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LEXICOGRAPHICAL OPERATOR EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduced and studied lexicographical operator equilibrium problem by using lexicographical cone in $\mathbb{R}^2$. Also, we prove an existence result for solution of this problem by using Kakutani fixed point theorem. As corollaries existence results for lexicographical operator variational inequalities and lexicographical operator minimization problems are also discussed. Our results in this paper are new which can be considered as significant extension of previously known results in the literature.

Keywords: lexicographical cone; operator equilibrium problems; pseudo-monotonicity; upper-semicontinuity.

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1. Introduction

The theory of variational inequalities has originally been introduced in the seventies of 20^{th} century as an innovative and effective method to solve a variety of nonlinear boundary value problems for partial differential equations of elliptic or parabolic type, for example, Signorini problem and the obstacle problem.The pioneer work in this area are due to Fichera [13] and Stampachhia [27]. This theory provides us a unified framework for dealing with a wide class of problems arising in elasticity, structural analysis, physical and engineering sciences, etc., see for example [1], [7], [8], [12], [27] and [29]. The generalization of variational inequalities

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known as vector variational inequalities in a finite dimensional Euclidean space was introduced by Giannessi [14]. Since then many authors have intensively studied the various extension of vector variational inequalities with different aspects, see for example [5], [11], [14], [19], [20], [23] and [30].

Equilibrium problems which was introduced and studied by Blum and Oettli [10] are generalized by many authors in different directions. Vector equilibrium problem is an extension of scalar equilibrium problem and provides us a platform to study vector saddle point problems etc. For more details of the study of vector equilibrium problems, we refer to [2], [5], [6], [14], [15], [16], [18], [21], [22], [25], [26]and [28]. Domkos and Kolumban [12] gave an interesting interpretation of variational inequalities and vector variational inequalities in Banach space settings in terms of variational inequalities with operator solutions (for short, OVVI).They designed (OVVI) to provides a suitable unified approach to several kinds of variational inequality problems in Banach space and successfully described those problems in wider context of (OVVI). On the other hand, Kazmi and Raouf [17] extended the notion of (OVVI) to operator equilibrium problems (for short, OEP) and established some existence results for this class of (OEP) by using Minty type lemma and KKM theorem. Recently, Ahmad et al [3] proved an existence result for solutions of operator mixed vector equilibrium problems by using KKM theorem and vector 0- diagonally quasi convexity and another result without using KKM theorem.

Most work on vector variational inequalities and vector equilibrium problems are based on orders induced by convex closed cones, i.e., they used various extensions of the Pareto order. However, it is known that the set of Pareto-optimal points is usually too large, so that one needs certain additional rules to reduce it. One of the possible approaches is to utilize the lexicographic order, which was investigated in connection with its applications in optimizations and decision making theory. Konnov [18] studied vector equilibrium problems using lexicographic order and showed that several classes of inverse lexicographic optimization problems can be reduced to lexicographic vector equilibrium problems. Bianchi, Konnov and Pini [9] established some existence results for lexicographic variational inequalities by using equivalence properties between various kinds of lexicographic variational inequalities and sequential one. They also discussed lexicographic equilibrium problems on a Hausdorff topological vector space and their relationships with some other vector equilibrium problems. Anh et al [4] studied lexicographic equilibrium problems on a Banach spaces and obtained an existence of solutions for such problems and then they investigate the Painleve-Kuratowski convergence of the solution sets for a family of perturbed problems in such a way that they are perturbed by sequences constrained sets and objective functions converging.

Motivated and inspired by research work mentioned above, the aim of this paper is to study operator solutions of lexicographic cone in $\mathbb{R}^2$ and prove an existence of solution for this problem by using Kakutani fixed point theorem.

The Problem under consideration is the following:

Let X be a finite dimensional Hausdorff topological vector space and $L(X,\mathbb{R}^2)$ be the space of all continous linear operators from X to $\mathbb{R}^2$.

