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THE WELL-POSEDNESS OF AN ELLIPTIC PROBLEM AND ITS SOLUTION USING THE FINITE ELEMENT METHOD

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Abstract: In this paper, we deal with an elliptic problem with the Dirichlet boundary condition. We operate in Sobolev spaces and the main analytic tool we use is the Lax-Milgram lemma. First, we present the variational approach of the problem which allows us to apply different functional analysis techniques. Then we study thoroughly the well-posedness of the problem. We conclude our work with a solution of the problem using numerical analysis techniques and the free software freefem++.

Keywords: the variational approach; weak solution; Lax-Milgram; freefem++.
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1. INTRODUCTION

Partial differential equations are one of the most efficient tools in the study of physics phenomena since many physics problems are modeled mathematically. The theory of modern PDE (partial differential equations) is closely related to functional analysis. The use of functional analysis tools is essential in the study of partial differential problems, especially the elliptic ones. Finding the classical solution for PDE problems with boundary conditions is a real challenge and to ease the burden of continuity and differentiability in classic PDE we deal with Sobolev spaces and weak

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solutions. We study the existence, the unicity, and the stability of the problem, its well-posedness with the variational approach. We use the celebrated theorem of Lax-Milgram that is formulated as follows:

Theorem 1.1 [2] (Lax - Milgram lemma). Let H be a Hilbert space and let: $H\ x\ H \to \mathbb{R}$ be a bilinear form.

1) B is continuous, it exists C > 0 such that $|B[u,v]| \leq C\|u\|\|v\|$, for every $u, v \epsilon H$.
2) B is coersive, it exists $\beta > 0$ such that $\beta\|u\|^2 \leq B[u,u]$, for every $u \epsilon H$.

then for every $f \epsilon H$, there exists a unique element $u \epsilon H$ such that

$$B[u,v] = (f,v) \text{ for every } v \epsilon H.$$

Moreover,

$$\|u\| \leq \beta^{-1}\|f\|.$$

Definition 1.1[2] Let H be a Hilbert space on real numbers. A linear operator $A: H \to H$ is called coersive if there exists a constant $\beta > 0$ such that

$$(Au, u) \geq \beta\|u\|^2.$$

Theorem 1.2 [2] Let H be a Hilbert space on real numbers and $A: H \to H$ linear, bounded and coercive operator. Then for every $f \epsilon H$ there exists a unique $u = A^{-1}f \epsilon H$, such that $Au = f$. The inverse operator satisfies the inequality

$$\|A^{-1}\| \leq \frac{1}{\beta}.$$

2. MATERIALS AND METHODS

Homogeneous second-order elliptic operators. [2] We begin by studying solutions to the elliptic boundary value problem

$$\begin{cases} Lu = f, \ x \epsilon \Omega & (1.1) \\ u = 0, \quad x\epsilon\ \partial\Omega & (1.2) \end{cases}$$

assuming that the differential operator L contains only second-order terms:

$$Lu = -\sum_{i.j=1}^{n}\left(a^{ij}(x)u_{x_i}\right)_{x_j} \qquad (1.3)$$

A weak solution of (1.1)-(1.2) is a function $u \epsilon H_0^1(\Omega)$ such that

$$B[u,v] = \langle f, v\rangle_{L^2} \text{ for all } v \in H_0^1(\Omega),$$

where $B: H_0^1(\Omega)\ x\ H_0^1(\Omega) \to \mathbb{R}$ is the continuous bilinear form

$$B[u,v] = \int_\Omega \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j}\ dx.$$

Lemma 2.1 [2] Let $\Omega \subset \mathbb{R}^{\boldsymbol{n}}$ be a bounded open set. Then for every $f \in L^2(\Omega)$, the boundary value problem (1.1) - (1.2) has a unique weak solution $u \in H_0^1(\Omega)$. The corresponding map $f \to u$ is a compact linear operator from $L^2(\Omega)$ into $H_0^1(\Omega)$.

Proof. By the Rellich-Kondrachov theorem, the canonical embedding $\imath : H_0^1(\Omega) \to L^2(\Omega)$ is compact. Hence, its dual operator $\imath^*$ is also compact. Since $H_0^1(\Omega)$ and $L^2(\Omega)$ are Hilbert spaces, they can be identified with their duals. We thus obtain the following diagram:

$$i^*: [L^2(\Omega]^* = L^2(\Omega) \to [H_0^1(\Omega)]^* = H_0^1(\Omega).$$

For each $f \in L^2(\Omega)$, the definition of dual operator yields

$$(i^* f, v)_{H^1} = (f, iv)_{L^2} = (f, v)_{L^2}, \forall f \in L^2(\Omega) \text{ and } v \in H_0^1(\Omega).$$

Theorem 2.1 (Unique solution of the elliptic boundary value problem) [2] Let $\Omega \subset \mathbb{R}^{\boldsymbol{n}}$ be a bounded open set. Let the operator (1.3) be uniformly elliptic, with coefficients $a^{ij} \in L^\infty(\Omega)$. Then for every $f \in L^2(\Omega)$ the boundary value problem (1.1)– (1.2) has a unique weak solution $u \in H_0^1(\Omega)$. The corresponding solution operator, denoted as $L^{-1} : f \to u$ is a compact linear operator from $L^2(\Omega)$ into $H_0^1(\Omega)$.

