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SOME NEW CONCEPTS IN SOFT NANO TOPOLOGICAL SPACES

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Abstract: The notions of soft nano subspaces, soft nano closure and soft nano interior in soft nano topological spaces are introduced. Also, a new classes of sets namely weakly soft nano g-closed sets, weakly soft nano g-open sets and corresponding-closure and interior are introduced and their properties are investigated.. Further, the inter-relationship between these new classes of soft nano sets with existing soft nano sets in soft nano topological spaces are studied.

Keywords: soft nano subspaces; weakly soft nano g-closed sets; weakly soft nano g-open sets.

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1. INTRODUCTION

Soft set theory was introduced by Molodtsov [12] to overcome the drawbacks of theory of probability, the interval mathematics and theory of fuzzy sets. A soft set over a universal set U is a structure (F, E) such that F: E→P(U), where E is a parameter set and P(U) is the power set of U. Shabir and Naz [13] introduced the concept of soft topological spaces and studied the notions of soft open sets, soft closed sets, soft closure, soft interior, soft neighbourhood of a point and soft separation axioms. Theoretical studies and developments are made by many researchers in the

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field of soft sets and soft topological spaces [1], [3], [7], [8], [9], [10], [11]. Thivagar [14] introduced the concept of nano topology, using approximation spaces. Based on these backgrounds, Benchalli et al. [7] initiated the notion of soft nano topological spaces, with the utilization of soft set equivalence relation on the universal set and soft approximation spaces on a soft subset of the universal set.

Let U be the initial universal set, E is the set of parameters whose elements are attributes, characteristics or properties of the objects in U. Then, [4] the triplet (U, R, E), where R is a soft equivalence relation on U, is said to be the soft approximation space for any subset X of U if:

(i) The soft lower approximation of X corresponding to R and E is the set of all objects, denoted and defined by $(L_R(X), E) = \cup \{R(x): R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by $x \in U$.

(ii) The soft upper approximation of X corresponding to R and E is the set of all objects, denoted and defined by $(U_R(X), E) = \cup \{R(x): R(x) \cap X \neq \emptyset\}$.

(iii) The soft boundary region of X corresponding to R and E is the set of all objects, denoted and defined by $(B_R(X), E) = (U_R(X), E) - (L_R(X), E)$.

The family, $\{\tau_R(X), U, E\} = \{\emptyset, U, (L_R(X), E), (U_R(X), E), (B_R(X), E)\}$ is called as the soft nano topology on (U, E) with respect to X. Elements of soft nano topology are known as the soft nano open sets and their complements are known as the soft nano closed sets.

In continuation, Benchalli et al. [5], [6] introduced the notions of soft nano semi-open, soft nano pre-open, soft nano α-open, soft nano β-open via δ-operation and soft nano generalized closed sets as well as their corresponding complement sets.

In the present work, the notions of soft nano subspaces, soft nano closure and soft nano interior in soft nano subspaces are initiated. The pivotal objective is to propose the concepts of weakly soft nano g-closed sets, weakly soft nano g-open sets and their properties are investigated. Further, the inter-relationship between these new classes of soft nano sets with existing soft nano sets in soft nano topological spaces are studied. In addition, the definitions of weakly soft nano g-neighbourhood of a point, weakly soft nano g-neighbourhood of a set, weakly soft nano g-interior points and weakly soft nano g-limit points are introduced.

Throughout this paper, let U denotes the initial universal set and SNO(U, E) denotes the family of all soft nano open sets in soft nano topological space ($\tau_R(X)$, U, E).

2. SOFT NANO SUBSPACES

In this section, we define soft nano subspaces and give some properties of soft nano subspaces.

Definition 2.1: Let ($\tau_R(X)$, U, E) be a soft nano topological space. Let Y be a non-empty subset of U such that $X \subseteq Y \subseteq U$ and $Y/R \subseteq U/R$. Then, $\{\tau_R^*(X), Y, E\} = \{\emptyset, Y, (L_R^*(X), E), (U_R^*(X), E), (B_R^*(X), E)\}$ is called a soft nano relative topology on Y,
where, $(L_R^*(X), E) = (L_R(X), E) \cap Y$, $(U_R^*(X), E) = (U_R(X), E) \cap Y$, $(B_R^*(X), E) = (B_R(X), E) \cap Y = (U_R^*(X), E) - (L_R^*(X), E)$ with $(L_R(X), E), (U_R(X), E), (B_R(X), E) \in$ SNO(U, E).

Then, the structure ($\tau_R^*(X)$, Y, E) is said to be a soft nano subspace of ($\tau_R(X)$, U, E). The elements of soft nano relative topology are known as soft nano Y-open sets and the family of all soft nano Y-open sets is denoted by SNO(Y, E). The complements of soft nano Y-open sets are known as soft nano Y-closed sets.

Remark 2.2: $\{\tau_R^*(X), Y, E\}$ is a soft nano topology on Y.

Examples 2.3: Let U = {a, b, c, d}, E = {m_1, m_2, m_3}, U/R = {{a}, {d}, {b, c}} and X = {a, c} $\subseteq$ U. Then $(L_R(X), E) = \{(m_1, \{a\}), (m_2, \{a\}), (m_3, \{a\})\}$, $(U_R(X), E) = \{(m_1, \{a, b, c\}), (m_2, \{a, b, c\}), (m_3, \{a, b, c\})\}$, $(B_R(X), E) = \{(m_1, \{b, c\}), (m_2, \{b, c\}), (m_3, \{b, c\})\}$. Therefore, $\{\tau_R(X), U, E\} = \{\emptyset, U, (L_R(X), E), (U_R(X), E), (B_R(X), E)\}$ is a soft nano topology on U. Let Y = {a, b, c}. Then, $X \subseteq Y \subseteq U$ and Y/R = {{a}, {b, c}} $\subseteq$ U/R. Hence, $\{\tau_R^*(X), Y, E\} = \{\emptyset, Y, \{(m_1, \{a\}), (m_2, \{a\}), (m_3, \{a\})\}, \{(m_1, \{a, b, c\}), (m_2, \{a, b, c\}), (m_3, \{a, b, c\})\}, \{(m_1, \{b, c\}), (m_2, \{b, c\}), (m_3, \{b, c\})\}\}$ is a soft nano relative topology on Y and ($\tau_R^*(X)$, Y, E) is a soft nano subspace of ($\tau_R(X)$, U, E).