Recall: The collection of zero and all non-zero vectors in $\mathbb{R}^n$ whose first non-zero coordinate is a positive number is a lexicographical cone in $\mathbb{R}^n$, denoted by C_{lex}. i.e

$$C_{lex} = \{0\} \cup \{x \in \mathbb{R}^n : x_1 = x_2 ......x_i = 0, x_{i+1} > 0, \text{ for some i, } 1 \leq i < n\}.$$

Throughout this paper, the cone C_{lex} defines a total order on $\mathbb{R}^n$ as follows:

$$x \geq_{C_{lex}} y \text{ iff } x - y \in C_{lex}$$

$$x >_{C_{lex}} y \text{ iff } x - y \in intC_{lex}$$

Let $K \subseteq L(X,\mathbb{R}^2)$ be a non-empty convex subset and $C : K \longrightarrow 2^{\mathbb{R}^2}$ be a constant map to lexicographical cone i.e $C(f) = C_{lex}$, $\forall f \in K$ is a lexicographical cone in R^2. Let $F : K \times K \longrightarrow \mathbb{R}^2$ be a bifunction such that $F(f,f) = 0, \forall f \in K$. Then the *lexicographical operator equilibrium problem* (for short, LOEP) is to find:

$$f^* \in K \text{ such that } F(f^*,g) \in C(f^*)_{lex}, \forall g \in K, \ (1.1)$$

where $C(f^*)_{lex}$ is a lexicographic cone in $\mathbb{R}^2$.

Remark 1.1. Lexicographical cone in $\mathbb{R}^n$ is convex, pointed, but neither open nor closed.

Remark 1.2. If $K \subseteq X$, then the above problem reduces to lexicographical vector equilibrium problem (LVEP) introduced and studied by Konnov [18].

Definition 1.3. A subset K of a vector space X is called *convex* if for all $x,y\in K,\lambda\in[0,1]$, we have $\lambda x+(1-\lambda)y\in K$.

Definition 1.4. [17] Let X and Y be Hausdorff topological vector spaces and K be a non-empty convex subset of $L(X,Y)$. Then a mapping $F:K\times K\to Y$ is said to be *C(f)-pseudo-monotone* if

$$F(f,g)\in C(f)\Longrightarrow F(g,f)\in -C(f)$$

Definition 1.5. [24] Let $C:K\to 2^Y$ be a set-valued mapping such that for each $f\in K$, $C(f)$ is a convex cone in Y. A bifunction $F:K\times K\to Y$ is said to be $C(f)-$ *quasi convex* if for all $f,g_1,g_2\in K$ and $\lambda\in[0,1]$, $g_\lambda=\lambda g_1+(1-\lambda)g_2$, we have

$$F(f,g_\lambda)\in F(f,g_1)-C(f)$$

or

$$F(f,g_\lambda)\in F(f,g_2)-C(f)$$

Definition 1.6. [21] Let X and Y be two topological spaces.Then a set-valued mapping $T:X\to 2^Y$ is said to be an upper-semicontinuous, if for each $x_0\in X$ and for any net $\{x_\lambda\}$ in X such that $x_\lambda\to x_0$ and for any net y_λ in Y with $y_\lambda\in T(x_\lambda)$ such that $y_\lambda\to y_0$, we have $y_0\in T(x_0)$.

Theorem 1.7. *(Kakutani fixed point theorem)*[15] *Let K be non-empty compact convex subset of a locally convex Hausdorff topological vector space X. Let $P:K\to 2^K$ be an upper-semicontinuous set-valued map with non-empty closed and convex values. Then P has a fixed point in K.*

2. Existence Result

In this section, we have found the existence result for the solution of Lexicographical operator equilibrium problem by using Kakutani fixed point theorem.

Theorem 2.1. *Let X be a finite dimensional Hausdorff topological vector space and $L(X,\mathbb{R}^2)$ be the space of all continuous linear operators from X to $\mathbb{R}^2$. Let $K\subseteq L(X,\mathbb{R}^2)$ be a non-empty convex and compact and $C:K\longrightarrow 2^{\mathbb{R}^2}$ be a constant map to lexicographical cone i.e $C(f)=C_{lex}$, $\forall f\in K$ is a lexicographical cone in R^2. Let $F:K\times K\longrightarrow\mathbb{R}^2$ be a bifunction such that $F(f,f)=0,\forall f\in K$. Assume the following assumptions holds:*

(i) *F is upper-semicontinuous on $K \times K$ in first and second argument,*

(ii) *F is C(f)-quasiconvex in the second argument,*

(iii) *F is pseudomonotone.*

Then $\exists f^* \in K$ such that $F(f^*,g) \in C(f^*)_{lex}, \forall g \in K$.

