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SOME OPERATORS ON μ-Pre*-CLOSED SETS IN GENERALIZED TOPOLOGICAL SPACES

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Abstract: Using the idea of μ-pre*-closed set in generalized topological space, the concept of μ-pre*-closure and μ-pre*-interior in generalized topological space have been studied and several of their properties are proved. In this paper we are introducing some new operators in μ-pre* closed sets in generalized topological space as μ-pre*-dervied, μ-pre*-border, μ-pre*-frontier and μ-pre*-exterior. Also the aim is to deal with some basic properties.

Keywords: μ-pre*-dervied set; μ-pre*-border; μ-pre*-frontier; μ-pre*-exterior.

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I. INTRODUCTION

A. Csaszar introduced generalized topological spaces and he listed out the features which distinguish general generalized topological space from other typical topologies. The features are these families of subsets are not closed under intersection and the whole space is not open. By making use of his concept, we introduced new types of concepts as μ - pre *- closed sets and μ - pre *- open sets in generalized topological space. In this paper, we define the

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μ - pre * - dervied, μ - pre * - border, μ - pre * - frontier and μ - pre * - exterior on generalized topological spaces with their nature as well.

2. PRELIMINARIES

First we recall some definitions and results to be used in the paper.

Definition 2.1:[4] Let X be a non - empty set and μ be a collection of subsets of X. Then the pair (X, μ) is called as a generalized topological space (short on GTS) on X if φ must be in μ and union of any members of μ is also in μ.

The members of μ are called μ - open sets and the complement of μ - open sets are called μ - closed sets.

In GTS (X, μ), M_μ is defined as $M_\mu = \bigcup_{i\in I} U_i$.

Definition 2.2:[2] Let (X, μ) be a GTS. A subset A of X is called a μ - generalized closed set (in short, μ - g - closed set) iff $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and U is a μ - open set in X. The complement of a μ - g - closed set is called a μ - g - open set.

Definition 2.3 [14]: A subset A of a GTS (X, μ) is called a μ - pre * - closed set if $c_\mu^*(i_\mu(A)) \subseteq A$. The collection of all μ - pre * - closed sets in X is denoted by $P^*C_\mu(X)$ or $pre^*C_\mu(X)$.

Definition 2.4 [14]: Let (X, μ) be a GTS. A subset A of X is called a μ - pre* - open set if X \ A is a μ - pre* - closed set. $pre^*O_\mu(X)$ or $P^*O_\mu(X)$ is the collection of all μ - pre* - open sets in X.

Lemma 2.1 [14]:

(i) Arbitrary intersection of μ - pre *- closed sets is μ - pre * - closed.

(ii) Arbitrary union of μ - pre* - open sets is μ - pre* - open.

(iii) Every μ - g - closed is μ - pre * - closed.

(iv) Every μ - closed is μ - pre * - closed.

Definition 2.5[14]: Let (X, μ) be a GTS and $A \subseteq X$. Then the μ- pre *-closure set of A, denoted by $pre^*c_\mu(A)$ or $p^*c_\mu(A)$ and is defined as, the intersection of all μ - pre * - closed sets containing A . $p^*c_\mu(A)$ is the smallest μ - pre * - closed set containing A.

ie) $pre^*c_\mu(A) = \bigcap \{F \,/\, A \subseteq F$ and F is μ - pre * - closed$\}$

Definition 2.6[14]: Let A be a subset of a GTS (X, μ). Then the μ - pre * - interior of A is defined by $\cup \{U / U \subseteq A$ and $U \in$ pre $*O_\mu (X)\}$ and denoted as pre $*i_\mu(A)$ or p * $i_\mu(A)$.

pre $*i_\mu(A)$ is the largest μ - pre * - open set contained in A.

Lemma 2.2: In GTS (X, μ), every μ - pre*- closed set contains a μ - pre*- closed set $X \setminus M_\mu$.

Proof: For every μ - pre*- open set U in X, $U \subseteq M_\mu$, so that $X \setminus M_\mu \subseteq X \setminus U$. Thus every μ - pre*- closed set includes $X \setminus M_\mu$.

Lemma 2.3[14]: Let A be a subset of a GTS (X, μ). Then the followings are valid.

(i) $\text{pre}^*c_\mu(X) = X$ and $\text{pre}^*i_\mu(\varphi) = \varphi$.

(ii) $\text{pre}^*c_\mu(\varphi) = X \setminus M_\mu$.

(iii) $\text{pre}^*i_\mu(X) = M_\mu$.

(iv) $\text{pre}^*i_\mu(A) \subseteq A \subseteq \text{pre}^* c_\mu(A)$.

(v) $A \subseteq \text{pre}^*c_\mu(A) \subseteq \text{pre}\, c_\mu(A) \subseteq c_\mu(A)$.

(vi) $i_\mu(A) \subseteq \text{pre}\, i_\mu(A) \subseteq \text{pre}^*i_\mu(A) \subseteq A$.

Lemma 2.4[14]: Let A be a subset of a GTS (X, μ). Then the followings are hold.

(i) Monotonicity property: If $A \subseteq B$ then $\text{pre}^*c_\mu(A) \subseteq \text{pre}^*c_\mu(B)$ and $\text{pre}^*i_\mu(A) \subseteq \text{pre}^*i_\mu(B)$.

(ii) Idempotent property: $\text{pre}^*c_\mu(\text{pre}^*c_\mu(A)) = \text{pre}^*c_\mu(A)$ and $\text{pre}^*i_\mu(\text{pre}^*i_\mu(A)) = \text{pre}^* i_\mu(A)$.

(iii) A is μ - pre * - closed if and only if $\text{pre}^*c_\mu(A) = A$.

(iv) A is μ - pre * - open if and only if $\text{pre}^*i_\mu(A) = A$.

(v) $\text{pre}^*c_\mu(A \cap B) \subseteq \text{pre}^*c_\mu(A) \cap \text{pre}^*c_\mu(B)$.

(vi) $\text{pre}^*i_\mu(A \cup B) \supseteq \text{pre}^*i_\mu(A) \cup \text{pre}^*i_\mu(B)$.

(vii) $\text{pre}^*i_\mu(X \setminus A) = X \setminus \text{pre}^*c_\mu(A)$ and $\text{pre}^*c_\mu(X \setminus A) = X \setminus \text{pre}^*i_\mu(A)$.

(viii) $\text{pre}^*i_\mu(A) = X \setminus \text{pre}^*c_\mu(X \setminus A)$ and $\text{pre}^*c_\mu(A) = X \setminus \text{pre}^*i_\mu(X \setminus A)$.

Lemma 2.5[14]: Let $x \in X$. Then $x \in \text{pre}^*c_\mu(A)$ if and only if $V \cap A \neq \varphi$ for every μ - pre * - open set containing x.

Theorem 2.1: Let A and B be subsets of a GTS (X, μ). Then $pre^*c_\mu(A \cup B) \supseteq pre^*c_\mu(A) \cup pre^*c_\mu(B)$.

Proof: Since A ∪ B contains A as well as B and by lemma 2.4(i), $pre^*c_\mu(A \cup B) \supseteq pre^*c_\mu(A)$ and $pre^*c_\mu(A \cup B) \supseteq pre^*c_\mu(B)$. Hence, $pre^*c_\mu(A \cup B) \supseteq pre^*c_\mu(A) \cup pre^*c_\mu(B)$.

Remark 2.1: In theorem 2.1, the inclusion may be strict and equal which can be seen in the ensuing example.

Let us consider X = {a, b, c, d} and μ = {φ, {b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d} {b, c, d}, X}.

Take A = {a} and B = {c}. Then A ∪ B = {a, c}. Here $pre^*c_\mu(A)$ = {a}, $pre^*c_\mu(B)$ = {c}, $pre^*c_\mu(A \cup B)$ = {a, c, d} and $pre^*c_\mu(A) \cup pre^*c_\mu(B)$ = {a, c}. Hence $pre^*c_\mu(A \cup B) \supset pre^*c_\mu(A) \cup pre^*c_\mu(B)$.

Take A = {a, b} and B = {a, d}. Then A ∪ B = {a, b, d}. Here $pre^*c_\mu(A)$ = {a, b}, $pre^*c_\mu(B)$ = {a, c, d}, $pre^*c_\mu(A \cup B)$ = X and $pre^*c_\mu(A) \cup pre^*c_\mu(B)$ = X. Hence,$pre^*c_\mu(A \cup B) = pre^*c_\mu(A) \cup pre^*c_\mu(B)$.