Also, when Y = {a, c, d}, $\{\tau_R^*(X), Y, E\}$ = {∅, Y, {(m_1, {a}), (m_2, {a}), (m_3, {a})}, {(m_1, {a, c}), (m_2, {a, c}), (m_3, {a, c})}, {(m_1, {c}), (m_2, {c}), (m_3, {c})}} is a soft nano relative topology on Y and (τ_R^*(X), Y, E) is a soft nano subspace of (τ_R(X), U, E).

Similarly, for Y = {a, c} and Y = U = {a, b, c, d}.

Definition 2.4: Let (τ_R^*(X), Y, E) be a soft nano subspace of (τ_R(X), U, E) and (A, E) be any soft set over Y. Then the soft nano closure of (A, E) in (τ_R^*(X), Y, E) is defined as SNCl(A, E) ∩ Y and is denoted by $SNCl_Y$(A, E), where SNCl(A, E) is soft nano closure of (A, E) in (τ_R(X), U, E).

Remarks 2.5: Let (τ_R^*(X), Y, E) be a soft nano subspace of (τ_R(X), U, E) and (A, E), (B, E) are any soft sets over Y. Then the following results hold good:

i) $SNCl_Y(\emptyset) = \emptyset$ and $SNCl_Y(Y) = Y$

ii) (A, E) ⊆ $SNCl_Y$(A, E)

iii) $SNCl_Y$($SNCl_Y$(A, E)) = $SNCl_Y$(A, E)

iv) (A, E) ⊆ (B, E) implies $SNCl_Y$(A, E) ⊆ $SNCl_Y$(B, E)

v) $SNCl_Y$[(A, E) ∪ (B, E)] = $SNCl_Y$(A, E) ∪ $SNCl_Y$(B, E)

vi) $SNCl_Y$[(A, E) ∩ (B, E)] ⊆ $SNCl_Y$(A, E) ∩ $SNCl_Y$(B, E)

Definition 2.6: Let (τ_R^*(X), Y, E) be a soft nano subspace of (τ_R(X), U, E). Then the soft nano interior of (A,E) ⊆ (Y, E) in (τ_R^*(X), Y, E), denoted by $SNInt_Y$(A, E) is defined as SNInt(A, E) ∩ Y, where SNInt(A, E) is soft nano interior of (A, E) in (τ_R(X), U, E).

Remarks 2.7: Let (τ_R^*(X), Y, E) be a soft nano subspace of (τ_R(X), U, E) and (A, E), (B, E) are any soft sets over Y. Then,

i) $SNInt_Y(\emptyset) = \emptyset$

ii) $SNInt_Y$(A, E) ⊆ (A, E)

iii) $SNInt_Y$($SNInt_Y$(A, E)) = $SNInt_Y$(A, E)

iv) $(A, E) \subseteq (B, E)$ implies $SNInt_Y(A, E) \subseteq SNInt_Y(B, E)$

v) $SNInt_Y(A, E) \cup SNInt(B, E) \subseteq SNInt_Y[(A, E) \cup (B, E)]$

vi) $SNInt_Y[(A, E) \cap (B, E)] = SNInt_Y(A, E) \cap SNInt_Y(B, E)$

3. WEAKLY SOFT NANO GENERALIZED CLOSED SETS

In this section, we define weakly soft nano generalized closed (in short WSNg-closed) sets by using soft nano open sets and study some properties of WSNg-closed sets.

Definition 3.1: A soft subset (G, E) over U is said to be a weakly soft nano g-closed (briefly WSNg-closed) set in $(\tau_R(X), U, E)$ if and only if $SNCl(SNInt(G, E)) \subseteq (A, E)$ whenever $(G, E) \subseteq (A, E)$ and (A, E) is soft nano open. The family of all WSNg-closed sets over U is denoted by WSNgC(U, E).

Example 3.2: Let $U = \{a, b, c, d\}$, $E = \{m_1, m_2, m_3\}$, $U/R = \{\{a\}, \{d\}, \{b, c\}\}$ and $X = \{a, c\} \subseteq U$. Then $(L_R(X), E) = \{(m_1, \{a\}), (m_2, \{a\}), (m_3, \{a\})\}$, $(U_R(X), E) = \{(m_1, \{a, b, c\}), (m_2, \{a, b, c\}), (m_3, \{a, b, c\})\}$ and $(B_R(X), E) = \{(m_1, \{b, c\}), (m_2, \{b, c\}), (m_3, \{b, c\})\}$. Now, $\{\tau_R(X), U, E\} = \{\emptyset, U, (L_R(X), E), (U_R(X), E), (B_R(X), E)\}$ is a soft nano topology on U and $(\tau_R(X), U, E)$ is a soft nano topological space.

Let $(G, E)_1 = \{(m_1, \{a\}), (m_2, \{a\}), (m_3, \{a\})\}$

$(G, E)_2 = \{(m_1, \{b\}), (m_2, \{b\}), (m_3, \{b\})\}$

$(G, E)_3 = \{(m_1, \{c\}), (m_2, \{c\}), (m_3, \{c\})\}$

$(G, E)_4 = \{(m_1, \{d\}), (m_2, \{d\}), (m_3, \{d\})\}$

$(G, E)_5 = \{(m_1, \{a, b\}), (m_2, \{a, b\}), (m_3, \{a, b\})\}$

$(G, E)_6 = \{(m_1, \{a, c\}), (m_2, \{a, c\}), (m_3, \{a, c\})\}$

$(G, E)_7 = \{(m_1, \{a, d\}), (m_2, \{a, d\}), (m_3, \{a, d\})\}$

$(G, E)_8 = \{(m_1, \{b, c\}), (m_2, \{b, c\}), (m_3, \{b, c\})\}$

$(G, E)_9 = \{(m_1, \{b, d\}), (m_2, \{b, d\}), (m_3, \{b, d\})\}$

$(G, E)_{10} = \{(m_1, \{c, d\}), (m_2, \{c, d\}), (m_3, \{c, d\})\}$

$(G, E)_{11} = \{(m_1, \{a, b, c\}), (m_2, \{a, b, c\}), (m_3, \{a, b, c\})\}$

$(G, E)_{12} = \{(m_1, \{a, b, d\}), (m_2, \{a, b, d\}), (m_3, \{a, b, d\})\}$

$(G, E)_{13} = \{(m_1, \{a, c, d\}), (m_2, \{a, c, d\}), (m_3, \{a, c, d\})\}$

$(G, E)_{14} = \{(m_1, \{b, c, d\}), (m_2, \{b, c, d\}), (m_3, \{b, c, d\})\}$.

