$,*

$$M(D_\alpha p, \rho) \le n \frac{(k^\mu + \rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}} (k^\mu + |\alpha| \rho^{\mu-1}) \times \left[1 - \frac{k^\mu (k-\rho) R}{S} \left\{1 - \left(\frac{k^\mu + r^\mu}{k^\mu + \rho^\mu}\right)^{\frac{n}{\mu}}\right\}\right] M(p,r). \tag{1.8}$$

*For $\mu = 1$, the result is best possible with $\alpha > 0$ and the extremal polynomial is $p(z) = (z+k)^n$.*

*Remark* 1.2. From Lemma 2.7, we have

$$1 - \left(\frac{k^\mu + r^\mu}{k^\mu + \rho^\mu}\right)^{\frac{n}{\mu}} \ge \frac{n}{\mu} \left(\frac{\rho^\mu - r^\mu}{k^\mu + \rho^\mu}\right) \left(\frac{k^\mu + r^\mu}{k^\mu + \rho^\mu}\right)^{\frac{n}{\mu}-1},$$

and hence under the same hypotheses of Theorem 1.1, we have the following result which gives a generalised extension of Theorem A to polar derivative.

$$M(D_\alpha p, \rho) \le n \frac{(k^\mu + \rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}} (k^\mu + |\alpha| \rho^{\mu-1}) \times \left[1 - \frac{k^\mu (k-\rho) n R}{\mu S} \left(\frac{\rho^\mu - r^\mu}{k^\mu + \rho^\mu}\right) \left(\frac{k^\mu + r^\mu}{k^\mu + \rho^\mu}\right)^{\frac{n}{\mu}-1}\right] M(p,r). \tag{1.9}$$

Further, for $r = \rho = 1$, inequality (1.8) of Theorem 1.1 reduces to inequality (2.17) of Lemma 2.10, which gives an extension of (2.1) due to Chan and Malik [7] to polar derivative.

For $\mu = 1$, and $r = \rho = 1$, Theorem 1.1 reduces to Theorem C.

*Remark* 1.3. If we divide both sides of (1.8) by $|\alpha|$ and make $|\alpha| \to \infty$, we have the following interesting generalization as well as an improvement of Theorem A.

**Corollary 1.4.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, then for $0 < r \le \rho \le k$,*

$$M(p', \rho) \le n \rho^{\mu-1} \frac{(k^\mu + \rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}} \times \left[1 - \frac{k^\mu (k-\rho) R}{S} \left\{1 - \left(\frac{k^\mu + r^\mu}{k^\mu + \rho^\mu}\right)^{\frac{n}{\mu}}\right\}\right] M(p,r). \tag{1.10}$$

*Inequality* (1.10) *is best possible for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where $n$ is a multiple of $\mu$.*

As mentioned earlier, when $\mu = 1$, inequality (1.8) improves upon (1.5) of Theorem A. Because by Lemma 2.7, we have

$$1-\left(\frac{k+r}{k+\rho}\right)^{n} \geq n\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}. \tag{1.11}$$

Corollary 1.4 gives an improvement of a result due to Aziz and Zargar [4]. Again, for $r = \rho = 1$, it reduces to a result of Chan and Malik [7]. Also, when $\mu = 1$ and $r = 1$, Corollary 1.4 improves upon inequality (1.4). Further, it reduces to inequality (1.3) due to Malik [14] for $\mu = 1$ and $r = \rho = 1$.

*Remark* 1.5. Dividing both sides of inequality (1.9) by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain the following generalization of Theorem A.

**Corollary 1.6.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, and for $0 < r \leq \rho \leq k$,*

$$\begin{aligned}M(p',\rho) \leq\ & n\rho^{\mu-1}\frac{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+r^\mu)^{\frac{n}{\mu}}}\\ &\times\left[1-\frac{k^\mu(k-\rho)nR}{\mu S}\left(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\right)\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}-1}\right]M(p,r).\end{aligned} \tag{1.12}$$

*This result is best possible and equality holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where $n$ is a multiple of $\mu$.*

Next, under the same set of hypotheses of Theorem 1.1, it is of interest to find better bound than that of Theorem 1.1, by involving $m(p,k) = \min_{|z|=k} |p(z)|$. In this context, more precisely, we are able to prove the following significant result, which gives improved extension of Theorem B to polar derivative.