Proof. 1. The continuity of B is clear. Indeed,

$$|B[u,v]| \le \sum_{i,j=1}^n \int_\Omega \left|a^{ij} u_{x_i} v_{x_j}\right| dx \le \sum_{i,j=1}^n \left\|a^{ij}\right\|_{L^\infty} \left\|u_{x_i}\right\|_{L^2} \left\|v_{x_j}\right\|_{L^2} \le C\|u\|_{H^1}\|v\|_{H^1}.$$

2. B is strictly positive definite, i.e., there exists $\beta > 0$ such that

$$B[u,u] \ge \beta\|u\|_{H^1(\Omega)}^2 \text{ for all } u \in H_0^1(\Omega).$$

Indeed, since Ω is bounded, Poincar$\acute{e}$ inequality yields the existence of a constant C_p such that $\|u\|_{L^2(\Omega)}^2 \le C_p \int_\Omega |\nabla u|^2 dx$, for all u $\in H_0^1(\Omega)$.

On the other hand, the uniform ellipticity condition implies

$B[u,u] = \int_\Omega \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} dx \ge \int_\Omega \theta \sum_{i=1}^n u_{x_i}^2 dx = \theta \int_\Omega |\nabla u|^2\, dx$, where $\nabla^2 u = \sum_{i=1}^n u_{x_i}^2$

and $\int_\Omega |\nabla u|^2 = \|\nabla u\|^2_{L^2(\Omega)}$ so $B[u,u] \geq \theta \|\nabla u\|^2_{L^2(\Omega)}$.

The two above inequalities yield to the following inequality

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq (C_p + 1)\|\nabla u\|^2_{L^2(\Omega)} \leq \frac{(C_p+1)}{\theta} B[u,u] .$$

This proves that the operator B is strictly positive definite for $\beta = \frac{\theta}{(C_p+1)}$.

From the Lax – Milgram lemma, for every $\tilde{f} \in H_0^1(\Omega)$ there exists a unique element $u \in H_0^1(\Omega)$, such that

$$B[u,v] = (\tilde{f}, v)_{H^1} \text{ for all } v \in H_0^1(\Omega).$$

Moreover, the map $\Lambda : \tilde{f} \to u$ is continuous,

$$\|u\|_{H^1} \leq \frac{1}{\beta} \|\tilde{f}\|_{H^1}.$$

Choosing $\tilde{f} = i^* f \in H_0^1(\Omega)$ where $i^*: L^2(\Omega) \to H_0^1(\Omega)$, $f \in L^2(\Omega)$ we have

$$B[u,v] = (i^* f, v)_{H^1} = (f, iv)_{L^2} = (f, v)_{L^2} \text{ for all } v \in H_0^1(\Omega).$$

By the definition, u is a weak solution of (1.1) – (1.2).

To prove that the solution operator $L^{-1}: L^2(\Omega) \to H_0^1(\Omega)$ is compact, we consider the operators

$$i^*: L^2(\Omega) \to H_0^1(\Omega),\ \Lambda : H_0^1(\Omega) \to H_0^1(\Omega).$$

By lemma 2.1 the linear operator i^* is compact. Moreover, Λ is continuous ($H_0^1(\Omega) \subset L^2(\Omega)$) and compact. Therefore, the composition $L^{-1} = \Lambda(i^*)$ is compact.

3. The Solution of an Elliptic Problem with Dirichlet Boundary Conditions

Let $\Omega = \{(x,y);\ x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$. Prove that, for every bounded measurable function $f = f(x,y)$

the problem $\begin{cases} u_{xx} + x u_{xy} + u_{yy} = f \text{ on } \Omega & (1.4) \\ u = 0 \quad\quad \text{ on } \partial\Omega & (1.5) \end{cases}$

has a unique weak solution.