Here, $\emptyset$, U, $(G, E)_1$, $(G, E)_8$, $(G, E)_{11} \in$ SNO(U, E)

and $\emptyset$, U, $(G, E)_2$, $(G, E)_3$, $(G, E)_4$, $(G, E)_7$, $(G, E)_9$, $(G, E)_{10}$, $(G, E)_{12}$, $(G, E)_{13}$, $(G, E)_{14} \in$ WSNgC(U, E).

Definition 3.3: Let $(\tau_R^*(X), Y, E)$ be a soft nano subspace of $(\tau_R(X), U, E)$. A soft subset (G, E) of Y is said to be WSNg-closed in $(\tau_R^*(X), Y, E)$ if and only if $SNCl_Y(SNInt_Y(G, E)) \subseteq (A, E)$ whenever $(G, E) \subseteq (A, E)$ and (A, E) is soft nano open in $(\tau_R^*(X), Y, E)$. The family of all WSNg-closed sets in $(\tau_R^*(X), Y, E)$ is denoted by WSNgC(Y, E).

Example 3.4: In Example 3.2, if Y = {a, c, d}, then WSNgC(Y, E) $=\{\emptyset, Y, \{(m_1, \{c\}), (m_2, \{c\}), (m_3, \{c\})\}, \{(m_1, \{d\}), (m_2, \{d\}), (m_3, \{d\})\}, \{(m_1, \{a, d\}), (m_2, \{a, d\}), (m_3, \{a, d\})\}, \{(m_1, \{c, d\}), (m_2, \{c, d\}), (m_3, \{c, d\})\}\}$.

Theorem 3.5: Every soft nano closed set in a soft nano topological space is WSNg-closed set.

Proof: Let (F, E) be a soft nano closed set in $(\tau_R(X), U, E)$ and $(F, E) \subseteq (A, E)$ with $(A, E) \in$ SNO(U, E). We have, $SNInt(F, E) \subseteq (F, E) \subseteq SNCl(F, E)$. Therefore, $SNInt(F, E) \subseteq SNCl(F, E)$, which implies $SNCl(SNInt(F, E)) \subseteq SNCl(SNCl(F, E)) = SNCl(F, E) = (F, E)$, as (F, E) is soft nano closed, that is, $SNCl(SNInt(F, E)) \subseteq (F, E) \subseteq (A, E)$. Thus, $SNCl(SNInt(F, E)) \subseteq (A, E)$. Hence, (F, E) is WSNg-closed.

Remark 3.6: Every WSNg-closed set is not a soft nano closed in general.

In Example 3.2, $(G, E)_2$, $(G, E)_3$, $(G, E)_9$, $(G, E)_{10}$, $(G, E)_{12}$, $(G, E)_{13}$ are WSNg-closed sets but are not soft nano closed.

Theorem 3.7: Every soft nano g-closed set is WSNg-closed.

Proof: Let (F, E) be a soft nano g-closed set in (τ_R(X), U, E) and (F, E) ⊆ (A, E), (A, E) be a soft nano open set in (τ_R(X), U, E). Then SNInt(F, E) ⊆ SNCl(F, E) ⊆ (A, E), which implies SNCl(SNInt(F, E)) ⊆ (A, E). Hence, (F, E) is WSNg-closed.

Remark 3.8: Every WSNg-closed set is not a soft nano g-closed in general.

In Example 3.2, $(G, E)_2$, $(G, E)_3$ are WSNg-closed sets but are not soft nano g-closed.

Theorem 3.9: If a soft subset (G, E) is WSNg-closed and (G, E) ∈ SNO(U, E) in (τ_R(X), U, E), then it is soft nano closed.

Proof: Let (G, E) ∈ WSNgC(U, E) and (G, E) ∈ SNO(U, E). Then, SNCl(SNInt(G, E)) ⊆ (A, E) whenever (G, E) ⊆ (A, E) and (A, E) ∈ SNO(U, E). Since (G, E) ∈ SNO(U, E), we have, SNCl(SNInt(G, E)) ⊆ (G, E), implies, SNCl(G, E) ⊆ (G, E). But, (G, E) ⊆ SNCl(G, E) is always true. Thus, SNCl(G, E) = (G, E).

Corollary 3.10: If a soft subset (G, E) is WSNg-closed and (G, E) ∈ SNO(U, E) in (τ_R(X), U, E), then it is both soft nano regular open and soft nano regular closed.

Proof: From Theorem 3.9, (G, E) is soft nano closed.

Since (G, E) is both soft nano open and soft nano closed, from [5] it follows that, SNInt(G, E) = (G, E) and SNCl(G, E) = (G, E). Therefore, SNInt(SNCl(G, E)) = (G, E) and SNCl(SNInt(G, E)) = (G, E).

Corollary 3.11: If a soft subset (G, E) is WSNg-closed and (G, E) ∈ SNO(U, E) in (τ_R(X), U, E), then it is soft nano g-closed.

Proof: The proof follows from the Theorem 3.9 and the result, every soft nano closed set is soft nano g-closed [7].

Theorem 3.12: If a soft subset (G, E) is both soft nano semi open and WSNg-closed, then it is soft nano g-closed in (τ_R(X), U, E).