From the above, we conclude that $pre^*c_\mu(A \cup B) \supseteq pre^*c_\mu(A) \cup pre^*c_\mu(B)$.

Theorem 2.2: Let A and B be subsets of X in a GTS (X, μ). Then $pre^*i_\mu(A \cap B) \subseteq pre^*i_\mu(A) \cap pre^*i_\mu(B)$.

Proof: Since A and B containing A ∩ B and by lemma 2.4(i), $pre^*i_\mu(A) \supseteq pre^*i_\mu(A \cap B)$ and $pre^*i_\mu(B) \supseteq pre^*i_\mu(A \cap B)$. Hence, $pre^*i_\mu(A \cap B) \subseteq pre^*i_\mu(A) \cap pre^*i_\mu(B)$.

The above inclusion may be strict and equal, which can be explained with an example.

Consider X = {a, b, c, d} and μ = {φ, {c}, {a, b}, {b, c}, {c, d}, {a, b, c}, {a, b, d}, {b, c, d}, X}.

Let A = {a, b} and B = {a, c}. Then A ∩ B = {a}. Here $pre^*i_\mu(A)$ = {a, b} , $pre^*i_\mu(B)$ = {a, c}, $pre^*i_\mu(A \cap B)$ = φ and $pre^*i_\mu(A) \cap pre^*i_\mu(B)$ = {a}. Hence, $pre^*i_\mu(A \cap B) \subset pre^*i(A) \cap pre^*i_\mu(B)$.

Let A = {b, c} and B = {c, d}. Then A ∩ B = {c}. Here $pre^*i_\mu(A)$ = {b, c}, $pre^*i_\mu(B)$ = {c, d}, $pre^*i_\mu(A \cap B)$ = {c} and $pre^*i_\mu(A) \cap pre^*i_\mu(B)$ = {c}. Hence $pre^*i_\mu(A \cap B) = pre^*i_\mu(A) \cap pre^*i_\mu(B)$.

3. μ -PRE*-DERIVED SET

The present section gives the definition of μ - pre* - Derived set and investigates some of its properties.

Definition 3.1: Let A be a subset of X in a GTS (X, μ). A point x ∈ X is said to be a μ - pre*- limit point of A if G ∩{A \ {x}} ≠ φ, for every $G \in P^*O_\mu(X)$ containing x.

The set of all μ - pre*- limit points of A is called μ - pre* - derived set of A and is denoted by $p * Dr_\mu(A)$.

Remark 3.1: We observe that $p * Dr_\mu(\varphi) = X \setminus M_\mu$. In particular, if GTS (X, μ) is strong then $p * Dr_\mu(\varphi) = \varphi$.

Theorem 3.1: A subset A of X in a GTS (X, μ), $p * Dr_\mu(A) \subseteq Dr_\mu(A)$, where $Dr_\mu(A)$ is the μ - derived set of A.

Proof: Let $x \in p * Dr_\mu(A)$. Suppose that $x \notin Dr_\mu(A)$, then implies G ∩{A \ {x}} = φ, for some μ - open set G containing x. By lemma 2.1 (iv), G ∩ {A \ {x}} = φ, where $G \in P^*O_\mu(X)$ and x ∈ G. This shows that $x \notin p * Dr_\mu(A)$, which is a contradiction to our assumption. Therefore, $x \in Dr_\mu(A)$. Hence $p * Dr_\mu(A) \subseteq Dr_\mu(A)$.

The following example can be explained, the reverse inclusion is not valid in the above theorem.

Example 1: Let us consider a GTS, X = {a, b, c, d} with μ = {φ, {b}, {a, b}, {a, d}, {a, b, d}}

(i)Take A = {c, d}. Then $p * Dr_\mu(A)$ = {c} and $Dr_\mu(A)$ = {c}. Hence $p * Dr_\mu(A) = Dr_\mu(A)$.

(ii) Take A = {a, b, d}. Then $p * Dr_\mu(A)$ = {c} and $Dr_\mu(A)$ = {a, c, d}. Hence $p * Dr_\mu(A) \subset Dr_\mu(A)$.

Theorem 3.2: If $x \in p * Dr_\mu(A)$ implies $x \in p * Dr_\mu(A \setminus \{x\})$, where A is a subset of X in a GTS (X, μ).

Proof: Suppose $x \in p * Dr_\mu(A)$, then $G \cap \{A \setminus \{x\}\} \neq \varphi$, for every $G \in P^*O_\mu(X)$. Then $G \cap [\{A \setminus \{x\}\} \setminus \{x\}] \neq \varphi$, for every $G \in P^*O_\mu(X)$ and $x \in G$. Therefore x is a μ - pre* - limit point of $A \setminus \{x\}$.

Theorem 3.3: Let A and B be subsets of a GTS (X, μ). Then the followings are valid:

(i) If $A \subseteq B$, then $p * Dr_\mu(A) \subseteq p * Dr_\mu(B)$.

(ii) $p * Dr_\mu(A) \cup p * Dr_\mu(B) \subseteq p * Dr_\mu(A \cup B)$

Proof: (i) Let $x \in p * Dr_\mu(A)$.Then $G \cap \{A \setminus \{x\}\} \neq \varphi$, for all $G \in P^*O_\mu(X)$ and $x \in G$. Since $A \subseteq B$, $A \setminus \{x\} \subseteq B \setminus \{x\}$ and hence $G \cap \{B \setminus \{x\}\} \neq \varphi$. Therefore, $x \in p * Dr_\mu(B)$. This proves (i).

(ii) We know that, $A \subseteq A \cup B$ and also $B \subseteq A \cup B$. By part (i) $p * Dr_\mu(A) \subseteq p * Dr_\mu(A \cup B)$ and $p * Dr_\mu(B) \subseteq p * Dr_\mu(A \cup B)$. From this, $p * Dr_\mu(A) \cup p * Dr_\mu(B) \subseteq p * Dr_\mu(A \cup B)$.

Remark 3.2: The converse of the above theorem (i) and (ii) is not true, which can be seen in the succeeding example.

Let us consider, X = {a, b, c, d, e, f} with μ = {φ, {b}, {d}, {e}, {a, b}, {a, c}, {b, d}, {b, e}, {b, f}, {d, e}, {a, b, c}, {a, b, d}, {a, b, e}, {a, b, f}, {a, c, d}, {a, c, e}, {b, d, e}, {b, d, f}, {b, e, f}, {a, b, c, d}, {a, b, c, e}, {a, b, c, f}, {a, b, d, e}, {a, b, d, f}, {a, b, e, f}, {a, c, d, e}, {b, d, e, f}, {a, b, c, d, e}, {a, b, c, d, f}, {a, b, c, e, f}, {a, b, d, e, f}, X}.

(i) Take A = {b} and B = {a, b, c, d}. Then $p * Dr_\mu(A) = \{f\}$ and $p * Dr_\mu(B) = \{c, f\}$. Therefore $p * Dr_\mu(B) \not\subset p * Dr_\mu(A)$

(ii) Take A = {a, c, d} and B = {b, c}. Then $A \cup B = X$. Here $p * Dr_\mu(A) = \varphi$, $p * Dr_\mu(B) = \{f\}$ and $p * Dr_\mu(A \cup B) = \{c, f\}$. Hence $p * Dr_\mu(A) \cup p * Dr_\mu(B) \subset p * Dr_\mu(A \cup B)$.

Take A = {b, d} and B = {b, e}. Then $A \cup B = \{b, d, e\}$. Here $p * Dr_\mu(A) = \{f\}$, $p * Dr_\mu(B) = \{f\}$ and $p * Dr_\mu(A \cup B) = \{f\}$. Hence $p * Dr_\mu(A) \cup p * Dr_\mu(B) = p * Dr_\mu(A \cup B)$.

Form this, $p * Dr_\mu(A \cup B) \not\subset p * Dr_\mu(A) \cup p * Dr_\mu(B)$

Theorem 3.4: Let A be a subset of X in a GTS (X, μ). Then $p * Dr_\mu[A \cup p * Dr_\mu(A)] \subseteq A \cup p * Dr_\mu(A)$.

Proof: Let $x \in p * Dr_\mu[A \cup p * Dr_\mu(A)]$.

Case 1: If $x \in A$, then the proof is complete.