Proof. Let $L_1 u = u_{xx} + x u_{xy} + u_{yy}$ be the linear, second-order differential operator. The partial differential equation can be written as $Lu = - f$ where

$$Lu = -((u_x)_x + (xu_x)_y + (u_y)_y),\ -f = f_1 \epsilon L^2(\Omega)$$

and $\Omega \subset \mathbb{R}^2$ is a bounded open set. First, we prove that the operator Lu is uniformly elliptic. There exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{2} a^{ij}(x)\, \xi_i \xi_j \geq \theta |\xi|^2,\ \forall\, x \epsilon\, \Omega\,, \xi \epsilon\, \mathbb{R}^2$$

$\sum_{i,j=1}^{2} a^{ij}(x)\, \xi_i \xi_j = a^{11}(x)\xi_1^2 + a^{12}(x)\xi_1\xi_2 + a^{21}(x)\xi_2\xi_1 + a^{22}(x)\xi_2^2 = \xi_1^2 + x\,\xi_1\xi_2 + \xi_2^2.$

It suffices to check that the quadratic form Q $(\xi_1, \xi_2) = \xi_1^2 + x\xi_1\xi_2 + \xi_2^2$ is strictly positive definite for $(x, y)\, \epsilon\, \Omega$.

The quadratic form above is given as Q(x_1, x_2)= ax_1^2 + 2b $x_1x_2 + cx_2^2$ where a = 1, b = $\frac{x}{2}$, c = 1.

Q$(\xi_1, \xi_2) = \xi_1^2 + x\xi_1\xi_2 + \xi_2^2 = (\xi_1, \xi_2)\begin{pmatrix} 1 & \frac{x}{2} \\ \frac{x}{2} & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ where A = $\begin{pmatrix} 1 & \frac{x}{2} \\ \frac{x}{2} & 1 \end{pmatrix}$ is the symmetric matrix of the quadratic form and the determinant $\begin{vmatrix} 1 & \frac{x}{2} \\ \frac{x}{2} & 1 \end{vmatrix} = 1 - \frac{x^2}{4}$ is called discriminant of Q. It is easy to see that

$$\xi_1^2 + x\xi_1\xi_2 + \xi_2^2 = 1(\xi_1 + \frac{\frac{x}{2}}{1}\,\xi_2)^2 + (\frac{1 - \frac{x^2}{4}}{1})\,\xi_2^2 = (\xi_1 + \frac{x}{2}\,\xi_2)^2 + (1 - \frac{x^2}{4})\,\xi_2^2.$$

We use the notation D_1 = a =1 and D_2 = ac – b^2 =$1 - \frac{x^2}{4}$. If $D_1 > 0$ $(1 > 0)$, $D_2 > 0$ $(1 - \frac{x^2}{4} > 0)$ then the form is of $x^2 + y^2$ type, so is positive definite for $\forall\ (x, y) \neq 0$.

We derive the variational formulation of the elliptic problem in the unit disc.

The set $C_0^\infty(\Omega)$ denotes the space of test functions , infinitely differentiable in Ω that vanish in some neighborhood of $\partial\Omega$[4]. Now we multiply the equation $Lu = f_1$ by an arbitrary v $\epsilon\, C_0^\infty(\Omega)$ and integrate over Ω. Then,

$$\int_\Omega (Lu)v\, dx\, dy = \int_\Omega f_1 v\, dxdy \quad \forall\, v\, \epsilon\, C_0^\infty(\Omega),$$

$$\int_\Omega -\left((u_x)_x + (xu_x)_y + \left(u_y\right)_y\right) v\, dxdy = \int_\Omega f_1 v\, dxdy \quad \forall\, v\, \epsilon\, C_0^\infty(\Omega),$$

$\int_\Omega (u_x)_x v\, dxdy = \int_\Omega u_{xx} v\, dxdy = \int_{\partial\Omega} v\, \frac{\partial u}{\partial n}\, ds - \int_\Omega u_x v_x dxdy = - \int_\Omega u_x v_x dxdy$ (from Green's first identity [1] where $\int_{\partial\Omega} v\, \frac{\partial u}{\partial n}\, ds = 0$ and v = 0 on $\partial\Omega$).

For $\int_\Omega (xu_x)_y \, v \, dxdy$, by integration by parts formula [1] we obtain

$\int_\Omega (xu_x)_y \, v \, dxdy = \int_{\partial\Omega} xu_x v \, n \; ds$ - $\int_\Omega \mathrm{x}u_x \, v_y \, dx\mathrm{dy}$= - $\int_\Omega \mathrm{x}u_x \, v_y \, dx \, dy$.

$\int_\Omega (u_y)_y \, v \, dxdy = \int_\Omega u_{yy} v \, dxdy \;\; = \;\; \int_{\partial\Omega} v \, \frac{\partial u}{\partial n} \, ds \;\; - \;\; \int_\Omega u_y v_y \, dxdy \; = -\int_\Omega u_y v_y \, dxdy$ (from Green's first identity).