Proof: SNCl(SNInt(G, E)) ⊆ (A, E) whenever (G, E) ⊆ (A, E) and (A, E) is soft nano open in

($\tau_R(X)$, U, E) and (G, E) ⊆ SNCl(SNInt(G, E)). Therefore, SNCl(G, E) ⊆ SNCl(SNInt(G, E)) ⊆ (A, E). Hence, SNCl(G, E) ⊆ (A, E) whenever (G, E) ⊆ (A, E) and (A, E) is soft nano open. Therefore, (G, E) is soft nano g-closed.

Theorem 3.13: If a soft subset (G, E) of ($\tau_R(X)$, U, E) is WSNg-closed then SNCl(SNInt(G, E)) – (G, E) contains no non-empty soft nano closed sets.

Proof: Let (A, E) be soft nano closed in ($\tau_R(X)$, U, E) such that (A, E) ⊆ SNCl(SNInt(G, E)) – (G, E). Therefore, (A, E) ⊆ SNCl(SNInt(G, E)) ∩ (G, E)′ which implies (A, E) ⊆ SNCl(SNInt(G, E)) and (A, E) ⊆ (G, E)′. Therefore, (G, E) ⊆ (A, E)′, where (A, E)′ ∈ SNO(U, E). Now, SNCl(SNInt(G, E)) ⊆ (A, E)′ as (G, E) is WSNg-closed. Thus, (A, E) ⊆ (SNCl(SNInt(G, E)))′, which implies, (A, E) ⊆ SNCl(SNInt(G, E)) ∩ (SNCl(SNInt(G, E)))′ = ∅. Hence, SNCl(SNInt(G, E)) – (G, E) contains no non-empty soft nano closed set.

Theorem 3.14: If (G, E) is WSNg-closed set and (G, E) ⊆ (F, E) ⊆ SNCl(SNInt(G, E)), then (F, E) is also WSNg-closed.

Proof: Let (F, E) ⊆ (A, E) and (A, E) be a soft nano open set. Then (G, E) ⊆ (A, E), where (A, E) is soft nano open. Since (G, E) is WSNg-closed, SNCl(SNInt((G, E)) ⊆ (A, E). Now, SNCl(SNInt(F, E)) ⊆ SNCl(F, E) ⊆ SNCl(SNInt(G, E)) ⊆ (A, E). Thus, SNCl(SNInt(F, E)) ⊆ (A, E) whenever (F, E) ⊆ (A, E) and (A, E) is soft nano open.

Theorem 3.15: If (G, E) is soft nano pre-closed then it is WSNg-closed.

Proof: Let (G, E) ⊆ (A, E) and (A, E) be a soft nano open set. Since (G, E) is soft nano pre-closed, SNCl(SNInt(G, E)) ⊆ (G, E). Therefore, SNCl(SNInt(G, E)) ⊆ (A, E) whenever (G, E) ⊆ (A, E) and (A, E) is soft nano open.

Remark 3.16: Every WSNg-closed set is not soft nano pre-closed in general.

Example 3.17: Let U = {a, b, c, d}, E = {m_1, m_2}, U/R = {{a, b}, {c, d}} and X = {a} ⊆ U

Then, (L_R(X), E) = ∅, (U_R(X), E) = {(m_1, {a, b}), (m_2, {a, b})}, (B_R(X), E) = {(m_1, {a, b}), (m_2, {a, b})}. Now, {τ_R(X), U, E} = {∅, U, (L_R(X), E), (U_R(X), E), (B_R(X), E)} is a soft nano topology on U and (τ_R(X), U, E) is a soft nano topological space.

Here {(m_1, {a, b, c}), (m_2, { a, b, c})}, {(m_1, {a, b, d}), (m_2, {a, b, d})} ∈ WSNgC(U, E) but they are not soft nano pre-closed.

Corollary 3.18: If (G, E) is soft nano regular closed then it is WSNg-closed.

Proof: Let (G, E) ⊆ (A, E) and (A, E) be a soft nano open set. Since (G, E) is soft nano regular closed, SNCl(SNInt(G, E)) = (G, E). Therefore, SNCl(SNInt(G, E)) ⊆ (G, E) which implies, (G, E) is soft nano pre-closed. Hence, by Theorem 3.15, (G, E) is WSNg-closed.

Remark 3.19: Every WSNg-closed set is not soft nano regular closed in general.

In Example 3.2, $(G, E)_2$, $(G, E)_3$, $(G, E)_4$, $(G, E)_9$, $(G, E)_{10}$, $(G, E)_{12}$, $(G, E)_{13}$ ∈ WSNgC(U, E) but they are not soft nano regular closed.

Theorem 3.20: If $(G, E)_1$, $(G, E)_2$ ∈ WSNgC(U, E) in (τ_R(X), U, E), then $(G, E)_1 \cap (G, E)_2$ ∈ WSNgC(U, E).

Proof: Let $(G, E)_1$ ⊆ (A, E) and $(G, E)_2$ ⊆ (B, E) and (A, E), (B, E) ∈ SNO(U, E). Since, $(G, E)_1$, $(G, E)_2$ ∈ WSNgC(U, E), we have, SNCl(SNInt$(G, E)_1$) ⊆ (A, E) and SNCl(SNInt$(G, E)_2$) ⊆ (B, E). Therefore, SNCl(SNInt$(G, E)_1$) ∩ SNCl(SNInt$(G, E)_2$) ⊆ (A, E) ∩ (B, E), which implies, SNCl(SNInt($(G, E)_1 \cap (G, E)_2$)) ⊆ (A, E) ∩ (B, E) where (A, E) ∩ (B, E) is a soft nano open set containing $(G, E)_1 \cap (G, E)_2$. Hence, $(G, E)_1 \cap (G, E)_2$ ∈ WSNgC(U, E) .

Remark 3.21: If $(G, E)_1$ and $(G, E)_2$ are WSNg-closed sets, then $(G, E)_1 \cup (G, E)_2$ need not be WSNg-closed in general.

In Example 3.2, {(m_1, {b}), (m_2, {b}), (m_3, {b})} and {(m_1, {c}), (m_2, {c}), (m_3, {c})} are WSNg-closed sets but {(m_1, {b, c}), (m_2, {b, c}), (m_3, {b, c})} is not WSNg-closed.