**Theorem 1.7.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, then for every real or complex number $\alpha$ with $|\alpha| \ge \rho$ and $0 < r \le \rho \le k$,*

$$
\begin{aligned}
M(D_\alpha p, \rho) \le\; & n\frac{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+r^\mu)^{\frac{n}{\mu}}}(k^\mu+|\alpha|\rho^{\mu-1}) \\
& \times \left[\left\{1-\frac{k^\mu(k-\rho)R}{S}\left(1-\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}}\right)\right\}M(p,r)-\frac{T}{S}\frac{(k^\mu+r^\mu)^{\frac{n}{\mu}}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}\right. \\
& \times \left.\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1-\frac{n}{\mu}\log\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)\right\}m(p,k)\right] \\
& -\frac{n(|\alpha|\rho^{\mu-1}-\rho^\mu)}{(k^\mu+\rho^\mu)}m(p,k). \qquad (1.13)
\end{aligned}
$$

*For $\mu = 1$, the result is best possible with $\alpha > 0$ and the extremal polynomial is $p(z) = (z+k)^n$.*

*Remark* 1.8. When $r = \rho = 1$, inequality (1.13) of Theorem 1.7 reduces to (2.18) of Lemma 2.10 due to Dewan and Singh [10].

Further, Theorem 1.7 gives an improvement of a result proved by Dewan and Singh [10, Theorem 2]. Moreover, when $\mu = 1$, and $r = \rho = 1$, Theorem 1.7 reduces to a result of Aziz and Shah [3], which improves upon Theorem C.

Dividing both sides of (1.13) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following interesting result, which improves upon Theorem B.

**Corollary 1.9.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, then for $0 < r \le \rho \le k$,*

$$
\begin{aligned}
M(p', \rho) \le\; & n\rho^{\mu-1}\frac{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+r^\mu)^{\frac{n}{\mu}}} \\
& \times \left[\left\{1-\frac{k^\mu(k-\rho)R}{S}\left(1-\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}}\right)\right\}M(p,r)-\frac{T}{S}\frac{(k^\mu+r^\mu)^{\frac{n}{\mu}}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}\right. \\
& \times \left.\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1-\frac{n}{\mu}\log\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)\right\}m(p,k)\right] \\
& -\frac{n\rho^{\mu-1}}{(k^\mu+\rho^\mu)}m(p,k). \qquad (1.14)
\end{aligned}
$$

*This result is best possible and equality occurs for* $p(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$, *where* $n$ *is a multiple of* $\mu$.

*Remark* 1.10. As mentioned earlier, Corollary 1.9 gives a sharper bound than that of Theorem B, because of the following facts:

(i) By Lemma 2.7, $1-\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}} \geq \frac{n}{\mu}\left(\frac{\rho^\mu-k^\mu}{k^\mu+\rho^\mu}\right)\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}-1}$.

(ii) $$(k^\mu+\rho^\mu)\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1-\frac{n}{\mu}\log\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)\right\}$$
$$\geq (k^\mu+r^\mu)\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1-\frac{n}{\mu}\log\left(\frac{\rho^\mu-r^\mu}{k^\mu+r^\mu}\right)\right\},$$
which is true by Lemma 2.10 and Lemma 2.12, and

(iii) The involvement of $-\frac{n\rho^{\mu-1}}{k^\mu}m(p,k)$ as a surplus term.

*Remark* 1.11. If we use the first two facts of Remark 1.10 to Theorem 1.7, we get the following result, which still gives an improved extension of Theorem B to polar derivative.

**Corollary 1.12.** *Under the same set of hypotheses of Theorem* 1.7, *we have*

$$\begin{aligned} M(D_\alpha,\rho) \leq{}& n\frac{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+r^\mu)^{\frac{n}{\mu}}}(k^\mu+|\alpha|\rho^{\mu-1}) \\ &\times\left[\left\{1-\frac{k^\mu(k-\rho)nR}{\mu S}\left(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\right)\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}-1}\right\}M(p,r)\right. \\ &\left.-\frac{(k^\mu+r^\mu)^{\frac{n}{\mu}+1}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}}}\left\{\frac{T}{S}\left(\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1-\frac{n}{\mu}\log\left(\frac{\rho^\mu-r^\mu}{k^\mu+r^\mu}\right)\right)\right\}m(p,k)\right] \\ &-\frac{n(|\alpha|\rho^{\mu-1}-\rho^\mu)}{(k^\mu+\rho^\mu)}m(p,k)\,. \end{aligned} \tag{1.15}$$

*This inequality is best possible for* $\mu=1$ *with* $\alpha>0$ *and the extremal polynomial being* $p(z)=(z+k)^n$, $k>0$.