$\int_\Omega -\left((u_x)_x + \mathrm{x}(u_x)_y + (u_y)_y \right) v \, dxdy = \int_\Omega (u_x v_x + xu_x v_y + u_y v_y) \, dxdy = \int_\Omega f_1 v \, dx \, dy$

$\forall \, v \in C_0^\infty(\Omega)$.

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ [1][3], the same integral identity remains valid for every $v \in H_0^1(\Omega)$

$$\int_\Omega (u_x v_x + xu_x v_y + u_y v_y) \, dxdy = \int_\Omega f_1 v \, dxdy \;\; \forall \, v \in H_0^1(\Omega).$$

Using the notation

$B[u,v]= \int_\Omega (u_x v_x + xu_x v_y + u_y v_y) \, dx \, dy$ for the bilinear form and $l_{f_1}(v) = \int_\Omega f_1 v \, dxdy = \langle f_1, v \rangle_{L^2}$ for the linear form in $H_0^1(\Omega)$, the variational or weak formulation of the (1.4)-(1.5) boundary value problem will be :

Find $u \in H_0^1(\Omega)$ such that

$$B[u,v] = l_{f_1}(v), \;\; \forall \, v \in H_0^1(\Omega), \;\; \text{where } B: H_0^1(\Omega) \; x \; H_0^1(\Omega) \to \mathbb{R} \, .$$

Let us prove now the existence and uniqueness of the weak solution to the elliptic boundary value problem (1.4)-(1.5). The bilinear form B satisfies all the assumptions of the Lax-Milgram lemma.

1. The bilinear form B is continuous. Indeed,

$|B[u,v]| = \left| \int_\Omega (u_x v_x + xu_x v_y + u_y v_y) dxdy \right| v \leq \int_\Omega |u_x v_x| \, dxdy + \int_\Omega |xu_x v_y| dxdy + \int_\Omega |u_y v_y| \, dxdy \leq \|u_x\|_{L^2} \|v_x\|_{L^2} + \|x\|_{L^\infty} \|u_x\|_{L^2} \|v_y\|_{L^2} + \|u_y\|_{L^2} \|v_y\|_{L^2} \leq C \|u\|_{H^1} \|v\|_{H^1}$.

2. B is strictly positive definite.

There exists $\beta > 0$ such that

$$B[u,u] \geq \beta \|u\|_{H^1(\Omega)}^2 \;\; \text{for all } u \in H_0^1(\Omega).$$

Since $\mathbf{\Omega}$ is bounded, from the Poincaré inequality there exists a constant C_p such that

$\|u\|_{L^2(\Omega)}^2 \leq C_p \int_\Omega |\nabla u|^2 dx$ for all $u \in H_0^1(\Omega)$, where $\int_\Omega |\nabla u|^2 dx = \|\nabla u\|^2{}_{L^2(\Omega)}$.

On the other hand, the uniform ellipticity condition implies

$B[u,u] = \int_\Omega (u_x u_x + x u_x u_y + u_y u_y)\, dxdy \geq \theta \int_\Omega \left(u_x^2 + u_y^2\right) dxdy =$

$\theta \int_\Omega |\nabla u|^2 dx = \theta \|\nabla u\|^2{}_{L^2(\Omega)}$.

Combining the two above inequalities we find

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq (C_p + 1)\|\nabla u\|_{L^2(\Omega)}^2 \leq \frac{(C_p+1)}{\theta}\,\mathrm{B}[u,u].$$

This proves B is strictly positive definite with $\beta = \frac{\theta}{(C_p+1)}$.

By the Lax-Milgram lemma, for every $\widetilde{f} \,\epsilon\, H_0^1(\Omega)$ there exists a unique element $u \,\epsilon\, H_0^1(\Omega)$ such that

$$B[u,v] = (\widetilde{f}\,, v)_{H^1} \text{ for all } v \,\epsilon\, H_0^1(\Omega).$$

Moreover

$$\|u\|_{H^1} \leq \frac{1}{\beta}\left\|\tilde{f}\right\|_{H^1}.$$

Choosing $\widetilde{f} = i^*(f_1) \,\epsilon\, H_0^1(\Omega)$ where $i^*: L^2(\Omega) \to H_0^1(\Omega)$ is a compact operator, we have

$$B[u,v] = (i^*(f_1), v)_{H^1} = (f_1, v)_{L^2} \text{ for all } v \,\epsilon\, H_0^1(\Omega).$$

So there exists a unique element $u \,\epsilon\, H_0^1(\Omega)$ such that

$$B[u,v] = (f_1, v)_{L^2}, \text{ for all v } \epsilon\, H_0^1(\Omega) \qquad (1.6)$$

$$\Leftrightarrow \quad \int_\Omega \left(u_x v_x + x u_x v_y + u_y v_y\right) dxdy = \int_\Omega f_1\, v\, dxdy \text{ for all v } \epsilon\, H_0^1(\Omega).$$

By the definition u is a weak solution of (1.4)-(1.5).