Case 2: If $x \notin A$. Since $x \in p * Dr_\mu[A \cup p * Dr_\mu (A)]$, $G \cap [[A \cup p * Dr_\mu (A)] \setminus \{x\}] \neq \varphi$, for every $G \in P^*O_\mu(X)$ and $x \in G$. Hence $G \cap \{A \setminus \{x\}\} \neq \varphi$ or $G \cap [p * Dr_\mu (A) \setminus \{x\}] \neq \varphi$. Suppose $G \cap \{A \setminus \{x\}\} \neq \varphi$ then $x \in p * Dr_\mu(A)$.Thus $x \in A \cup p * Dr_\mu (A)$. Otherwise, if $G \cap [p * Dr_\mu (A) \setminus \{x\}] \neq \varphi$ then $y \in G \cap [p * Dr_\mu(A) \setminus \{x\}]$ for every $G \in P^*O_\mu(X)$ and $x \in G$. Then $y \in G$ and $y \in [p * Dr_\mu(A) \setminus \{x\}]$ with $y \neq x$. Therefore we have $y \in p * Dr_\mu (A)$ and so $G \cap \{A \setminus \{y\}\} \neq \varphi$. Let $z \in G \cap \{A \setminus \{y\}\}$, $z \neq y$ and $x \notin A$. Therefore, $z \in A \setminus \{x\}$ and so $z \in G \cap \{A \setminus \{x\}\}$ which implies $G \cap \{A \setminus \{x\}\} \neq \varphi$.Thus $x \in A \cup p * Dr_\mu (A)$. Hence $p * Dr_\mu[A \cup p * Dr_\mu (A)] \subseteq A \cup p * Dr_\mu (A)$.

Theorem 3.5: For any subset A of X in a GTS (X, μ), we have $p * Dr_\mu (A) \subseteq p * c_\mu (A)$.

Proof: Let $x \in p * Dr_\mu(A)$. Then $G \cap \{A \setminus \{x\}\} \neq \varphi$ for every $G \in P^*O_\mu(X)$ and $x \in G$. Hence $x \in G \Rightarrow G \cap A \neq \varphi$ and by lemma 2.5, $x \in p * c_\mu(A)$. Hence $p * Dr_\mu (A) \subseteq p * c_\mu (A)$.

The reverse inclusion of theorem 3.5 is not true as shown in the following illustration. Let $X = \{a, b, c, d, e\}$ and $\mu = \{\varphi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, e\}, \{c, e\}, \{a, b, c\}, \{a, b, e\}, \{a, c, e\}, \{b, c, e\},\{a, b, c, e\}\}$. Take $A = \{b, d\}$. $p * Dr_\mu (A) = \{d\}$ and $p * c_\mu (A) = \{b, d\}$.Hence $p * c_\mu (A) \nsubseteq p * Dr_\mu (A)$.

Note 3.1: We observe that if A is a μ- pre*-closed set, then A contains the set of all μ - pre*- limit points. Hence A contains $p * Dr_\mu (A)$.

Theorem 3.6: For any subset A of X in a GTS (X, μ), $p * c_\mu (A) = A \cup p * Dr_\mu (A)$.

Proof: By theorem 3.5, $p * Dr_\mu (A) \subseteq p * c_\mu (A)$.Since $A \subseteq p * c_\mu (A)$, $A \cup p * Dr_\mu (A) \subseteq p * c_\mu (A)$. On the other hand, let $x \in p * c_\mu(A)$

Case 1: Suppose $x \in A$. Obviously, $x \in A \cup p * Dr_\mu (A)$.

Case 2: Suppose $x \notin A$. By lemma 2.5, $G \cap A \neq \varphi$, for every $G \in P^*O_\mu(X)$ and $x \in G$. Also $G \cap \{A \setminus \{x\}\} \neq \varphi$ which implies $x \in p * Dr_\mu(A)$.Therefore $x \in A \cup p * Dr_\mu (A)$. In both cases $p * c_\mu(A) \subseteq A \cup p * Dr_\mu (A)$.

Theorem 3.7: Let A be a subset of X in a GTS (X, μ). Then $p * i_\mu (A) = A \setminus p * Dr_\mu(X \setminus A)$.

Proof: Let $x \in p * i_\mu (A)$. Since $p * i_\mu (A)$ and $(X \setminus A)$ are independent to each other and also $p * i_\mu (A)$ is the greatest μ - pre* - open set contained in A. Hence $x \notin p * Dr_\mu(X \setminus A)$ and so $x \in A$. Thus we have $x \in A \setminus p * Dr_\mu(X \setminus A)$.

On the other hand if $x \in A \setminus p * Dr_\mu(X \setminus A)$, this implies that $x \notin p * Dr_\mu(X \setminus A)$. Therefore, $G \cap \{(X \setminus A) \setminus \{x\}\} = \varphi$, for some $G \in P^*O_\mu(X)$ and $x \in G$. Then $x \in G \subseteq A$ and $G \in P^*O_\mu(X)$ which gives $x \in p * i_\mu (A)$. Hence $p * i_\mu (A) = A \setminus p * Dr_\mu(X \setminus A)$.

Theorem 3.8: $p * Dr_\mu[p * Dr_\mu(A)] \setminus A \subseteq p * Dr_\mu(A)$, where A is a subset of X in a GTS (X, μ).

Proof: Let $x \in p * Dr_\mu[p * Dr_\mu(A)] \setminus A$. Then $G \cap [p * Dr_\mu (A) \setminus \{x\}] \neq \varphi$, for every $G \in P^*O_\mu(X)$ and $x \in G$ and $x \notin A$. Let $y \in G \cap [p * Dr_\mu (A) \setminus \{x\}]$ which gives $y \in G$ and $y \in p * Dr_\mu(A)$ with $y \neq x$ and hence $G \cap \{A \setminus \{y\}\} \neq \varphi$. Take $z \in G \cap \{A \setminus \{y\}\}$. Now $z \in A \setminus \{y\}$ which gives $z \in A$, $z \neq y$ and so $z \neq x$ because $x \notin A$. Therefore $G \cap A \setminus \{x\} \neq \varphi$ and hence $x \in p * Dr_\mu(A)$. Thus $p * Dr_\mu[p * Dr_\mu(A)] \setminus A \subseteq p * Dr_\mu(A)]$.

The reverse inclusion does not valid, which can be seen in the succeeding example.

Example 2: Consider the GTS (X, μ), $X = \{1, 2, 3, 4, 5\}$ with $\mu = \{\varphi, \{2\}, \{3, 4\}, \{4, 5\}, \{3, 5\}, \{2, 3, 4\}, \{3, 4, 5\}, \{2, 4, 5\}, \{2, 3, 5\}, \{2, 3, 4, 5\}\}$.

Let $A = \{1\}$ Here $p * Dr_\mu(A) = \{1\}$ and $p * Dr_\mu[p * Dr_\mu(A)] \setminus A = \{1\} \setminus \{1\} = \varphi$. Hence $p * Dr_\mu[p * Dr_\mu(A)] \setminus A \subset p * Dr_\mu(A)$.

Let $A = \{2, 3\}$. Here $p * Dr_\mu(A) = \{1\}$ and $p * Dr_\mu[p * Dr_\mu(A)] \setminus A = \{1\} \setminus \{2, 3\} = \{1\}$. Hence $p * Dr_\mu[p * Dr_\mu(A)] \setminus A = p * Dr_\mu(A)$.

4. μ-PRE*-BORDER

This section introduces the concept of μ-pre*-Border and studies some of their properties.

Definition 4.1: Let A be a subset of a GTS (X, μ). Then $p * B_\mu(A) = A \setminus p * i_\mu(A)$ is called the μ - pre* - Border of A.

Theorem 4.1: For a subset A of X, the following statements are hold:

(i) $p * B_\mu(A) \subseteq A$.

(ii) $p * B_\mu(A) \subseteq B_\mu(A)$, where $B_\mu(A)$ is the μ - Border of A.

(iii) $p * B_\mu(\varphi) = \varphi$ and $p * B_\mu(X) = X \setminus M_\mu$.

(iv) $p * B_\mu(A) = A \cap p * c_\mu(X \setminus A)$.

(v) $p * i_\mu(A) \cap p * B_\mu(A) = \varphi$.

(vi) $p * B_\mu(p * i_\mu(A)) = \varphi$.

(vii) $p * i_\mu(p * B_\mu(A)) = \varphi$.

(viii) Idempotent property hold in μ - pre* - Border.

Proof: (i) and (iii) are obvious.