Proof. For each $g \in K$, Let

$$P(g) = \{f \in K : F(f,g) \in C(f)_{lex}, \forall g \in K\}.$$

First, we shall check that $P(g)$ is convex. For this, let $f_1, f_2 \in P(g)$, then $F(f_1,g) \in C(f_1)_{lex}$ and $F(f_2,g) \in C(f_2)_{lex}$. This implies $F(g,f_1) \in -C(f_1)_{lex}$ and $F(g,f_2) \in -C(f_2)_{lex}$. Let $f^* = \lambda f_1 + (1-\lambda)f_2$. Then by quasi-convexity of F, we have

$$\begin{aligned} F(g,f^*) &\in F(g,f_1) - C(g)_{lex} \in -C(f_1)_{lex} - C(g)_{lex} \\ &= -C_{lex} - C_{lex} \end{aligned}$$

$\implies F(g,f^*) \in -C_{lex} = -C(f^*)_{lex}$

$\implies F(g,f^*) \in -C(f^*)_{lex}$

and hence $F(f^*,g) \in C(f^*)_{lex}$.

Consequently, $F(\lambda f_1 + (1-\lambda)f_2, g) \in C(f_1)_{lex}$ and so $f^* \in P(g)$. Hence $P(g)$ is convex.

Next we shall show that $P(g)$ is closed. For this, let $\{f_\alpha : \alpha \in \Delta\}$ be an arbitrary net in $P(g)$ such that $f_\alpha \to f$. We have to show that $f \in P(g)$. Now $\{f_\alpha\} \in P(g)$ implies $F(f_\alpha, g) \in C(f_\alpha)_{lex}, \forall g \in K$. So, by the upper-semicontinuity of F in the first argument, we have $F(f,g) \in C(f)_{lex}$. So, $P(g)$ is closed.

Finally, we will show that P is upper-semicontinuous on K. If possible, suppose P is not upper-semicontinuous at some point f_0 in K. Then there exists some neighborhood, say U of $P(f_0)$ such that there is a net $w_\alpha \to f_0$ in K and $h_\alpha \in P(w_\alpha)$ such that $h_\alpha \to h_0$ but $h_0 \notin P(f_0)$.

Now since $h_\alpha \in P(w_\alpha)$ implies $F(h_\alpha, w_\alpha) \in C(h_\alpha) = C_{lex}$. So, by the upper-semicontinuity of F, we have $F(h_0, f_0) \in C(h_0) = C_{lex}$ implies $h_0 \in P(f_0)$, which is not true. Therefore, our supposition is wrong. So, P is upper-semicontinuous on K.

Thus by Kakutani fixed point Theorem , there exists $f \in K$ such that $P(f) = f$. Hence there exists $f \in K$ such that $F(f,g) \in C(f)$. This proves that LOEP has a solution. ∎

3. Lexicographical Operator Variational Inequalities and Lexicographical Operator Minimization Problems

In this section, we have introduced lexicographical operator variational inequality and lexicographical operator minimization problems. Also, we have proved the existence results for the solution of these problems.

Let X be a finite dimensional Hausdorff topological vector space and $K \subseteq L(X,\mathbb{R}^2)$ a nonempty convex set. Let $C : K \to 2^{\mathbb{R}^2}$ be a constant set-valued mapping such that for each $f \in K, C(f) = C_{lex}$ is a lexicographical cone in $\mathbb{R}^2$. Let $\langle f,x\rangle$ the value of an operator $f \in L(X,\mathbb{R}^2)$ at $x \in X$ and let $T : K \to X$ be a given mapping. The *lexicographical operator variational inequality problem* (for short, LOVIP) is to find $f \in K$ such that

$$\langle g-f,T(f)\rangle \in C(f)_{lex}, \forall g \in K.$$

Corollary 3.1. *Let $K \subseteq L(X,Y)$ be a nonempty closed convex set. Let $T : K \to X$ be a mapping. Assume the following condition holds:*