The two formulations (1.4)-(1.5) and (1.6) are equivalent. Assume that (1.6) is true. Integrating by parts in the reverse order,

$\int_\Omega \left(u_x v_x + x u_x v_y + u_y v_y\right) dxdy = \int_\Omega \left\{-(u_x)_x \mathrm{v} - \mathrm{x}(u_x)_y \mathrm{v} - \left(u_y\right)_y \mathrm{v}\right\} dx\, dy = \int_\Omega f_1\, v\, dx\, dy$

$\forall\, v \,\epsilon\, H_0^1(\Omega)$,

$\int_\Omega \left\{-(u_x)_x \mathrm{v} - \mathrm{x}(u_x)_y \mathrm{v} - \left(u_y\right)_y \mathrm{v} + f\, v\right\} dxdy = 0 \quad \forall\, v \,\epsilon\, C_0^\infty(\Omega)$,

$\int_\Omega \left\{-(u_x)_x - \mathrm{x}(u_x)_y - \left(u_y\right)_y + f\right\} v\, dxdy = 0$.

By using the fundamental lemma of the calculus of variations [5] we recover

$\left\{-(u_x)_x - \mathrm{x}(u_x)_y - \left(u_y\right)_y + f\right\} v = 0.$

The arbitrarity of v implies

$-(u_x)_x - \mathrm{x}(u_x)_y - \left(u_y\right)_y + f = 0$ in Ω

$-(u_x)_x - \mathrm{x}(u_x)_y - \left(u_y\right)_y = -f \ \ (Lu = -f = f_1)$

$(u_x)_x + \mathrm{x}(u_x)_y + \left(u_y\right)_y = \ f$ in Ω

$\mathrm{u}_{xx} + \mathrm{x}\mathrm{u}_{xy} + \mathrm{u}_{yy} = \ f$ in Ω.

The boundary condition $u = 0$ on $\partial\Omega$ means that $u \in H_0^1(\Omega)$.

So, for every bounded, measurable function $f = (x, y)$ for the boundary value problem

$$\begin{cases} u_{xx} + xu_{xy} + u_{yy} = f \ on \ \Omega \\ u = 0 \qquad\qquad\qquad on \ \partial\Omega \end{cases}$$

there exists a unique weak solution.

The solution depends continuously on the initial data.

Let $l_{f_1}(v) = \int_\Omega f_1 v \, dxdy = \langle f_1, v\rangle_{L^2}$ be the linear form generated from $f_1 \in L^2(\Omega)$. It is easy to see that l_{f_1} is bounded in $H_0^1(\Omega)$. Using Schwarz and Poincaré inequality we obtain

$$\left|l_{f_1}(v)\right| = \left|\int_\Omega f_1 v \, dxdy\right| \leq \|f_1\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq C_p\|f_1\|_{L^2(\Omega)}\|\nabla v\|_{L^2} = C_p\|f_1\|_{L^2(\Omega)}\|\mathrm{v}\|_{H_0^1(\Omega)}$$

$$\left|l_{f_1}(v)\right| \leq C\|\mathrm{v}\|_{H_0^1(\Omega)}.$$

The space of bounded linear functionals on $H_0^1(\Omega)$ is given by $(H_0^1(\Omega))^* = H^{-1}(\Omega)$ [6] and from the definition of the norm in $H^{-1}(\Omega)$ we find

$$\left\|l_{f_1}\right\|_{H^{-1}} = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\left|\langle f_1, v\rangle_{L^2}\right|}{\|v\|_{H_0^1(\Omega)}}.$$

Using the inequality above we have

$$\left\|l_{f_1}\right\|_{H^{-1}} \leq C_p\|f_1\|_{L^2(\Omega)}.$$

Now let us show that for $l_{f_1} \in (H_0^1(\Omega))^*$ the problem is well-posed.

$$\left\|l_{f_1}\right\|_{H^{-1}} = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{l_{f_1}(v)}{\|v\|_{H_0^1(\Omega)}} = sup \frac{B[u,v]}{\|v\|_{H_0^1(\Omega)}} \geq \frac{B[u,u]}{\|u\|_{H_0^1(\Omega)}} \geq \beta\|u\|_{H_0^1(\Omega)}$$

and therefore

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\beta}\left\|l_{f_1}\right\|_{H^{-1}}.$$

By the definition of the well-posedness [7], the elliptic problem (1.4)-(1.5) is well-posed, more specifically it has a uniquely determined solution that depends continuously on its initial data.