(ii) $p * B_\mu(A) = A \setminus p * i_\mu(A) \subseteq A \setminus i_\mu(A) = B_\mu(A)$. Thus, $p * B_\mu(A) \subseteq B_\mu(A)$.

(iv) $p * B_\mu(A) = A \setminus p * i_\mu(A) = A \cap (X \setminus p * i_\mu(A)) = A \cap p * c_\mu(X \setminus A)$.

(v) By definition 4.1 $p * B_\mu(A)$ and $p * i_\mu(A)$ are independent, $p * i_\mu(A) \cap p * B_\mu(A) = \varphi$.

(vi)By definition 4.1and idempotent property of μ - pre* - interior, $p * B_\mu(p * i_\mu(A)) = \varphi$.

(vii) Let $x \in X$ and assume that $x \in p * i_\mu(p * B_\mu(A))$. By part (i) and monotonicity of μ - pre* - interior, $p * i_\mu(p * B_\mu(A)) \subseteq p * i_\mu(A)$. Hence $x \in p * i_\mu(A)$. Since $p * i_\mu(p * B_\mu(A)) \subseteq p * B_\mu(A)$, $x \in p * B_\mu(A)$. Form this $x \in p * i_\mu(A) \cap p * B_\mu(A)$. By part (v), $p * i_\mu(A) \cap p * B_\mu(A) = \varphi$. Therefore, we get a contradiction to our assumption. Since $x \in X$ is arbitrary, $p * i_\mu(p * B_\mu(A)) = \varphi$.

(viii) $p * B_\mu(p * B_\mu(A)) = p * B_\mu(A) \setminus p * i_\mu (p * B_\mu(A))$. By part (vii), $p * B_\mu(p * B_\mu(A)) = p * B_\mu(A) \setminus \varphi = p * B_\mu(A)$.

The reverse inclusion of part (ii) in theorem 4.1 is not valid, as shown in Example 3.

Example 3: Let $X = \{a, b, c, d, e\}$ and $\mu = \{\varphi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, e\}, \{c, e\}, \{a, b, c\}, \{a, b, e\}, \{a, c, e\}, \{b, c, e\},\{a, b, c, e\}\}$. (i) Let $A = \{a, d, e\}$, then $p * B_\mu(A) = \{d\}$ and $B_\mu(A) = \{d, e\}$. Then $p * B_\mu(A) \subset B_\mu(A)$. (ii) Let $B = \{c, d\}$, then $p * B_\mu(B) = \{d\}$ and $B_\mu(B) = \{d\}$. Then $p * B_\mu(B) = B_\mu(B)$. From this, we can conclude that $B_\mu(A) \not\subset p * B_\mu(A)$.

Corollary 4.1: For any subset A of a GTS (X, μ), $p * B_\mu(A) \subseteq p * c_\mu(X \setminus A)$

Proof: By theorem 4.1 part (iv), the result is true.

Theorem 4.2: Let A be a subset of a GTS (X, μ). Then $A = p * i_\mu(A) \cup p * B_\mu(A)$.

Proof: Let $x \in A$.

Case 1: Suppose $x \in p * i_\mu(A)$ then $x \in p * i_\mu(A) \cup p * B_\mu(A)$

Case 2: Suppose $x \notin p * i_\mu(A)$. By definition 4.1, $x \in p * B_\mu(A)$. From both cases $A \subseteq p * i_\mu(A) \cup p * B_\mu(A)$.

On the other hand, $p * i_\mu(A) \cup p * B_\mu(A) = p * i_\mu(A) \cup [A \setminus p * i_\mu(A)] = p * i_\mu(A) \cup [A \cap p * c_\mu(X \setminus A)] \subseteq p * i_\mu(A) \cup A \subseteq A$. Hence $A = p * i_\mu(A) \cup p * B_\mu(A)$.

Theorem 4.3: A subset A in a GTS (X, μ) is μ-pre*-open if and only if $p * B_\mu(A) = \varphi$.

Proof: Suppose that $A \in P^*O_\mu(X)$. By definition 4.1 $p * B_\mu(A) = A \setminus p * i_\mu(A) = A \setminus A = \varphi$. Conversely, suppose $p * B_\mu(A) = \varphi$. Then $A \setminus p * i_\mu(A) = \varphi$. Since $p * i_\mu(A) \subseteq A$, $A = p * i_\mu(A)$. Hence A is μ - pre* - open.

Theorem 4.4: For any subset A of a GTS (X, μ), $p * B_\mu(A) = A \cap p * Dr_\mu(X \setminus A)$

Proof: By Part (iv) in theorem 4.1, $p * B_\mu(A) = A \cap p * c_\mu(X \setminus A) = A \cap [(X \setminus A) \cup p * Dr_\mu(X \setminus A)]$ (by theorem 3.6) $= [A \cap (X \setminus A)] \cup [A \cap p * Dr_\mu(X \setminus A)] = \varphi \cup [A \cap p * Dr_\mu(X \setminus A)] = [A \cap p * Dr_\mu(X \setminus A)]$.

Theorem 4.5: Let A and B be subsets of GTS (X, μ). Then

(i) $p * B_\mu(A \cup B) \subseteq p * B_\mu(A) \cup p * B_\mu(B)$.

(ii) $p * B_\mu(A \cap B) \supseteq p * B_\mu(A) \cap p * B_\mu(B)$.

Proof: (i)$p * B_\mu(A \cup B) = (A \cup B) \setminus p * i_\mu (A \cup B) = (A \cup B) \cap [X \setminus p * i_\mu (A \cup B)] = (A \cup B) \cap [p * c_\mu ((X \setminus A) \cap (X \setminus B))] \subseteq (A \cup B) \cap [p * c_\mu (X \setminus A) \cap p * c_\mu (X \setminus B)] = [A \cap [p * c_\mu (X \setminus A) \cap p * c_\mu (X \setminus B)]] \cup [B \cap [p * c_\mu (X \setminus A) \cap p * c_\mu (X \setminus B)]] = [[A \cap p * c_\mu (X \setminus A)] \cap [A \cap p * c_\mu (X \setminus B)]] \cup [[B \cap [p * c_\mu (X \setminus A)] \cap [B \cap p * c_\mu (X \setminus B)]] \subseteq [A \cap p * c_\mu (X \setminus A)] \cup [B \cap p * c_\mu (X \setminus B)]] = p * B_\mu(A) \cup p * B_\mu(B)$.

(ii)$p * B_\mu(A \cap B) = (A \cap B) \setminus p * i_\mu (A \cap B) = (A \cap B) \cap [X \setminus p * i_\mu (A \cap B)] = (A \cap B) \cap [p * c_\mu ((X \setminus A) \cup (X \setminus B))] \supseteq (A \cap B) \cap [p * c_\mu (X \setminus A) \cup p * c_\mu (X \setminus B)] = [A \cap [p * c_\mu (X \setminus A) \cup p * c_\mu (X \setminus B)]] \cap [B \cap [p * c_\mu (X \setminus A) \cup p * c_\mu (X \setminus B)]] = [[A \cap p * c_\mu (X \setminus A)] \cup [A \cap p * c_\mu (X \setminus B)]] \cap [[B \cap [p * c_\mu (X \setminus A)] \cup [B \cap p * c_\mu (X \setminus B)]] \supseteq [A \cap p * c_\mu (X \setminus A)] \cap [B \cap p * c_\mu (X \setminus B)] = p * B_\mu(A) \cap p * B_\mu(B)$.

The inclusion of the above theorem (i) and (ii) may be strict or equal. Let X = {a, b, c, d, e, f} and μ = {φ, {b}, {d}, {e}, {a, b}, {a, c}, {b, d}, {b, e}, {b, f}, {d, e}, {a, b, c}, {a, b, d}, {a, b, e}, {a, b, f}, {a, c, d}, {a, c, e}, {b, d, e}, {b, d, f}, {b, e, f}, {a, b, c, d}, {a, b, c, e}, {a, b, c, f}, {a, b, d, e}, {a, b, d, f}, {a, b, e, f}, {a, c, d, e}, {b, d, e, f}, {a, b, c, d, e}, {a, b, c, d, f}, {a, b, c, e, f}, {a, b, d, e, f}, X}.

(i)Take A = {a, b} and B = {c, f}. Then A ∪ B = {a, b, c, f}. Here $p * B_\mu(A) = \varphi$, $p * B_\mu(B) =$ {c, f} and $P* B_\mu(A \cup B) = \varphi$. Therefore, $P* B_\mu(A \cup B) \subset p * B_\mu(A) \cup p * B_\mu(B)$.