(i) *Suppose for arbitrary nets $\{f_\alpha\},\ \{g_\alpha\} \in K$, such that $f_\alpha \to f,\ g_\alpha \to g$ and $\langle g_\alpha - f_\alpha, T(f_\alpha)\rangle \in C(f_\alpha)_{lex} \implies \langle g-f,T(f)\rangle \in C(f)_{lex},\ \forall g \in K$,*

(ii) *Suppose for $f,g \in K$, $\langle g-f,T(f)\rangle \in C(f)_{lex} \implies \langle f-g,T(g)\rangle \in -C(f)_{lex}$.*

Then there exists

$$f \in K \text{ such that } \langle g-f,T(f)\rangle \in C(f)_{lex},\ \forall g \in K.$$

Proof. Let $F : K \times K \to \mathbb{R}^2$ be defined by $F(f,g) = \langle g-f,T(f)\rangle$. Since $0 \in C(f)_{lex}$, we have

$$\begin{aligned}
F(f,\lambda g_1+(1-\lambda)g_2) &= \langle \lambda g_1+(1-\lambda)g_2-f,T(f)\rangle \\
&= \langle \lambda g_1+(1-\lambda)g_2-\lambda f-(1-\lambda)f,T(f)\rangle \\
&= \lambda\langle g_1-f,T(f)\rangle+(1-\lambda)\langle g_2-f,T(f)\rangle \\
&= \lambda F(f,g_1)+(1-\lambda)F(f,g_2) \in co\{F(f,g_1),F(f,g_2)\}-C(f)_{lex}, \\
&\forall \quad f,g_1,g_2 \in K, \forall\lambda \in [0,1].
\end{aligned}$$

Therefore, F is $C(f)-$quasi convex. Also by (i), we have, for arbitrary nets $\{h_\alpha\},\{w_\alpha\} \in K$ such that $h_\alpha \to h_0,\ w_\alpha \to f_0$ and $\langle w_\alpha - h_\alpha, T(h_\alpha)\rangle \in C(h_\alpha) \implies \langle f_0 - h_0, T(h_0)\rangle \in C(h_0)$. Thus all the hypotheses of Theorem 2.1 are satisfied. Therefore,

$$\exists f \in K \text{ such that } \langle g - f, T(f)\rangle \in C(f)_{lex},\ \forall g \in K.$$

∎

Next, we have the lexicographical operator minimization problem which is as follows:

Let X be a finite dimensional Hausdorff topological vector space and $K \subseteq L(X,\mathbb{R}^2)$ a nonempty convex set. Let $C : K \to 2^{\mathbb{R}^2}$ be a constant set-valued mapping such that for each $f \in K$, $C(f) = C_{lex}$ is a lexicographical cone in $\mathbb{R}^2$. Let $\phi : K \to \mathbb{R}^2$ be a given mapping. The *lexicographical operator minimization problem* (for short, LOMP) is to find $f \in K$ such that

$$\phi(f) - \phi(g) \in C(f)_{lex}, \forall g \in K.$$

Corollary 3.2. *Let $K \subseteq L(X,Y)$ be a nonempty closed convex set. Let $\phi : K \to \mathbb{R}^2$ be a given mapping. Assume the following conditions holds:*

(i) *ϕ is upper-semicontinuous on K,*

(ii) *ϕ is $C(f)-$ quasiconvex,*

(iii) *ϕ is $C(f)-$ pseudomonotone.*

Then there exist $f \in K$ such that $\phi(f) - \phi(g) \in C(f)_{lex},\ \forall g \in K$.

Proof. To prove this, let $F : K \times K \to \mathbb{R}^2$ be defined by $F(f,g) = \phi(f) - \phi(g)$. Then it is easy to see that all the hypotheses of Theorem 2.1 are satisfied. Thus (LOMP) has a solution. ∎

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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