4. THE FINITE ELEMENT METHOD

In this section, we focus on the numerical resolution of the elliptic problem by the Finite Element Method (FEM) and the construction of it. Later on, we shall illustrate the power of this method by solving this problem with the free software freefem++.

The first step in the construction of a finite element method for the elliptic boundary value problem is done, we have already converted the problem into its weak formulation:

Find $u \in H_0^1(\Omega)$ such that

$$B[u,v] = l_{f_1}(v)\ , \quad \forall\ v \in H_0^1(\Omega) \tag{1.7}$$

where $B[u,v]$= $\int_\Omega\left(u_x v_x + x u_x v_y + u_y v_y\right)\, dx\, dy$and $l_{f_1}(v) = \int_\Omega f_1 v\, dxdy.$

Let $V = H_0^1(\Omega)$ donate the Hilbert Space. From the Lax-Milgram lemma, we know that the weak solution u to the elliptic problem exists and is unique. The approximate solution can be found by using this particular class of numerical techniques.

The second step in the construction of the FEM is to replace V (which is infinite-dimensional) [8][10] by a finite – dimensional subspace $V_h \subset V$ which consists of continuous piecewise polynomial functions of a fixed degree associated with a subdivision of the computational domain [9]. In V_h we can solve the variational problem and hence define a finite element approximation u_h.

4.1 Construction of a triangulation

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$, thus Ω can be exactly covered by a finite number of triangles [9]. Let $\mathcal{T}_h$ be a triangulation of Ω. $\mathcal{T}_h$ is a set T_i , i =1,2,….. n_t (n_t the numbers of triangles) of triangles such that $\Omega = \bigcup_{i=1}^{n_t} T_i$. The corners of the triangles are called the nodes. In freefem++ we call $\mathcal{T}_h$ by mesh.

4.2 Construction of a piecewise polynomial function space on this domain (FE-space)

Let T be a triangle, the space of linear functions on T, $\mathcal{P}_1(T)$ is defined by

$$\mathcal{P}_1(T) = \{v: v = c_0 + c_1 x + c_2 y;\ (x,y) \in T,\ c_0, c_1, c_2 \in \mathbb{R}\}.$$

On each triangle T_i, i =1,2,….. n_t, a function v_h is simply required to belong to $\mathcal{P}_1(T_i)$. Requiring also continuity of v_h between neighboring triangles, we obtain the space of all continuous piecewise linear polynomials V_h which is a finite-dimensional subspace of V defined by

$$V_h = \{v_h: v_h \in C(\Omega), v_h|_{T_i} \in \mathcal{P}_1(T_i), \forall\, T_i \in \mathcal{T}_h, v_h|_{\partial\Omega} = 0\}.$$

With this choice of approximation space, the finite element method takes the form:

Find $u_h \in V_h$ such that

$$B[u_h, v_h] = l_{f_1}(v_h)\,,\quad \forall\, v_h \in V_h \tag{1.8}$$

$$\int_\Omega \big((u_h)_x (v_h)_x + x(u_h)_x (v_h)_y + (u_h)_y (v_h)_y\big)\, dxdy = \int_\Omega f_1 v_h\, dxdy,\quad \forall\, v_h \in V_h.$$

4.3 Derivation of a Linear System of Equation

Suppose that

$$\dim(V_h) = n_p \ \text{ and } \ V_h = span\{\varphi_1, \varphi_2, \ldots\ldots \varphi_{n_p}\}$$

where φ_j, $j = 1,2, \ldots n_p$ (linearly independent) are the basis functions and

$$\varphi_j(N_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad i,j = 1, \ldots\ldots n_p$$

where N_i , $i = 1, \ldots\ldots n_p$ are the interior nodes in the mesh.

With each interior node we associate a basis function which is equal to 1 at that node and equal to 0 at all the other nodes. Each basis function φ_j is a continuous function on Ω and linear in each of the triangles (piecewise linear). These basis functions are also called hat functions. Here n_p is the number of internal nodes in the mesh since the functions of V_h vanish on the boundary.

Now, using the hat function basis we note that any function v_h in V_h can be written as

$$v_h = \sum_{i=1}^{n_p} \alpha_i \varphi_i\,,\ i = 1, \ldots\ldots n_p,$$

where $\alpha_i = v(N_i)$, i = 1,2, …, n_p are the nodal values of v_h.