Theorem 3.22: Let ($\tau_R^*(X)$, Y, E) be a soft nano subspace of ($\tau_R(X)$, U, E). If a soft subset (G, E) of Y is WSNg-closed in ($\tau_R(X)$, U, E) then it is WSNg-closed in ($\tau_R^*(X)$, Y, E).

Proof: Let (G, E) ⊆ (A, E) and (A, E) ∈ SNO(U, E). Then, SNCl(SNInt(G, E)) ⊆ (A, E). Also, since, (G, E) ⊆ (A, E) and (G, E) ⊆ Y, (G, E) ⊆ (A, E) ∩ Y where (A, E) ∩ Y is soft nano open in ($\tau_R^*(X)$, Y, E). Now, SNCl(SNInt(G, E)) ∩ Y ⊆ (A, E) ∩ Y, which implies, $SNCl_Y(SNInt_Y$(G, E)) ⊆ (A, E) ∩ Y where (G, E) ⊆ (A, E) ∩ Y and (A, E) ∩ Y is soft nano open in ($\tau_R^*(X)$, Y, E). Therefore, (G, E) is WSNg-closed in ($\tau_R^*(X)$, Y, E).

4. Weakly Soft Nano g-Open Sets

Definition 4.1: A soft subset (G, E) over U is said to be a weakly soft nano g-open (briefly WSNg-open) set in ($\tau_R(X)$, U, E) if and only if (G, E)′ is WSNg-closed.

The family of all WSNg-open sets over U is denoted by WSNgO(U, E).

Example 4.2: In Example 3.2, WSNgO(U, E) = {∅, U, $(G, E)_1$, $(G, E)_2$, $(G, E)_3$, $(G, E)_5$, $(G, E)_6$, $(G, E)_8$, $(G, E)_{11}$, $(G, E)_{12}$, $(G, E)_{13}$}.

Definition 4.3: Let ($\tau_R^*(X)$, Y, E) be a soft nano subspace of ($\tau_R(X)$, U, E). A soft subset (G, E) of Y is said to be WSNg-open in ($\tau_R^*(X)$, Y, E) if and only if (G, E)′ is WSNg-closed in ($\tau_R^*(X)$, Y, E). The family of all WSNg-open sets in ($\tau_R^*(X)$, Y, E) is denoted by WSNgO(Y, E).

Example 4.4: In Example 3.4, WSNgO(Y, E) = {∅, Y, {(m_1, {a}), (m_2, {a}), (m_3, {a})}, {(m_1, {c}), (m_2, {c}), (m_3, {c})}, {(m_1, {a, c}), (m_2, {a, c}), (m_3, {a, c})}, {(m_1, {a, d}), (m_2, {a, d}), (m_3, {a, d})}}.

Proposition 4.5: Every soft nano open set (respectively soft nano g-open set) is WSNg-open but not conversely.

In Example 3.2, $(G, E)_2$, $(G, E)_3$, $(G, E)_5$, $(G, E)_6$, $(G, E)_{12}$, $(G, E)_{13}$ are WSNg-open sets, but are not soft nano open.

Further, $(G, E)_{12}$ and $(G, E)_{13}$ are WSNg-open sets but not soft nano g-open.

Theorem 4.6: If $(G, E)_1, (G, E)_2 \in$ WSNgO(U, E) in a soft nano topological space ($\tau_R(X)$, U, E), then $(G, E)_1 \cup (G, E)_2 \in$ WSNgO(U, E).

Proof: Let $(G, E)_1, (G, E)_2 \in$ WSNgO(U, E). Then $(G, E)_1', (G, E)_2' \in$ WSNgC(U, E). By Theorem 3.20, $(G, E)_1' \cap (G, E)_2' \in$ WSNgC(U, E), which implies, $(G, E)_1 \cup (G, E)_2 \in$ WSNgO(U, E).

Remark 4.7: If $(G, E)_1$ and $(G, E)_2$ are WSNg-open sets, then $(G, E)_1 \cap (G, E)_2$ need not be WSNg-open in general.

In Example 3.2, $\{(m_1, \{a, c, d\}), (m_2, \{a, c, d\}), (m_3, \{a, c, d\})\}$ and $\{(m_1, \{a, b, d\}), (m_2, \{a, b, d\}), (m_3, \{a, b, d\})\}$ are WSNg-open sets but $\{(m_1, \{a, d\}), (m_2, \{a, d\}), (m_3, \{a, d\})\}$ is not WSNg-open.

5. Weakly Soft Nano g-Closure And Weakly Soft Nano g-Interior

Definition 5.1: Let (G, E) be any soft subset over U. Then weakly soft nano g-closure of (G, E) in ($\tau_R(X)$, U, E) is defined as the intersection of all weakly soft nano g-closed sets containing (G, E) and is denoted by WSNgCl(G, E).

Thus, WSNgCl(G, E) = $\cap$ {(A, E): (A, E) $\in$ WSNgC(U, E) and (G, E) $\subseteq$ (A, E)}.

Example 5.2: In Example 3.2,

$\text{WSNgCl}(G, E)_1 = \{(m_1, \{a, d\}), (m_2, \{a, d\}), (m_3, \{a, d\})\}$

$\text{WSNgCl}(G, E)_2 = \{(m_1, \{b\}), (m_2, \{b\}), (m_3, \{b\})\}$

$\text{WSNgCl}(G, E)_3 = \{(m_1, \{c\}), (m_2, \{c\}), (m_3, \{c\})\}$

$\text{WSNgCl}(G, E)_4 = \{(m_1, \{d\}), (m_2, \{d\}), (m_3, \{d\})\}$

$\text{WSNgCl}(G, E)_5 = \{(m_1, \{a, b, d\}), (m_2, \{a, b, d\}), (m_3, \{a, b, d\})\}$

$\text{WSNgCl}(G, E)_6 = \{(m_1, \{a, c, d\}), (m_2, \{a, c, d\}), (m_3, \{a, c, d\})\}$