Take A = {a} and B = {b, c, f}. Then A ∪ B = {a, b, c, f}. Here $p * B_\mu(A) = \varphi$, $p * B_\mu(B) = \varphi$ and $P* B_\mu(A \cup B) = \varphi$. Therefore, $P* B_\mu(A \cup B) = p * B_\mu(A) \cup p * B_\mu(B)$.

(ii) Take A = {a, b, e, f} and B = {a, c, d, f}. Then A ∩ B = {a, f}. Here $p * B_\mu(A) = \varphi$, $p * B_\mu(B) =$ {f} and $P* B_\mu(A \cap B) =$ {f}. Therefore, $P* B_\mu(A \cap B) \supset p * B_\mu(A) \cap p * B_\mu(B)$.

Take A = {c, d, f} and B = {c, e, f}. Then A ∩ B = {c, f}. Here $p * B_\mu(A) =$ {c, f}, $p * B_\mu(B) =$ {c, f} and $P* B_\mu(A \cap B) =$ {c, f}. Therefore, $P* B_\mu(A \cap B) = p * B_\mu(A) \cap p * B_\mu(B)$.

5. μ-PRE*-FRONTIER

In this section, we introduce μ-pre*-Frontier and investigate some of their characterization.

Definition 5.1: Let A be a subset of a GTS (X, μ). Then the μ - pre* - frontier of A is $p * Fr_\mu(A) = p * c_\mu(A) \setminus p * i_\mu(A)$.

Note 5.1: From definition of μ - pre* - frontier, we have $p * Fr_\mu(A) \subseteq p * c_\mu(A)$ for any subset A of a GTS (X, μ).

Theorem 5.1: If A is a subset of a GTS (X, μ), then $p * Fr_\mu(A) = p * c_\mu(A) \cap p * c_\mu(X \setminus A)$.

Proof: $p * Fr_\mu(A) = p * c_\mu(A) \setminus p * i_\mu(A) = p * c_\mu(A) \cap [X \setminus p * i_\mu(A)] = p * c_\mu(A) \cap p * c_\mu(X \setminus A)$.

Theorem 5.2: $p * B_\mu(A) \subseteq p * Fr_\mu(A) \subseteq p * c_\mu(A)$, where A is a subset of a GTS (X, μ)

Proof: $p * B_\mu(A) = A \setminus p * i_\mu(A) \subseteq p * c_\mu(A) \setminus p * i_\mu(A) \subseteq p * c_\mu(A)$. Hence $p * B_\mu(A) \subseteq p * Fr_\mu(A) \subseteq p * c_\mu(A)$.

The following example shows the strict inclusion of the above theorem.

Example 4: Let X = {a, b, c, d, e, f} with μ = {φ, {b}, {d}, {e}, {a, b}, {a, c}, {b, d}, {b, e}, {b, f}, {d, e}, {a, b, c}, {a, b, d}, {a, b, e}, {a, b, f}, {a, c, d}, {a, c, e}, {b, d, e}, {b, d, f}, {b, e, f}, {a, b, c, d}, {a, b, c, e}, {a, b, c, f}, {a, b, d, e}, {a, b, d, f}, {a, b, e, f}, {a, c, d, e}, {b, d, e, f}, {a, b, c, d, e}, {a, b, c, d, f}, {a, b, c, e, f}, {a, b, d, e, f}, X}.

Take A = {a, b, c, d}. Here $p * B_\mu(A) = \varphi$, $p * Fr_\mu(A)$ = {f} and $p * c_\mu(A)$ = {a, b, c, d, f}. Therefore, $p * B_\mu(A) \subset p * Fr_\mu(A) \subset p * c_\mu(A)$.

Take B = {c, d, e, f}. Here $p * B_\mu(B)$ = {c, f}, $p * Fr_\mu(B)$ = {c, f} and $p * c_\mu(B)$ = {c, d, e, f}. Therefore, $p * B_\mu(B) = p * Fr_\mu(B) \subset p * c_\mu(B)$.

Take C = {c, f}. Here $p * B_\mu(C)$ = {c, f}, $p * Fr_\mu(C)$ = {c, f} and $p * c_\mu(C)$ = {c, f}. Therefore, $p * B_\mu(C) = p * Fr_\mu(C) = p * c_\mu(C)$.

Theorem 5.3: $p * Fr_\mu(A) \subseteq Fr_\mu(A)$, where $Fr_\mu(A)$ is the μ - frontier of A.

Proof: $p * Fr_\mu(A) = p * c_\mu(A) \setminus p * i_\mu(A) \subseteq c_\mu(A) \setminus i_\mu(A) = Fr_\mu(A)$.

Remark 5.1: In theorem 5.3, the inclusion may be strict and equal which can be shown in the ensuing example.

Let us consider, X = { i, j, k, l, m} with μ = {φ, {i}, {k}, {i, j}, {i, k}, {j, m}, {k, m}, {i, j, k}, {i, j, m}, {i, k, m}, {j, k, m}, {i, j, k, m}}

Take A = {i, m}. Then $p * Fr_\mu(A)$ ={l} and $Fr_\mu(A)$ = {j, l, m}. This shows that $p * Fr_\mu(A) \subset Fr_\mu(A)$.

Take A = {i, j, m}. Then $p * Fr_\mu(A)$ ={l} and $Fr_\mu(A)$ = {l}. This shows that $p * Fr_\mu(A) = Fr_\mu(A)$.

Theorem 5.4: In GTS (X, μ), $p * Fr_\mu(\varphi) = p * Fr_\mu(X) = X \setminus M_\mu$.

Proof: By lemma 2.2, the proof is straightforward.

Remark 5.2: For any subset A of a GTS (X, μ), μ - pre*- frontier of A and X \ A are equal. (ie) $p * Fr_\mu(A) = p * Fr_\mu(X \setminus A)$.

Remark 5.3: Monotonic property is not valid in μ - pre*- frontier which can be explained from the succeeding example.

Let us consider X = {a, b, c, d} with μ = {φ, {c}, {a, b}, {b, c}, {c, d}, {a, b, c}, {a, b, d}, {b, c, d}, X}. Take A = {a} and B = {a, b, c}. Then $p * Fr_\mu(A)$ = {a} and $p * Fr_\mu(B)$ = {d}. Here $A \subseteq B$ but $p * Fr_\mu(A) \nsubseteq p * Fr_\mu(B)$.

Theorem 5.5: In any generalized topological space (X, μ), the followings are equivalent

(i) $X \setminus p * Fr_\mu(A) = p * i_\mu(A) \cup p * i_\mu(X \setminus A)$.

(ii) $p * c_\mu(A) = p * i_\mu(A) \cup p * Fr_\mu(A)$

(iii)$p * Fr_\mu(A) = p * c_\mu(A) \cap p * c_\mu(X \setminus A)$.

(iv) $X = p * i_\mu(A) \cup p * i_\mu(X \setminus A) \cup p * Fr_\mu(A)$.

Proof: (i) ⇒ (ii). From (i) $X \setminus p * Fr_\mu(A) = p * i_\mu(A) \cup p * i_\mu(X \setminus A) \Rightarrow p * Fr_\mu(A) = X \setminus [p * i_\mu(A) \cup p * i_\mu(X \setminus A)] = [X \setminus p * i_\mu(A)] \cap [X \setminus p * i_\mu(X \setminus A)]$. Therefore $p * i_\mu(A) \cup p * Fr_\mu(A) = p * i_\mu(A) \cup [[X \setminus p * i_\mu(A)] \cap [X \setminus p * i_\mu(X \setminus A)]] = [p * i_\mu(A) \cup [X \setminus p * i_\mu(A)]] \cap [p * i_\mu(A) \cup p * c_\mu(A)] = X \cap p * c_\mu(A) = p * c_\mu(A)$.

(ii) ⇒ (iii) From (ii) $X \cap p * c_\mu(A) = p * i_\mu(A) \cup p * Fr_\mu(A) \Rightarrow [p * i_\mu(A) \cup [X \setminus p * i_\mu(A)]] \cap p * c_\mu(A) = p * i_\mu(A) \cup p * Fr_\mu(A) \Rightarrow [p * i_\mu(A) \cap p * c_\mu(A)] \cup [p * c_\mu(X \setminus A) \cap p * c_\mu(A)] = p * i_\mu(A) \cup p * Fr_\mu(A) \Rightarrow p * i_\mu(A) \cup [p * c_\mu(X \setminus A) \cap p * c_\mu(A)] = p * i_\mu(A) \cup p * Fr_\mu(A) \Rightarrow p * c_\mu(X \setminus A) \cap p * c_\mu(A) = p * Fr_\mu(A)$.