For $\mu=1$, Corollary 1.12 gives an improved extension to polar derivative of a result due to Aziz and Zargar [5]. The bound of Corollary 1.12 improves upon a result due to Dewan and Singh [10, Corollary 1]. Further for $\mu=1$, it gives an improvement of inequality (1.5) of Theorem A.

## 2. Lemmas

The following lemmas are needed for the proofs of the theorems.

**Lemma 2.1.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$M(p',1) \le \frac{n}{1+k^\mu} M(p,1). \tag{2.1}$$

The above result is due to Chan and Malik [7].

**Lemma 2.2.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$M(p',1) \le \frac{n}{1+k^\mu} \{M(p,1) - m(p,k)\}. \tag{2.2}$$

This result was proved by Aziz and Shah [2] (see also [11]).

**Lemma 2.3.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$M(p',1) \le n \left\{ \frac{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{\mu+1} + k^{2\mu})} \right\} M(p,1) \tag{2.3}$$

*and*

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^\mu \le 1\,. \tag{2.4}$$

These results are due to Qazi [16].

**Lemma 2.4.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \le \rho \le k$,*

$$M(p,r) \ge \left(\frac{r^\mu + k^\mu}{\rho^\mu + k^\mu}\right)^{\frac{n}{\mu}} M(p,\rho), \tag{2.5}$$

*and*

$$M(p,r) \ge \left(\frac{r^\mu + k^\mu}{\rho^\mu + k^\mu}\right)^{\frac{n}{\mu}} M(p,\rho) + \left\{1 - \left(\frac{r^\mu + k^\mu}{\rho^\mu + k^\mu}\right)^{\frac{n}{\mu}}\right\} m(p,k). \tag{2.6}$$

Inequality (2.5) is due to Jain [12] and (2.6) was proved by Dewan et al. [9].

We are interested to prove the following lemma concerning the validity of inequality (2.4) of Lemma 2.3 for any $k>0$, because this has subsequent uses.

**Lemma 2.5.** *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\le\mu\le n$, is a polynomial of degree $n$ having no zero in $|z|<k$, $k>0$, then*

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu\le 1\,. \tag{2.7}$$

**Proof of Lemma 2.5.** Since $p(z)\neq 0$ in $|z|<k$, $k>0$, the polynomial $P(z)=p(tz)\neq 0$ in $|z|<\frac{k}{t}$, $\frac{k}{t}\ge 1$ where $0<t\le k$. Hence applying inequality (2.4) of Lemma 2.3 to $P(z)$, we get

$$\frac{\mu}{n}\frac{|a_\mu|t^\mu}{|a_0|}\left(\frac{k}{t}\right)^\mu\le 1\,,$$

and which proves the lemma. □

**Lemma 2.6.** *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\le\mu\le n$, is a polynomial of degree $n$ having no zero in $|z|<k$, $k>0$, then the function*

$$f(t)=\frac{(n|a_0|t+\mu|a_\mu|k^{\mu+1})(t^\mu+k^\mu)}{(t^{\mu+1}+k^{\mu+1})n|a_0|+\mu|a_\mu|(k^{\mu+1}t^\mu+k^{2\mu}t)} \tag{2.8}$$

*is a non-decreasing function of $t$ in $(0,k]$.*

**Proof of Lemma 2.6.** We prove this by derivative test. Now, we have

$$\begin{aligned}f'(t)=&\frac{(n|a_0|-\mu|a_\mu|k^\mu)}{\{(t^{\mu+1}+k^{\mu+1})n|a_0|+\mu|a_\mu|(k^{\mu+1}t^\mu+k^{2\mu}t)\}^2}\\&\times\{(k-t)t^{\mu-1}\mu k^\mu(n|a_0|t+\mu|a_\mu|k^{\mu+1})+(n|a_0|+\mu|a_\mu|k^\mu)(k^{\mu+1}t^\mu+k^{2\mu+1})\},\end{aligned}$$

which is non-negative, since by Lemma 2.5, $(n|a_0|-\mu|a_\mu|k^\mu)\ge 0$, and the fact that $t\le k$. □