$\text{WSNgCl}(G, E)_7 = \{(m_1, \{a, d\}), (m_2, \{a, d\}), (m_3, \{a, d\})\}$

$\text{WSNgCl}(G, E)_8 = \{(m_1, \{b, c, d\}), (m_2, \{b, c, d\}), (m_3, \{b, c, d\})\}$

$WSNgCl(G, E)_9 = \{(m_1, \{b, d\}), (m_2, \{b, d\}), (m_3, \{b, d\})\}$

$WSNgCl(G, E)_{10} = \{(m_1, \{c, d\}), (m_2, \{c, d\}), (m_3, \{c, d\})\}$

$WSNgCl(G, E)_{11} = U$

$WSNgCl(G, E)_{12} = \{(m_1, \{a, b, d\}), (m_2, \{a, b, d\}), (m_3, \{a, b, d\})\}$

$WSNgCl(G, E)_{13} = \{(m_1, \{a, c, d\}), (m_2, \{a, c, d\}), (m_3, \{a, c, d\})\}$

$WSNgCl(G, E)_{14} = \{(m_1, \{b, c, d\}), (m_2, \{b, c, d\}), (m_3, \{b, c, d\})\}$

WSNgCl(U) = U

WSNgCl(∅) = ∅.

Theorem 5.3: Let (G, E) be any soft subset of (τ_R(X), U, E). Then,

i) WSNgCl(G, E) is a WSNg-closed set containing (G, E).

ii) WSNgCl(G, E) is the smallest WSNg-closed set containing (G, E).

iii) (G, E) is WSNg-closed if and only if (G, E) = WSNgCl(G, E).

Proof: (i) Follows from the Definition 5.1 and Theorem 3.20.

(ii) Let (B, E) be a WSNg-closed set containing (G, E). Then, ∩ {(A, E): (A, E) ∈ WSNgC(U, E) and (G, E) ⊆ (A, E)} ⊆ (B, E). That is, WSNgCl(G, E) ⊆ (B, E), which is true for all WSNg-closed set (B, E) containing (G, E). Therefore, WSNgCl(G, E) is the smallest WSNg-closed set containing (G, E).

(iii) From (i), we have, (G, E) ⊆ WSNgCl(G, E). Also, since (G, E) is WSNg-closed containing itself and WSNgCl(G, E) is the smallest WSNg-closed set containing (G, E), it follows that WSNgCl (G, E) ⊆ (G, E). Therefore, (G, E) = WSNgCl(G, E).

Conversely, if (G, E) = WSNgCl(G, E). Since, WSNgCl(G, E) is a WSNg-closed set, we have, (G, E) is also a WSNg-closed set.

Remarks 5.4: In (τ_R(X), U, E), for any two soft subsets $(G, E)_1$ and $(G, E)_2$ over U, the following results hold true

(i) WSNgCl(U, E) = WSNgCl(U) = U.

(ii) WSNgCl(∅, E) = WSNgCl(∅) = ∅.

(iii)If $(G, E)_1 \subseteq (G, E)_2$, then $WSNgCl(G, E)_1 \subseteq WSNgCl(G, E)_2$.

(iv)$WSNgCl(G, E)_1 \cup WSNgCl(G, E)_2 \subseteq WSNgCl[(G, E)_1 \cup (G, E)_2]$.

(v) $WSNgCl[(G, E)_1 \cap (G, E)_2] \subseteq WSNgCl(G, E)_1 \cap WSNgCl(G, E)_2$.

Theorem 5.5: Let (G, E) be a soft subset over U, then (G, E) ⊆ WSNgCl(G, E) ⊆ SNCl(G, E).

Proof: We know that (G, E) ⊆ WSNgCl(G, E) and (G, E) ⊆ SNCl(G, E). Suppose, WSNgCl(G, E) ⊈ SNCl(G, E), then there exist x ∈ WSNgCl(G, E) such that x ∉ SNCl(G, E). Now, x ∈ WSNgCl(G, E) = ∩ {(A, E): (A, E) ∈ WSNgC(U, E) and (G, E) ⊆ (A, E)}, implies that x ∈ (A, E) for all WSNg-closed set (A, E) containing (G, E). …(1)

Also, x ∉ SNCl(G, E) = ∩ {(B, E): (B, E) is a soft nano closed set and (G, E) ⊆ (B, E)}, implies that x ∉ (B_0, E) for some soft nano closed set (B_0, E) containing (G, E). By Theorem 3.5, (B_0, E) is a WSNg-closed set, which is a contradiction to (1). Therefore, WSNgCl(G, E) ⊆ SNCl(G, E) and hence (G, E) ⊆ WSNgCl(G, E) ⊆ SNCl(G, E).

Definition 5.6: Let (G, E) be any soft subset over U. Then weakly soft nano g-interior of (G, E) in $(\tau_R(X), U, E)$ is defined as the union of all weakly soft nano g-open sets contained in (G, E) and is denoted by WSNgInt(G, E).

Thus, WSNgInt(G, E) = ∪ {(A, E): (A, E) ∈ WSNgO(U, E) and (A, E) ⊆ (G, E)}.

Example 5.7: In example 3.2,

$WSNgInt(G, E)_1 = \{(m_1, \{a\}), (m_2, \{a\}), (m_3, \{a\})\}$

$WSNgInt(G, E)_2 = \{(m_1, \{b\}), (m_2, \{b\}), (m_3, \{b\})\}$

$WSNgInt(G, E)_3 = \{(m_1, \{c\}), (m_2, \{c\}), (m_3, \{c\})\}$

$WSNgInt(G, E)_4 = \emptyset$

$WSNgInt(G, E)_5 = \{(m_1, \{a, b\}), (m_2, \{a, b\}), (m_3, \{a, b\})\}$

$WSNgInt(G, E)_6 = \{(m_1, \{a, c\}), (m_2, \{a, c\}), (m_3, \{a, c\})\}$

$WSNgInt(G, E)_7 = \{(m_1, \{a\}), (m_2, \{a\}), (m_3, \{a\})\}$

$WSNgInt(G, E)_8 = \{(m_1, \{b, c\}), (m_2, \{b, c\}), (m_3, \{b, c\})\}$

$WSNgInt(G, E)_9 = \{(m_1, \{b\}), (m_2, \{b\}), (m_3, \{b\})\}$

$WSNgInt(G, E)_{10} = \{(m_1, \{c\}), (m_2, \{c\}), (m_3, \{c\})\}$

$WSNgInt(G, E)_{11} = \{(m_1, \{a, b, c\}), (m_2, \{a, b, c\}), (m_3, \{a, b, c\})\}$