(iii) ⇒ (i) From (iii) we have that, $X \setminus p * Fr_\mu(A) = X \setminus [p * c_\mu(X \setminus A) \cap p * c_\mu(A)] = [X \setminus p * c_\mu(X \setminus A)] \cup [X \setminus p * c_\mu(A)] = p * i_\mu(A) \cup p * i_\mu(X \setminus A)$.

(ii) ⇔ (iv) Obvious.

Theorem 5.6: $p * i_\mu(A) = A \setminus p * Fr_\mu(A)$, where A is a subset of a GTS (X, μ).

Proof: $A \setminus p * Fr_\mu(A) = A \setminus [p * c_\mu(A) \cap p * c_\mu(X \setminus A)] = A \cap [[X \setminus p * c_\mu(A)] \cup [X \setminus p * c_\mu(X \setminus A)]] = [A \cap p * i_\mu(X \setminus A)] \cup [A \cap [X \setminus p * c_\mu(X \setminus A)]] \subseteq [A \cap (X \setminus A)] \cup [A \cap p * i_\mu(A)] = \varphi \cup p * i_\mu(A) = p * i_\mu(A)$. Hence $A \setminus p * Fr_\mu(A) \subseteq p * i_\mu(A)$. On the other hand, suppose $x \notin A \setminus p * Fr_\mu(A)$ then $x \notin A$ or $x \in p * Fr_\mu(A)$. This implies that $x \notin A$ or $x \notin p * i_\mu(A)$. In both the cases, we can conclude that $x \notin p * i_\mu(A)$. Thus $p * i_\mu(A) \subseteq A \setminus p * Fr_\mu(A)$. Hence the proof.

Theorem 5.7: If A is a subset of a GTS (X, μ), then $p * Fr_\mu(A) = p * B_\mu(A) \cup [p * Dr_\mu(A) \setminus p * i_\mu(A)]$ and hence $p * Fr_\mu(A) \subseteq p * B_\mu(A) \cup p * Dr_\mu(A)$

Proof: $p * Fr_\mu(A) = p * c_\mu(A) \setminus p * i_\mu(A) = [A \cup p * Dr_\mu(A)] \setminus p * i_\mu(A)$ (by theorem 3.6) $= [A \cup p * Dr_\mu(A)] \cap [X \setminus p * i_\mu(A)] = [A \cap (X \setminus p * i_\mu(A))] \cup [p * Dr_\mu(A) \cap (X \setminus p * i_\mu(A))] = [A \setminus p * i_\mu(A)] \cup [p * Dr_\mu(A) \setminus p * i_\mu(A)] = p * B_\mu(A) \cup [p * Dr_\mu(A) \setminus p * i_\mu(A)]$.

Clearly, $p * Fr_\mu(A) \subseteq p * B_\mu(A) \cup p * Dr_\mu(A)$.

Theorem 5.8: Let A be a subset of a GTS (X, μ), then

(i) $p * Fr_\mu(p * c_\mu(A)) \subseteq p * Fr_\mu(A)$.

(ii) $p * Fr_\mu(p * i_\mu(A) \subseteq p * Fr_\mu(A)$.

(iii) $A \cup p * Fr_\mu(A) = p * c_\mu(A)$.

Proof: (i) $p * Fr_\mu(p * c_\mu(A)) = [p * c_\mu(p * c_\mu(A))] \setminus [p * i_\mu(p * c_\mu(A))] \subseteq p * c_\mu(A) \setminus p * i_\mu(A) = p * Fr_\mu(A)$.

(ii) $p * Fr_\mu(p * i_\mu(A)) = [p * c_\mu(p * i_\mu(A))] \setminus [p * i_\mu(p * i_\mu(A))] \subseteq p * c_\mu(A) \setminus p * i_\mu(A) = p * Fr_\mu(A)$.

(iii) Now $A \cup p * Fr_\mu(A) = A \cup [p * c_\mu(A) \cap p * c_\mu(X \setminus A)] = [A \cup p * c_\mu(A)] \cap [A \cup p * c_\mu(X \setminus A)] = p * c_\mu(A) \cap [A \cup p * c_\mu(X \setminus A)] \subseteq p * c_\mu(A)$. Conversely, let $x \in p * c_\mu(A)$. Suppose $x \in A$, clearly $p * c_\mu(A) \subseteq A \cup p * Fr_\mu(A)$. Suppose $x \notin A$ then $x \notin p * i_\mu(A)$. Therefore $x \in p * Fr_\mu(A)$. Hence $A \cup p * Fr_\mu(A) = p * c_\mu(A)$.

The converse of the above theorem (i) and (ii) is not true as shown by the succeeding illustration.

Let X = {a, b, c, d} and μ = {φ, {b}, {a, c}, {a, d}, {a, b, c},{a, b, d}, {a, c, d}, b, c, d}, X}.

(i)Let A = {a, d}. Then $p * Fr_\mu(A) = \{c\}$, $p * c_\mu(A) = \{a, c, d\}$ and $p * Fr_\mu(\{a, c, d\}) = \varphi$. Therefore $p * Fr_\mu(A) \not\subset p * Fr_\mu(p * c_\mu(A))$.

(ii) Let A = {b, c}. Then $p * Fr_\mu(A) = \{c\}$, $p * i_\mu(A) = \{b\}$ and $p * Fr_\mu(\{b\}) = \varphi$. Therefore $p * Fr_\mu(A) \not\subset p * Fr_\mu(p * i_\mu(A))$.

Theorem 5.9: Let (X, μ) be a generalized topological space. If $A \in P^*O_\mu(X)$, then the following statements are hold

(i) $p * Fr_\mu(A) = p * B_\mu(X \setminus A)$.

(ii) $p * Fr_\mu(A) \subseteq p * Dr_\mu(A)$.

Proof: (i) Assume that $A \in P^*O_\mu(X)$. Then $p * i_\mu(A) = A$ and by theorem 4.1 (v), $p * B_\mu(A) = \varphi$. By theorem 5.7, $p * Fr_\mu(A) = p * B_\mu(A) \cup [p * Dr_\mu(A) \setminus p * i_\mu(A)] = \varphi \cup [p * Dr_\mu(A) \setminus p * i_\mu(A)] = p * Dr_\mu(A) \setminus A = p * Dr_\mu(A) \cap [X \setminus A] = p * B_\mu(X \setminus A)$ (by theorem 4.4).

(ii) By part (i), $p * Fr_\mu(A) = p * Dr_\mu(A) \setminus A \subseteq p * Dr_\mu(A)$.

The converse of the above theorem (ii) is not true as shown by the following example.

Let us consider, $X = \{a, b, c, d\}$ with $\mu = \{\varphi, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$. Take $A = \{a, b, c\}$. Then $p * Fr_\mu(A) = \{d\}$ and $p * Dr_\mu(A) = \{a, d\} \Rightarrow p * Dr_\mu(A) \not\subset p * Fr_\mu(A)$.

Theorem 5.10: Let A be a subset of a generalized topological space (X, μ). Then $A \in P^*O_\mu(X)$ if and only if $A \cap p * Fr_\mu(A) = \varphi$.

Proof: Let $A \in P^*O_\mu(X)$. Then $X \setminus A$ is μ - pre* - closed and $p * c_\mu(X \setminus A) = X \setminus A$. Now, $A \cap p * Fr_\mu(A) = A \cap [p * c_\mu(A) \cap p * c_\mu(X \setminus A)] = A \cap [p * c_\mu(A) \cap (X \setminus A)] = [A \cap p * c_\mu(A)] \cap [A \cap (X \setminus A)] = A \cap \varphi = \varphi$. Hence $A \cap p * Fr_\mu(A) = \varphi$.

Conversely, assume that $A \cap p * Fr_\mu(A) = \varphi \Rightarrow A \cap [p * c_\mu(A) \cap p * c_\mu(X \setminus A)] = \varphi$ which implies that $A \cap p * c_\mu(X \setminus A) = \varphi \Rightarrow p * c_\mu(X \setminus A) \subseteq X \setminus A$. Since $X \setminus A \subseteq p * c_\mu(X \setminus A)$, $p * c_\mu(X \setminus A) = X \setminus A$. This shows that $X \setminus A \in P^*C_\mu(X)$. Hence $A \in P^*O_\mu(X)$.