**Lemma 2.7.** *For $x\in(0,1]$ and any $r\ge 1$, we have*

$$1-x^r\ge r(1-x)x^{r-1}\,. \tag{2.9}$$

**Proof of Lemma 2.7.** For $r=1$, the result follows trivially. Hence to prove Lemma 2.7, it is sufficient to show that for $r>1$,

$$(1-r)x^r+rx^{r-1}\le 1\,. \tag{2.10}$$

Suppose $f(x) = (1-r)x^r + rx^{r-1}$.

Now, $f'(x) = r(r-1)x^{r-2}(1-x) \geq 0$ for all $x \in (0,1]$.

Hence $f(x)$ is a non-decreasing function of $x$ in $(0,1]$ and therefore for $x \in (0,1]$, we have

$$f(x) \leq f(1) = 1\,,$$

which is inequality (2.10) and the proof of the lemma is complete. □

**Lemma 2.8.** *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zero in $|z| < k$, $k > 0$, then for $0 < r \leq \rho \leq k$,*

$$\begin{aligned} M(p,\rho) \leq \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} & \left[\left\{1 - \frac{k^\mu(k-\rho)R}{S}\left(1-\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}}\right)\right\} M(p,r)\right. \\ & \left. - \frac{T}{S}\frac{(k^\mu+r^\mu)^{\frac{n}{\mu}}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} - 1 - \frac{n}{\mu}\log\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)\right\} m(p,k)\right]. \end{aligned} \tag{2.11}$$

*The result is best possible and inequality holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where $n$ is a multiple of $\mu$.*

**Proof of Lemma 2.8.** Since the polynomial $p(z)$ has no zero in $|z| < k$, $k > 0$, the polynomial $P(z) = p(tz)$ where $0 < t \leq k$, has no zero in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$. Hence applying (2.3) of Lemma 2.3 to $P(z)$, we get

$$M(P',1) \leq n\left[\frac{1 + \frac{\mu}{n}\left|\frac{t^\mu a_\mu}{a_0}\right|\left(\frac{k}{t}\right)^{\mu+1}}{1 + \left(\frac{k}{t}\right)^{\mu+1} + \frac{\mu}{n}\left|\frac{t^\mu a_\mu}{a_0}\right|\left\{\left(\frac{k}{t}\right)^{\mu+1} + \left(\frac{k}{t}\right)^{2\mu}\right\}}\right] M(P,1),$$

which implies

$$M(p',t) \leq nt^{\mu-1}\left\{\frac{n|a_0|t + \mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+t^{\mu+1}) + \mu|a_\mu|(k^{2\mu}t + k^{\mu+1}t^\mu)}\right\} M(p,t). \tag{2.12}$$

Now, for $0 < r \leq \rho \leq k$ and $0 \leq \theta < 2\pi$, we have

$$|p(\rho e^{i\theta}) - p(re^{i\theta})| \leq \int_r^\rho |p'(te^{i\theta})|dt\,.$$

This gives

$$|p(\rho e^{i\theta})| \leq |p(re^{i\theta})| + \int_r^\rho |p'(te^{i\theta})|dt\,,$$

from which it follows that

$$M(p,\rho) \leq M(p,r) + \int_r^{\rho} M(p',t)dt\,. \tag{2.13}$$

Using (2.12) to (2.13), we get

$$\begin{aligned} M(p,\rho) &\leq M(p,r) \\ &+ \int_r^{\rho} nt^{\mu-1}\left\{\frac{n|a_0|t+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+t^{\mu+1})+\mu|a_\mu|(k^{2\mu}t+k^{\mu+1}t^{\mu})}\right\}M(p,t)dt. \end{aligned} \tag{2.14}$$