$WSNgInt(G, E)_{12} = \{(m_1, \{a, b, d\}), (m_2, \{a, b, d\}), (m_3, \{a, b, d\})\}$

$WSNgInt(G, E)_{13} = \{(m_1, \{a, c, d\}), (m_2, \{a, c, d\}), (m_3, \{a, c, d\})\}$

$WSNgInt(G, E)_{14} = \{(m_1, \{b, c\}), (m_2, \{b, c\}), (m_3, \{b, c\})\}$

WSNgInt(U) = U

WSNgInt(∅) = ∅.

Theorem 5.8: Let (G, E) be any soft subset of (τ_R(X), U, E). Then,

i) WSNgInt(G, E) is a WSNg-open set contained in (G, E).

ii) WSNgInt(G, E) is the largest WSNg-open set contained in (G, E).

iii) (G, E) is WSNg-open if and only if (G, E) = WSNgInt(G, E).

Proof: (i) Follows from the Definition 5.6 and Theorem 4.6.

(ii) Let (B, E) be a WSNg-open set contained in (G, E). Then, (B, E) ⊆ ∪ {(A, E): (A, E) ∈ WSNgO(U, E) and (A, E) ⊆ (G, E)}. That is, (B, E) ⊆ WSNgInt(G, E), which is true for all WSNg-open set (B, E) contained in (G, E). Therefore, WSNgInt(G, E) is the largest WSNg-open set contained in (G, E).

(iii) From (i), we have, WSNgInt(G, E) ⊆ (G, E). Also, since (G, E) is a WSNg-open set contained in itself and WSNgInt(G, E) is the largest WSNg-open set contained in (G, E), it follows that (G, E) ⊆ WSNgInt(G, E). Therefore, (G, E) = WSNgInt(G, E).

Conversely, if (G, E) = WSNgInt(G, E). Since, WSNgInt(G, E) is a WSNg-open set, we have, (G, E) is also a WSNg-open set.

Remarks 5.9: In (τ_R(X), U, E), for any two soft subsets $(G, E)_1$ and $(G, E)_2$ over U, the following results hold true:

i) WSNgInt(U, E) = WSNgInt(U) = U.

ii) WSNgInt(∅, E) = WSNgInt(∅) = ∅.

iii) If $(G, E)_1 \subseteq (G, E)_2$, then WSNgInt$(G, E)_1 \subseteq$ WSNgInt$(G, E)_2$.

iv) WSNgInt$(G, E)_1 \cup$ WSNgInt$(G, E)_2 \subseteq$ WSNgInt$[(G, E)_1 \cup (G, E)_2]$.

v) WSNgInt$[(G, E)_1 \cap (G, E)_2] \subseteq$ WSNgInt$(G, E)_1 \cap$ WSNgInt$(G, E)_2$.

Theorem 5.10: Let (G, E) be a soft subset over U, then SNInt(G, E) ⊆ WSNgInt(G, E) ⊆ (G, E).

Proof: We know that SNInt(G, E) ⊆ (G, E) and WSNgInt(G, E) ⊆ (G, E). Suppose, SNInt(G, E) ⊈ WSNgInt(G, E), then there exist x ∈ SNInt(G, E) such that x ∉ WSNgInt(G, E). Now, x ∈ SNInt(G, E) = ∪ {(A, E): (A, E) a soft nano open set and (A, E) ⊆ (G, E)}, implies that x ∈ (A_0, E) for some soft nano open set (A_0, E) contained in (G, E). …(1)

Also, x ∉ WSNgInt(G, E) = ∪ {(B, E): (B, E) ∈ WSNgO(U, E) and (B, E) ⊆ (G, E)}, implies that x ∉ (B, E) for any WSNg-open set (B, E) contained in (G, E). By Theorem 4.5, (A_0, E) is a WSNg-open set, which is a contradiction to (1). Therefore, SNInt(G, E) ⊆ WSNgInt(G, E) and hence SNInt(G, E) ⊆ WSNgInt(G, E) ⊆ (G, E).

Result 5.11: From theorems 5.5 and 5.10, we can conclude that,

SNInt(G, E) ⊆ WSNgInt(G, E) ⊆ (G, E) ⊆ WSNgCl(G, E) ⊆ SNCl(G, E).

Definition 5.12: A soft subset (A, E) of a soft nano topological space $(\tau_R(X), U, E)$ is called a weakly soft nano g-neighbourhood (briefly WSNg-nbd) of a point x of U, if there exists a WSNg-open set (B, E) such that x ∈ (B, E) ⊆ (A, E).

Example 5.13: In Example 3.2, consider a ∈ U. A soft subset (A, E) = $\{(m_1, \{a, d\}), (m_2, \{a, d\}), (m_3, \{a, d\})\}$ of U is a WSNg-nbd of 'a'. Because, there exist a WSNg-open set (B, E) = $\{(m_1, \{a\}), (m_2, \{a\}), (m_3, \{a\})\}$ such that a ∈ (B, E) ⊆ (A, E).

Theorem 5.14: A soft subset of a soft nano topological space $(\tau_R(X), U, E)$ is WSNg-open if and only if it is a WSNg-nbd of each of its points.

Proof: Let (A, E) be a WSNg-open set and x ∈ (A, E). Now, (A, E) is a WSNg-open set containing x. Therefore, a ∈ (A, E) ⊆ (A, E), for any arbitrary point x of (A, E). Hence, (A, E) is a WSNg-nbd of each of its points.

Conversely, let (A, E) be a WSNg-nbd of each of its points. Therefore, (A, E) is a WSNg-open set containing each of its points, that is, (A, E) is WSNg-open.