We note that the finite element method (1.8) is equivalent to:

$$B[u_h, \varphi_i] = l_{f_1}(\varphi_i), \text{ i} = 1, 2, \ldots, n_p \tag{1.9}$$

Expressing the finite element approximation u_h in terms of the basis functions $\{\varphi_j\}_{j=1}^{n_p}$ we can write

$$u_h = \sum_{j=1}^{n_p} \xi_j \varphi_j \tag{1.10}$$

where ξ_j , j = 1,2, …,n_p, are to be determined.

Inserting (1.10) into (1.9) we get

$$B\left[\sum_{j=1}^{n_p} \xi_j \varphi_j \ , \varphi_i\right] = l_{f_1}(\varphi_i), \ i = 1, 2, \ldots, n_p$$

$$\int_\Omega \left((\textstyle\sum_{j=1}^{n_p} \xi_j \varphi_j)_x (\varphi_i)_x + x(\sum_{j=1}^{n_p} \xi_j \varphi_j)_x (\varphi_i)_y + (\sum_{j=1}^{n_p} \xi_j \varphi_j)_y (\varphi_i)_y \right) dxdy$$

$$= \int_\Omega f_1 \varphi_i \, dxdy$$

$$= \sum_{j=1}^{n_p} \xi_j \int_\Omega \big((\varphi_j)_x (\varphi_i)_x + x(\varphi_j)_x (\varphi_i)_y + (\varphi_j)_y (\varphi_i)_y \big) \, dxdy.$$

Using the notation

$$A_{ij} = \int_\Omega \big((\varphi_j)_x (\varphi_i)_x + x(\varphi_j)_x (\varphi_i)_y + (\varphi_j)_y (\varphi_i)_y \big) \, dxdy, \ i, j = 1, 2, \ldots, n_p$$

$$b_i = \int_\Omega f_1 \varphi_i \, dxdy, \ i = 1, 2, \ldots, n_p$$

we have

$$b_i = \sum_{j=1}^{n_p} A_{ij} \xi_j \ \text{ i} = 1, 2, \ldots, n_p \tag{1.11}$$

which is a linear system for the unknowns ξ_j.

Solving the linear system (1.11) we obtain the unknowns ξ_j, and thus u_h.

5. PRACTICAL IMPLEMENTATION

Using the software program freefem++ to solve the problem numerically

The generation of a good mesh and the definition of the corresponding finite element space may be, in practice, a very difficult task [10], so an easy way to do it is by using the free software freefem++, that is based on the finite element method and executes all the usual steps required by this method we described on section 4. We note that part of the material of this section has been adapted from [11,12].

The standart process in freefem++ for solving the problem (1.8) is the following:

Step 1 (Define the geometry). For defining the geometry of the given domain, below is used the parametric method. The boundary $C = \partial\Omega$ is

$$\mathrm{C} = \{(x, y), x = cost, y = sint, 0 \leq t \leq 2\pi\}$$

and is defined by the analytic description such as

$$border\ C(t = 0, 2 * pi)\{x = cos(\mathrm{t}); y = sin(\mathrm{t}); \}.$$

We define the boundary of Ω in freefem++ by using the keyword border (line 1 below).

Step 2 (Mesh Generation) The triangulation T_h of Ω is automatically generated by using the keyword buildmesh.

$$mesh\ \mathrm{Th} = buildmesh\big(\mathrm{C}(50)\big);$$

The parameter 50 dictates the number of uniform discretization points taken on the curve C as in Fig.1.1. Refinement of the mesh are done by increasing the number of points on C [11].

Once the mesh is built we use the command below to visualize and save it :

$$plot(\mathrm{Th}, ps = \text{"malla.eps"}).$$

The name $\mathcal{T}_h$ referes to the family of triangles shows in Fig. 1.1.

Step 3 (Construct and solve the problem). First, we define a finite element space (where we want to solve the problem) on the constructed mesh by using the command

$$fespace\ \mathrm{Vh}(\mathrm{Th}, P1).$$

$P1$ means that we use the $P1$ finite elements (continuous piecewise linear on $\mathcal{T}_h$).

Once we have a finite element space we can define variables in this space.

$$\mathrm{Vh\ uh, vh;}$$

That means that the unknown function u_h and test functions v_h belongs to Vh.

We next define the given function f_1 by using the command

$$func\ f_1 = x*y;$$

The function f_1 is defined analytically by using the keyword func.