Using inequality (2.6) of Lemma 2.4 with $\rho = t$ and noting that $0 < r \leq t \leq \rho \leq k$, it follows that

$$\begin{aligned} M(p,\rho) &\leq M(p,r) + \int_r^{\rho} nt^{\mu-1}\left\{\frac{n|a_0|t+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+t^{\mu+1})+\mu|a_\mu|(k^{2\mu}t+k^{\mu+1}t^{\mu})}\right\} \\ &\quad\times\left(\frac{k^\mu+t^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}\left[M(p,r)-\left\{1-\left(\frac{k^\mu+r^\mu}{k^\mu+t^\mu}\right)^{\frac{n}{\mu}}\right\}m(p,k)\right]dt \\ &\leq M(p,r) + \frac{k^\mu+\rho^\mu}{(k^\mu+r^\mu)^{\frac{n}{\mu}}}\left\{\frac{n|a_0|\rho+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^{\mu})}\right\} \\ &\quad\times\int_r^{\rho} nt^{\mu-1}(k^\mu+t^\mu)^{\frac{n}{\mu}-1}\left[M(p,r)-\left\{1-\left(\frac{k^\mu+r^\mu}{k^\mu+t^\mu}\right)^{\frac{n}{\mu}}\right\}m(p,k)\right]dt \end{aligned} \tag{2.15}$$

(By Lemma 2.6, the factor in the integrand in (2.15) which is same as $f(t)$ of Lemma 2.6, is a non-decreasing function of $t$ in $(0,k]$)

$$\begin{aligned} &= \left[1+(k^\mu+\rho^\mu)\left\{\frac{n|a_0|\rho+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho)}\right\}\right. \\ &\quad\left.\times\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1\right\}\right]M(p,r)-(k^\mu+\rho^\mu) \\ &\quad\times\left\{\frac{n|a_0|\rho+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^{\mu})}\right\} \\ &\quad\times\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}-1-\frac{n}{\mu}\log\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)\right\} \end{aligned}$$

$$= \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} \left[\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}} \left\{1-(k^\mu+\rho^\mu)\frac{n|a_0|\rho+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\}\right.$$

$$\left. + (k^\mu+\rho^\mu)\left\{\frac{n|a_0|\rho+\mu|a_\mu|k^{\mu+1}}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\}\right]$$

$$\times M(p,r) - \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} \left[\frac{T}{S}\frac{(k^\mu+r^\mu)^{\frac{n}{\mu}}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}\right.$$

$$\left.\times\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} - 1 - \frac{n}{\mu}\log\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)\right\}\right] m(p,k)$$

$$= \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} \left[\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}} \left\{\frac{k^\mu(k-\rho)(n|a_0|-\mu|a_\mu|k^\mu)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\}\right.$$

$$\left. +1-\left\{\frac{k^\mu(k-\rho)(n|a_0|-\mu|a_\mu|k^\mu)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\}\right] M(p,r)$$

$$- \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} \left[\frac{T}{S}\frac{(k^\mu+r^\mu)^{\frac{n}{\mu}}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} - 1 - \frac{n}{\mu}\log\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)\right\}\right] m(p,k)$$

$$= \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} \left[1-\frac{k^\mu(k-\rho)R}{S}\left\{1-\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}}\right\}\right] M(p,r)$$

$$- \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} \left[\frac{T}{S}\frac{(k^\mu+r^\mu)^{\frac{n}{\mu}}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}-1}}\left\{\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} - 1 - \frac{n}{\mu}\log\left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)\right\}\right] m(p,k),$$

which proves Lemma 2.8. □

**Lemma 2.9.** *If $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\le\mu\le n$, is a polynomial of degree $n$ having no zero in $|z|<k$, $k>0$, then for $0<r\le\rho\le k$,*

$$M(p,\rho) \le \left(\frac{k^\mu+\rho^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}} \left[1-\frac{k^\mu(k-\rho)R}{S}\left\{1-\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}}\right\}\right] M(p,r)\,. \tag{2.16}$$

*The result is best possible and equality holds for $p(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$, where $n$ is a multiple of $\mu$.*

**Proof of Lemma 2.9.** The proof of this lemma follows on the same lines as that is in Lemma 2.8, but instead of applying (2.6) of Lemma 2.4, we use (2.5) of the same lemma. We omit the details. □

**Lemma 2.10.** *If $p(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then for every real or complex number $\alpha$ with $|\alpha| \ge 1$,*

$$M(D_\alpha p, 1) \le \frac{n(k^\mu + |\alpha|)}{1+k^\mu} M(p,1) \tag{2.17}$$

*and*

$$M(D_\alpha p, 1) \le \frac{n}{1+k^\mu}\{(k^\mu + |\alpha|)M(p,1) - (|\alpha|-1)m(p,k)\}. \tag{2.18}$$

These two results are due to Dewan and Singh [10].