Remark 5.4: For any subset A of a GTS (X, μ), then $A \in P^*O_\mu(X)$ if and only if $p * Fr_\mu(A) \subseteq X / A$.

Proof: Suppose $A \in P^*O_\mu(X)$. Now $p * Fr_\mu(A) = p * c_\mu(A) \cap p * c_\mu(X \setminus A) = p * c_\mu(A) \cap X \setminus A \subseteq X / A$. Conversely, $p * Fr_\mu(A) \subseteq X / A \Rightarrow A \cap p * Fr_\mu(A) = \varphi$. Then by theorem 5.10, $A \in P^*O_\mu(X)$.

Corollary 5.1: For any subset A of a GTS (X, μ) then $A \in P^*C_\mu(X)$ if and only if $p * Fr_\mu(A) \subseteq A$.

Theorem 5.11: A subset A of X in a GTS (X, μ) is μ - pre* - closed if and only if $p * Fr_\mu(A) = p * B_\mu(A)$.

Proof: We have $A \in P^*C_\mu(X)$ if and only if $A = p * c_\mu(A)$. Now, $p * Fr_\mu(A) = p * c_\mu(A) \setminus p * i_\mu(A) = A \setminus p * i_\mu(A) = p * B_\mu(A)$. Conversely, $p * Fr_\mu(A) = p * B_\mu(A) \Rightarrow p * c_\mu(A) \setminus p * i_\mu(A) = A \setminus p * i_\mu(A) \Rightarrow p * c_\mu(A) = A$. Hence $A \in P^*C_\mu(X)$.

Theorem 5.12: $p * Fr_\mu(A)$ is always μ - pre* - closed and hence $p * c_\mu(p * Fr_\mu(A)) = p * Fr_\mu(A)$.

Proof: $p * c_\mu(p * Fr_\mu(A)) = p * c_\mu[p * c_\mu(A) \cap p * c_\mu(X \setminus A)] \subseteq [p * c_\mu[p * c_\mu(A)]] \cap [p * c_\mu[p * c_\mu(X \setminus A)]] = p * c_\mu(A) \cap p * c_\mu(X \setminus A) = p * Fr_\mu(A)$. On the other hand, $A \subseteq p * c_\mu(A)$. Replace both sides A by $p * Fr_\mu(A)$ we have $p * Fr_\mu(A) \subseteq p * c_\mu(p * Fr_\mu(A))$. Hence the proof.

Lemma 5.1: If A and B are subsets of a GTS (X, μ) such that $A \cap B = \varphi$ and $A \in P^*O_\mu(X)$ then $p * i_\mu(A) \cap p * c_\mu(B) = \varphi$.

Proof: Suppose that $A \in P^*O_\mu(X)$, $X \setminus A \in P^*C_\mu(X)$ so that $p * c_\mu(X \setminus A) = X \setminus A$. Given $A \cap B = \varphi$, $A \subseteq X \setminus B$ and $B \subseteq X \setminus A$. This implies that $p * c_\mu(A) \subseteq p * c_\mu(X \setminus B)$ and $p * c_\mu(B) \subseteq p * c_\mu(X \setminus A) = X \setminus A$, $A \cap p * c_\mu(B) = \varphi$. Thus $p * i_\mu(A) \cap p * c_\mu(B) = \varphi$.

Theorem 5.13: If A and B are subsets of a GTS (X, μ) such that $A \cap B = \varphi$ and $A \in P^*O_\mu(X)$ then $p * i_\mu(A) \cap p * Fr_\mu(B) = \varphi$.

Proof: By lemma 5.1, $p * i_\mu(A) \cap p * c_\mu(B) = \varphi$. Since $p * Fr_\mu(B) \subseteq p * c_\mu(B)$, we have $p * i_\mu(A) \cap p * Fr_\mu(B) = \varphi$.

Theorem 5.14: Let A and B be two subsets of a GTS (X, μ). Then

(i) $p * Fr_\mu(A \cup B) \subseteq p * c_\mu(A \cup B) \cap p * c_\mu(X \setminus A) \cap p * c_\mu(X \setminus B)$

Proof: $p * Fr_\mu(A \cup B) = p * c_\mu(A \cup B) \cap p * c_\mu(X \setminus (A \cup B)) = p * c_\mu(A \cup B) \cap p * c_\mu[(X \setminus A) \cap (X \setminus B)] \subseteq p * c_\mu(A \cup B) \cap [p * c_\mu(X \setminus A) \cap p * c_\mu(X \setminus B)] = p * c_\mu(A \cup B) \cap p * c_\mu(X \setminus A) \cap p * c_\mu(X \setminus B)$.

Remark 5.5: If A is a μ - pre* - clopen in (X, μ), then X \ A is μ - pre* - clopen in (X, μ).

Theorem 5.15: For a subset A of a GTS (X, μ), $p * Fr_\mu(A)) = \varphi$ if and only if A is μ - pre* - clopen.

Proof: Let $p * Fr_\mu(A)) = \varphi$. Then $p * c_\mu(A) \setminus p * i_\mu(A) = \varphi$. This shows that $p * c_\mu(A) = p * i_\mu(A) = A$, since $p * i_\mu(A) \subseteq p * c_\mu(A)$. Hence A is both μ - pre* - open and μ - pre* - closed.

Conversely, assume that A is μ - pre* - clopen. Then A is both μ - pre* - open and μ - pre* - closed which implies that $p * c_\mu(A) = A = p * i_\mu(A)$. Now $p * Fr_\mu(A) = p * c_\mu(A) \setminus p * i_\mu(A) = A \setminus A = \varphi$.

6. μ-PRE*-EXTERIOR

In this section, we introduce μ-pre*-Exterior and investigate some properties of them.

Definition 6.1: If A is a subset of a GTS (X, μ). Then μ - pre*- Exterior of A is the μ - pre* - interior of X \ A and is denoted by $p *Ext_\mu(A)$.

(ie) $p *Ext_\mu(A) = p * i_\mu(X \setminus A)$ or $X \setminus p * c_\mu(A)$.

Theorem 6.1: Let A be a subset of a GTS (X, μ). Then the following statements are valid

(i) $p *Ext_\mu(X) = \varphi$.

(ii) $A \subseteq B \Rightarrow p *Ext_\mu(B) \subseteq p *Ext_\mu(A)$.

(iii) $p *Ext_\mu(A)$ is μ - pre* - open and hence $p * i_\mu[p *Ext_\mu(A)] = p *Ext_\mu(A)$.

(iv) $Ext_\mu(A) \subseteq p *Ext_\mu(A) \subseteq X\setminus A$, where $Ext_\mu(A)$ is μ - Exterior of A.

(v) $p *Ext_\mu(A) \cap p * i_\mu(A) = \varphi$.

(vi) $p *Ext_\mu(A) \cap p * Fr_\mu(A) = \varphi$.

(vii) $p *Ext_\mu(A) \cup p * Fr_\mu(A) = p * c_\mu(X \setminus A)$.

Proof: (i) Obviously true by definition 6.1.

(ii) $A \subseteq B \Rightarrow X \setminus A \supseteq X \setminus B$. This implies that, $p * i_\mu(X \setminus A) \supseteq p * i_\mu(X \setminus B) \Rightarrow p *Ext_\mu(A) \supseteq p *Ext_\mu(B)$.

(iii)Since $p * i_\mu(X \setminus A)$ is the union of all μ - pre* - open sets contained in X \ A. Thus $p *Ext_\mu(A)$ is μ - pre* - open. By lemma 2.4, $p * i_\mu[p *Ext_\mu(A)] = p *Ext_\mu(A)$.

(iv) $Ext_\mu(A) = i_\mu(X \setminus A) \subseteq p * i_\mu(X \setminus A) \subseteq X \setminus A$.

(v) Since $p * i_\mu(X \setminus A)$ and $p * i_\mu(A)$ are independent, $p *Ext_\mu(A) \cap p * i_\mu(A) = \varphi$.

(vi) $p *Ext_\mu(A) \cap p * Fr_\mu(A) = p *Ext_\mu(A) \cap p * Fr_\mu(X \setminus A)$ (by remark 5.2) $= p * i_\mu(X \setminus A) \cap [p * c_\mu(X \setminus A) \setminus p * i_\mu(X \setminus A)] = p * i_\mu(X \setminus A) \cap [p * c_\mu(X \setminus A) \cap [X \setminus p * i_\mu(X \setminus A)]] = p * i_\mu(X \setminus A) \cap p * c_\mu(X \setminus A) \cap [X \setminus p * i_\mu(X \setminus A)] = \varphi$.