(Variational problem) 9th – 12th lines in the code written below, defines the bilinear and linear form of equation (1.4) and its Dirichlet boundary condition (1.5) and (1.8) is written with $dx(u_h) = {}^{\partial u_h}/_{\partial x}, dy(u_h) = {}^{\partial u_h}/_{\partial y}$ and

$$\int_\Omega \big((u_h)_x (v_h)_x + x(u_h)_x (v_h)_y + (u_h)_y (v_h)_y\big)\, dxdy$$

$$\rightarrow\ int2d(\mathrm{Th})\big(dx(\mathrm{uh}) * dx(\mathrm{vh}) +\ x * dx(\mathrm{uh}) * dy(\mathrm{vh}) + dy(\mathrm{uh}) * dy(\mathrm{vh})\big)$$

$\int_\Omega f_1 v_h\, dxdy\ \rightarrow\ int2d(\mathrm{Th})(\mathrm{f}_1 * \mathrm{vh})$.

We declare and solve the problem (at the previously defined space and mesh) at the same time by using the keyword solve as in line 9 below.

Step 4 (Visualize the result) Now that the variable uh contains the numerical result of our problem we can visualize it by just typing $plot(uh)$ and we can see the obtained result as illustrated in Fig 1.2.

```
1.  border C(t=0,2*pi){x=cos(t);y=sin(t);} // boundary of the domain
2.  mesh Th=buildmesh(C(50)); // mesh with 50 points on the boundary
3.  savemesh(Th,"mesh_mallal.msh" ); // to save the mesh data
4.  plot(Th,ps="malla.eps");// to plot and save the mesh
5.  fespace Vh(Th,P1); // P1 Lagrange finite elements
6.  Vh uh,vh; // uh,vh belong to Vh
7.  func f1=x*y; // source term
8.  //  solving the variational formulation of the problem
9.  solve Problem(uh,vh)=
10. int2d(Th)(dx(uh)*dx(vh)+ x*dx(uh)*dy(vh)+dy(uh)*dy(vh))  //bilinear form
11. -int2d(Th)(f1*vh)  // linear form
12. +on(C,uh=0); // Dirichlet condition
13. plot(uh);
```

Code implemented in freefem++.

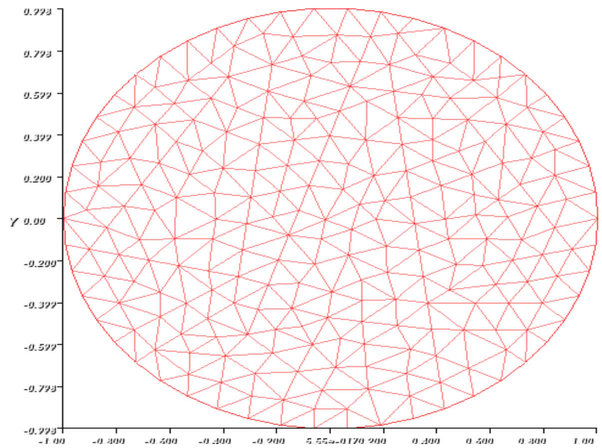

Figure 1.1 The triangulation of the unit disc

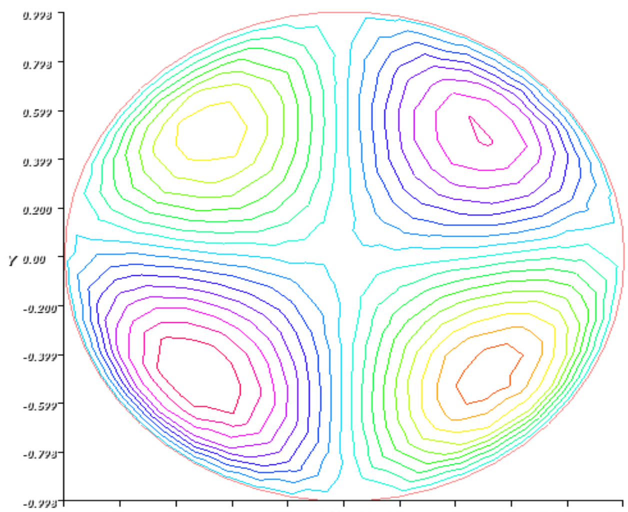

Figure 1.2 Solution using freefem++

6. CONCLUSION

In this paper, we presented an elliptic problem in the unit disc domain with Dirichlet boundary condition. We used the variational method to prove that the solution exists and depends continuously on the initial data. The proof was given in detail to be as helpful as possible for the new researchers in this field. We concluded the theoretical study proving the well-posedness of the given elliptic problem with Dirichlet boundary conditions in the unit disc.

Then we moved forward in our research presenting an approximate solution using numerical analysis methods, more specifically the finite element method but we didn't intend to solve the problem using FEM. Emphasis was placed on the numerical resolution of it by using the free software freefem++, which executes all the steps presented on this method and allows us to obtain quickly and in an easy way the numerical result of the elliptic problem.

CONFLICT OF INTERESTS

The authors declare there is no conflict of interests.

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