**Lemma 2.11.** *For $0 < r \le \rho \le k$ and any $\mu \ge 0$, we have*

$$\log\left(\frac{k^\mu + \rho^\mu}{k^\mu + r^\mu}\right) \le \left(\frac{\rho^\mu - r^\mu}{k^\mu + r^\mu}\right). \tag{2.19}$$

**Proof of Lemma 2.11.** For $\mu = 0$, the result follows trivially. Hence it is sufficient to prove for $\mu > 0$.

Now, for $t \in [r, \rho]$, we have

$$0 < \frac{\mu t^{\mu-1}}{k^\mu + t^\mu} \le \frac{\mu t^{\mu-1}}{k^\mu + r^\mu}. \tag{2.20}$$

Integrating both sides of (2.20) with respect to $t$ in $[r, \rho]$, we immediately obtain the required inequality and the lemma is proved. $\square$

**Lemma 2.12.** *If $\mu$, $n$ are positive integers such that $1 \le \mu \le n$, then for $0 < r \le \rho \le k$, the quantity*

$$U = \left(\frac{k^\mu + \rho^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}} - 1 - \frac{n}{\mu}\left(\frac{\rho^\mu - r^\mu}{k^\mu + r^\mu}\right), \tag{2.21}$$

*is non-negative.*

**Proof of Lemma 2.12.** For $t \in [r, \rho]$, the function

$$g(t) = \frac{1}{k^\mu + r^\mu}\left\{\left(\frac{k^\mu + t^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}-1} - 1\right\} n t^{\mu-1},$$

is always non-negative for $\dfrac{k^\mu + t^\mu}{k^\mu + r^\mu} \ge 1$ and $\dfrac{n}{\mu} \ge 1$.

Integrating $g(t)$ with respect to $t$ from $r$ to $\rho$, we shall obtain the quantity $U$, which is necessarily non-negative. $\square$

**Lemma 2.13.** *If $\mu$, $n$ are positive integers such that $1 \leq \mu \leq n$, then for $0 < r \leq \rho \leq k$, we have*

$$\left(\frac{k^\mu + \rho^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}} - 1 - \frac{n}{\mu}\log\left(\frac{k^\mu + \rho^\mu}{k^\mu + r^\mu}\right) \geq U,$$

*where $U$ is defined in Lemma* 2.12.

**Proof of Lemma 2.13.** The proof follows readily from Lemma 2.11. □

## 3. Proof of the Theorems

We first prove Theorem 1.7.

**Proof of Theorem 1.7.** Since $p(z)$ has no zero in $|z| < k$, $k > 0$, the polynomial $P(z) = p(\rho z)$ has no zero in $|z| < \frac{k}{\rho}$ where $\frac{k}{\rho} \geq 1$. Applying inequality (2.18) of Lemma 2.10 to $P(z)$ and noting that $\frac{|\alpha|}{\rho} \geq 1$, we have

$$M\left(D_{\frac{\alpha}{\rho}} p, 1\right) \leq \frac{n}{1 + \left(\frac{k}{\rho}\right)^\mu}\left[\left\{\left(\frac{k}{\rho}\right)^\mu + \frac{|\alpha|}{\rho}\right\} M(p,\rho) - \left(\frac{|\alpha}{\rho} - 1\right)\min_{|z|=\frac{k}{\rho}} |p(z)|\right],$$

which is equivalent to

$$M(D_\alpha p, \rho) \leq \frac{n}{k^\mu + \rho^\mu}[(k^\mu + |\alpha|\rho^{\mu-1})M(p,\rho) - (|\alpha|\rho^{\mu-1} - \rho^\mu)m(p,k)]. \tag{3.1}$$