(vii) $p*Ext_\mu(A) \cup p*Fr_\mu(A) = p*Ext_\mu(A) \cup p*Fr_\mu(X \setminus A)$ (by remark 5.2) $= p*i_\mu(X \setminus A) \cup [p*c_\mu(X \setminus A) \setminus p*i_\mu(X \setminus A)] = p*i_\mu(X \setminus A) \cup [p*c_\mu(X \setminus A) \cap [X \setminus p*i_\mu(X \setminus A)] = [p*i_\mu(X \setminus A) \cup p*c_\mu(X \setminus A)] \cap [p*i_\mu(X \setminus A) \cup [X \setminus p*i_\mu(X \setminus A)]] = p*c_\mu(X \setminus A) \cap X = p*c_\mu(X \setminus A)$.

The reverse inclusion of theorem 6.1 (iv) is not true as shown in the succeeding example. Consider $X = \{a, b, c, d\}$ with $\mu = \{\varphi, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$. Take $A = \{b\}$. Then $p*Ext_\mu(A) = \{a, c, d\}$ and $Ext_\mu(A) = \{c, d\}$. Hence $p*Ext_\mu(A) \not\subset Ext_\mu(A)$.

Note that in a GTS (X, μ), $p*Ext_\mu(\varphi) = M_\mu$. Since $p*c_\mu(\varphi)$ is a μ – pre* - closed set contains $X \setminus M_\mu$, $p*Ext_\mu(\varphi) = X \setminus p*c_\mu(\varphi) = M_\mu$.

Theorem 6.2: If A is a subset of a GTS (X, μ), then

(i) $p*Ext_\mu[p*Ext_\mu(A)] = p*i_\mu[p*c_\mu(A)]$

(ii) $p*Ext_\mu[X \setminus p*Ext_\mu(A)] = p*Ext_\mu(A)$

Proof: (i) Obviously true by definition 6.1.

(ii) $p*Ext_\mu[X \setminus p*Ext_\mu(A)] = p*Ext_\mu[p*c_\mu(A)] = p*i_\mu[X \setminus p*c_\mu(A)] = p*i_\mu[p*i_\mu(X \setminus A)] = p*i_\mu(X \setminus A) = p*Ext_\mu(A)$.

Theorem 6.3: In any GTS (X, μ), the following are equivalent

(i) $X \setminus p*Fr_\mu(A) = p*i_\mu(A) \cup p*i_\mu(X \setminus A)$.

(ii) $p*c_\mu(A) = p*i_\mu(A) \cup p*Fr_\mu(A)$

(iii) $p*Fr_\mu(A) = p*c_\mu(A) \cap p*c_\mu(X \setminus A)$

(iv) $X = p*i_\mu(A) \cup p*Ext_\mu(A) \cup p*Fr_\mu(A)$

Proof: Follows from theorem 5.5.

Theorem 6.4: In a GTS (X, μ), $A \in p*C_\mu(X)$ if and only if $p*Ext_\mu[(A)] = X \setminus A$.

Proof: A is μ-pre*-closed $\Leftrightarrow p*c_\mu(A) = A \Leftrightarrow X \setminus p*c_\mu(A) = X \setminus A \Leftrightarrow p*Ext_\mu(A)] = X \setminus A$.

Corollary 6.1: Let A be a subset of a GTS (X, μ). Then $A \in P*O_\mu(X)$ if and only if $p*Ext_\mu[(X \setminus A)] = A$.

Proof: Assume that $A \in P*O_\mu(X)$. By theorem 6.4, $p*Ext_\mu[(X \setminus A)] = A$. Conversely, assume that $p*Ext_\mu[(X \setminus A)] = A$. Thus $p*i_\mu[X \setminus (X \setminus A)] = A$. Hence $A \in P*O_\mu(X)$.

Corollary 6.2: Let (X, μ) be a GTS and A, B be subsets of X. If A, B ∈ $P^*O_\mu(X)$ then $p *Ext_\mu(X \setminus (A \cup B)] = A \cup B$.

Proof: Since arbitrary union of μ-pre*- open sets is μ-pre*- open and by Corrolary: 6.1, we have $p *Ext_\mu[(X \setminus (A \cup B)] = A \cup B$.

In general, $p *Ext_\mu[(X \setminus \cup_{i \in N} U_i] = \cup_{i \in N} U_i$, when U_i is μ-pre*- open.

Remark 6.1: If $A \subseteq B$ but $p * Ext_\mu(A) \nsubseteq p * Ext_\mu(B)$, which is shown by the following example. Let us consider, X = {a, b, c, d} and μ = {φ, {a, b}, {a, c}, {b, c}, {a, b, c}}. Take A = {a, b}, B = {a, b, c}. Then $p * Ext_\mu(A) = \{c\}$ and $p * Ext_\mu(B) = \varphi$. Here $A \subseteq B$ but $p * Ext_\mu(A) \nsubseteq p * Ext_\mu(B)$.

Remark 6.2: Idempotent property of μ - pre* - Exterior is not true in general, which can be illustrated in the following example.

Consider, X= {a, b, c, d} and μ = {φ, {b}, {a, b}, {a, d}, {a, b, d}}. Take A = {a, c}. Then $p * Ext_\mu(A) = \{b, d\}$ and $p * Ext_\mu[p * Ext_\mu(A)] = \{a\}$. Hence $p * Ext_\mu[p * Ext_\mu(A)] \neq p *Ext_\mu(A)$.

Theorem 6.5: Let A and B be subsets of a GTS (X, μ). Then the following statements are hold

(i) $p *Ext_\mu(A \cup B) \subseteq p *Ext_\mu(A) \cup p *Ext_\mu(B)$.

(ii) $p *Ext_\mu(A \cap B) \supseteq p *Ext_\mu(A) \cap p *Ext_\mu(B)$.

Proof: (i)$p *Ext_\mu(A \cup B) = p * i_\mu(X \setminus (A \cup B)) = p * i_\mu([X \setminus A] \cap [X \setminus B]) \subseteq p * i_\mu(X \setminus A) \cap p * i_\mu(X \setminus B) = p *Ext_\mu(A) \cap p *Ext_\mu(B) \subseteq p *Ext_\mu(A) \cup p *Ext_\mu(B)$

(ii)$p *Ext_\mu(A \cap B) = p * i_\mu(X \setminus (A \cap B)) = p * i_\mu([X \setminus A] \cup [X \setminus B]) \supseteq p * i_\mu(X \setminus A) \cup p * i_\mu(X \setminus B) = p *Ext_\mu(A) \cup p *Ext_\mu(B) \supseteq p *Ext_\mu(A) \cap p *Ext_\mu(B)$.

Remark 6.3: The reverse inclusion of the above theorem is not true as shown by the succeeding example. Let X = {a, b, c, d} with μ = {φ, {b}, {a, b}, {a, d}, {a, b, d}}.

(i) Take A = {b} and B = {c}. Then $A \cup B = \{b, c\}$. Here $p *Ext_\mu(A \cup B) = \{a, d\}$, $p *Ext_\mu(A) = \{a, d\}$ and $p *Ext_\mu(B) = \{a, b, d\}$. Hence $p *Ext_\mu(A) \cup p *Ext_\mu(B) \not\subset p *Ext_\mu(A \cup B)$.

(ii) Take A = {a, b} and B = {a, c}. Then $A \cap B = \{a\}$. Here $p *Ext_\mu(A \cap B) = \{b, d\}$, $p *Ext_\mu(A) = \{d\}$ and $p *Ext_\mu(B) = \{b, d\}$. Hence $p *Ext_\mu(A \cap B) \not\subset p *Ext_\mu(A) \cap p *Ext_\mu(B)$.

From the above sections, we can conclude that the monotonicity and idempotent property exists in μ - pre*- derived and μ - pre*- frontier respectively. The sub additive holds in μ - pre*- border and μ - pre*- exterior.

7. CONCLUSION

In this paper we have discussed about μ- pre *- derived, border, frontier and exterior sets in generalized topological spaces. Some of their basic attributes are discussed such as monotoincity property, idempotent property, intersection, union, relation between these operators and got interesting results.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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