Definition 5.15: Let (A, E) be a soft subset of a soft nano topological space (τ_R(X), U, E). Then a soft subset (B, E) of (τ_R(X), U, E) is said to be a WSNg-nbd of (A, E), if there exist a WSNg-open set (G, E) such that (A, E) ⊆ (G, E) ⊆ (B, E). Note that, (B, E) is a WSNg-nbd of (A, E) if and only if it is a WSNg-nbd of all points of (A, E).

Example 5.16: In Example 3.2, consider (A, E) = {(m_1, {d}), (m_2, {d}), (m_3, {d})}. Then (B, E) = {(m_1, {a, b, d}), (m_2, {a, b, d}), (m_3, {a, b, d})} is a WSNg-nbd of (A, E).

Definition 5.17: A point x ∈ U is called a weakly soft nano g-interior (briefly WSNg-interior) point of a soft subset (A, E) of (τ_R(X), U, E) if and only if (A, E) is a WSNg-nbd of x, that is, there exist a WSNg-open set (B, E) such that x ∈ (B, E) ⊆ (A, E).

Example 5.18: In Example 5.16, 'a' is a WSNg-interior point of (A, E) = {(m_1, {a, d}), (m_2, {a, d}), (m_3, {a, d})}.

Theorem 5.19: Let (A, E) be a soft subset of (τ_R(X), U, E). Then x ∈ WSNgCl(A, E) if and only if for any WSNg-nbd (N_x, E) of x in (τ_R(X), U, E), (A, E) ∩ (N_x, E) ≠ ∅.

Proof: Let x ∈ WSNgCl(A, E) and (N_x, E) be a WSNg-nbd of x. Suppose, (A, E) ∩ (N_x, E) = ∅. Since (N_x, E) is a WSNg-nbd of x, there exist a WSNg-open set (V_x, E) in (τ_R(X), U, E), such that x ∈ (V_x, E) ⊆ (N_x, E). Therefore, (A, E) ∩ (V_x, E) = ∅, implies that (A, E) ⊆ (V_x, E)′. Since (V_x, E)′ is a WSNg-closed set containing (A, E), we have, WSNgCl(A, E) ⊆ (V_x, E)′. Now, x ∉ WSNgCl(A, E), which is a contradiction. Therefore, (A, E) ∩ (N_x, E) ≠ ∅.

Conversely, let (N_x, E) be a WSNg-nbd of x such that $(A, E) \cap (N_x, E) \neq \emptyset$. Suppose, $x \notin$ WSNgCl(A, E). Then there exist a WSNg-closed set (F, E) containing (A, E) such that $x \notin (F, E)$. Therefore, $x \in (F, E)'$, where $(F, E)'$ is a WSNg-open set. That is, $(F, E)'$ is a WSNg-nbd of x. By hypothesis, $(A, E) \cap (F, E)' \neq \emptyset$, which is a contradiction to $(A, E) \subseteq (F, E)$. Hence, $x \in$ WSNgCl(A, E).

Definition 5.20: A point $x \in U$ is said to be a WSNg-limit point of a soft subset (A, E) of $(\tau_R(X), U, E)$ if every WSNg-nbd of x contains at least one point of (A, E) other than x.

Example 5.21: In Example 3.2, let $(A, E) = \{(m_1, \{a, b, c\}), (m_2, \{a, b, c\}), (m_3, \{a, b, c\})\}$ be a soft subset of $(\tau_R(X), U, E)$. Then $b, c, d \in U$ are the WSNg-limit points of (A, E).

CONCLUSION

The concept of soft nano subspaces of soft nano topological spaces is very useful to construct further results in the domain of soft nano topological spaces. Also, the key result of the current work is, the class of WSNg-closed sets satisfies the property that

SNInt(G, E) $\subseteq$ WSNgInt(G, E) $\subseteq$ (G, E) $\subseteq$ WSNgCl(G, E) $\subseteq$ SNCl(G, E).

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] M.I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir, On some new operations in soft set theory, Computers Math. Appl. 57 (2009), 1547–1553.

[2] A. Aygünoğlu, H. Aygün, Some notes on soft topological spaces, Neural Comput. Appl. 21 (2012), 113–119.

[3] K.V. Babitha, J.J. Sunil, Soft set relations and functions, Computers Math. Appl. 60 (2010), 1840–1849.

[4] S.S. Benchalli, P.G. Patil, N.S. Kabbur, On soft γ-operations in soft topological spaces, J. New Theory, 6 (2015), 20-32.

[5] S.S. Benchalli, P.G. Patil, N.S.Kabbur, On some weaker forms of soft closed sets in soft topological spaces, Int. J. Appl. Math. 28(3) (2015), 223-235.

[6] S.S. Benchalli, P.G. Patil, N.S. Kabbur, J. Pradeepkumar, Weaker forms of soft nano open sets, J. Computer Math. Sci. 8(11) (2017), 589-599.

[7] S.S. Benchalli, P.G. Patil, N.S. Kabbur, J. Pradeepkumar, On δ-operation in soft nano topological spaces, J. Computer Math. Sci. 9(8) (2018), 1001-1016.

[8] S.S. Benchalli, P.G. Patil, N.S. Kabbur, J. Pradeepkumar, On soft nano continuity in soft nano topological spaces and its applications, Ann. Fuzzy Math. Inform. 15(3) (2018), 265-283.

[9] B. Chen, Soft semi-open sets and related properties in soft topological spaces, Appl. Math. Inform. Sci. 7(1) (2013), 287-294.

[10] P.K. Maji, R. Biswas, A.R. Roy, Soft set theory, Computers Math. Appl. 45 (2003), 555-562.

[11] W.K. Min, A note on soft topological spaces, Computers Math. Appl. 62 (2011), 3524-3528.

[12] D. Molodtsov, Soft set theory-first results, Computers Math. Appl. 37 (1999), 19-31.

[13] M. Shabir, M. Naz, On soft topological spaces, Computers Math. Appl. 61 (2011), 1786-1799.

[14] M.L. Thivagar, C. Richard, On nano continuity, Math. Theory Model. 3(7) (2013), 32-37.