Combining (3.1) with (2.11) of Lemma 2.8, we get

$$\begin{aligned} M(D_\alpha p, \rho) \leq{}& n\frac{(k^\mu + \rho^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}}(k^\mu + |\alpha|\rho^{\mu-1}) \\ &\times\left\{1 - \frac{k^\mu(k-\rho)R}{S}\left(1 - \left(\frac{k^\mu + r^\mu}{k^\mu + \rho^\mu}\right)^{\frac{n}{\mu}}\right)\right\} M(p,r) - \frac{T}{S}\frac{(k^\mu + r^\mu)^{\frac{n}{\mu}}}{(k^\mu + r^\mu)^{\frac{n}{\mu}-1}} \\ &\times\left\{\left(\frac{k^\mu + \rho^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}} - 1 - \frac{n}{\mu}\log\left(\frac{k^\mu + r^\mu}{k^\mu + r^\mu}\right)\right\} m(p,k) \\ &- \frac{n(|\alpha|\rho^{\mu-1} - \rho^\mu)}{(k^\mu + \rho^\mu)} m(p,k), \end{aligned}$$

which proves Theorem 1.7. □

**Proof of Theorem 1.1.** The proof of this theorem follows on the same lines as that of Theorem 1.7, but instead of applying inequalities (2.18) of Lemma 2.10 and (2.11) of 2.8 we respectively use inequalities (2.17) of Lemma 2.10 and (2.16) of Lemma 2.9. We omit the details. □

## Conflict of Interests

The authors declare that there is no conflict of interests.

## References


[1] A. Aziz, Inequalities for the polar derivative of a polynomial, J. Approx. Theory, 55 (1988), 183-193.

[2] A. Aziz, W.M. Shah, $L^p$ inequalities for polynomials with restricted zeros, Glasnik Mat. 32(52) (1997),247-258.

[3] A. Aziz, W.M. Shah, Inequalities for the polar derivative of a polynomial, Indian J. Pure Appl. Math. 29(2) (1998), 163-173.

[4] A. Aziz, B.A. Zargar, Inequalities for a polynomial and its derivative, Math. Inequal. Appl. 1(4) (1998), 543-550.

[5] A. Aziz, B.A. Zargar, Inequalities for the maximum modulus of the derivative of a polynomial, J. Inequal. Pure Appl. Math. 8(2) (2007), 37.

[6] M. Bidkham, K.K. Dewan, Inequalities for a polynomial and its derivative, J. Math. Anal. Appl. 166(1992), 319-324.

[7] T.N. Chan, M.A. Malik, On Erdös-Lax theorem, Proc. Indian Acad. Sci. (Math. Sci.) 92(3) (1983),191-193.

[8] K.K. Dewan, A. Mir, On the maximum modulus of a polynomial and its derivatives, Int. J. Math. Math. Sci. 16 (2005), 2641-2645.

[9] K.K. Dewan, R.S. Yadav, M.S. Pukhta, Inequalities for a polynomial and its derivative, Math. Inequal. Appl. 2(2)(1999),203-205.

[10] K.K. Dewan, B. Singh, Inequalities for the polar derivative of a polynomial, J. Comb. Inform. Syst. Sci. 31(1-4) (2006), 83-90.

[11] N.K. Govil, On growth of polynomials, J. Inequal. Appl. 7(5) (2002), 623–631.

[12] V.K. Jain, On maximum modulus of polynomials with zeros outside a circle, Glasnik Mat. 29(49) (1994), 267–274.

[13] P.D. Lax, Proof of a conjecture of P. Erdös on the derivative of a polynomial, Bull. Amer. Math. Soc. 50 (1944), 509–513.

[14] M.A. Malik, On the derivative of a polynomial, J. London Math. Soc. 1 (1969), 57–60.

[15] A. Mir, S.A. Baba, M.S. Pukhta, Inequalities concerning maximum modulus of polynomials, Thai J. Math. 9(2) (2011), 297–304.

[16] M.A. Qazi, On the maximum modulus of polynomials, Proc. Amer. Math. Soc. 115(2) (1992), 337–343.

[17] A.C. Schaeffer, Inequalities of A Markoff and S. Bernstein for polynomials and related functions, Bull. Amer. Math. Soc. 47 (1941), 565–579.