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$\overrightarrow{P}_5$ - FACTORIZATION OF SYMMETRIC COMPLETE BIPARTITE MULTI-DIGRAPH

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Abstract. P_{2p} -factorization of a complete bipartite graph for p, an integer was studied by Wang [1]. Further, Beiling [2] extended the work of Wang[1], and studied the P_{2k} -factorization of complete bipartite multigraphs. For even value of k in P_k -factorization the spectrum problem is completely solved [1, 2, 3]. However for odd value of k i.e. P_3 , P_5, P_7 and P_9, the path factorization have been studied by a number of researchers [4, 5, 6, 7]. Again $\overrightarrow{P}_3$ -factorization of complete bipartite multigraphs and symmetric complete bipartite multi-digraphs was studied by Wang and Beiling [8]. In the present paper, we study $\overrightarrow{P}_5$-factorization of symmetric complete bipartite multi-digraphs and show that the necessary and sufficient conditions for the existence of $\overrightarrow{P}_5$ -factorization of symmetric complete bipartite multi-digraphs are:

(1) $3m \geq 2n$,

(2) $3n \geq 2m$,

(3) $m + n \equiv 0(mod5)$,

(4) $5\lambda mn/[2(m + n)]$ is an integer.

Keywords: Complete bipartite graph, factorization of graph

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1. Introduction

Let $\overrightarrow{P}_5$ be the directed path on five vertices and $K^*_{m,n}$ be the symmetric complete bipartite di-graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$. Further $\lambda K^*_{m,n}$ is $K^*_{m,n}$ in which every edge is taken λ times. A spanning subgraph $\overrightarrow{F}$ of $\lambda K^*_{m,n}$ is called a $\overrightarrow{P}_5$-factor if each such factor is isomorphic to $\overrightarrow{P}_5$. If $\lambda K^*_{m,n}$ is expressed as an edge disjoint sum of $\overrightarrow{P}_5$-factors, then this sum is called a $\overrightarrow{P}_5$- factorization of $\lambda K^*_{m,n}$.
In this paper the necessary and sufficient conditions for the existence of a $\overrightarrow{P}_5$-factorization of the symmetric complete bipartite multi-digraph $\lambda K^*_{m,n}$ are studied.

2. Mathematical Analysis

Theorem 2.1: Let m and n be positive integers. Then $\lambda K^*_{m,n}$ has $\overrightarrow{P}_5$-factorization if and only if

(1) $3m \geq 2n$,

(2) $3n \geq 2m$,

(3) $m + n \equiv (mod 5)$ and

(4) $5\lambda mn/[2(m + n)]$is an integer.

To prove this theorem the following number theoretic result is used.

Lemma 2.1: Let$g$, h, p and q be any positive integers. If gcd $(p, q) = 1$, then gcd $(p.q, p + g.q) = gcd(p, g)$. Similarly, if gcd $(gp, hq) = 1$ then gcd $(gp + hq, pq) = 1$.

The following existence theorems (theorem 2.2 and 2.3) will also be used in the proof of theorem 2.1.

Theorem 2.2: If $\lambda K^*_{m,n}$ has $\overrightarrow{P}_5$-factorization, then $\lambda K^*_{sm,sn}$ has $\overrightarrow{P}_5$-factorization for every positive integer s.

Proof: Let $K_(s, s)$ is 1-factorable [9], and $\{H_1, H_2, ..., H_s\}$ be a one factorization of it. For each i with $1 \leq i \leq s$, replace every edge of H_i with a $\lambda K^*_{m,n}$ to get a spanning sub graph G_i of $\lambda K^*_{sm,sn}$ such that the G_is $\{1 \leq i \leq s\}$ are pair wise edge disjoint, and there sum is $\lambda K^*_{sm,sn}$. Since $\lambda K^*_{m,n}$ is $\overrightarrow{P}_5$-factorable, therefore G_i is also $\overrightarrow{P}_5$-factorable, and hence, $\lambda K^*_{sm,sn}$ is also $\overrightarrow{P}_5$-factorable.

Theorem 2.3: If $\lambda K^*_{m,n}$ has a $\overrightarrow{P}_5$-factorization, then $\lambda s K^*_{sm,sn}$ has $\overrightarrow{P}_5$-factorization for every positive integer s.

Proof: Let us construct a $\vec{P}_5$-factorization of $\lambda K^*_{m,n}$ repeatedly s times. Then obviously, we have a $\vec{P}_5$-factorization of $\lambda s K^*_{sm,sn}$.

Now we will prove the main result (theorem 2.1). There are three cases to consider:

Case (I) 3m = 2n: Using conditions (4) of theorem 2.1 with theorem 2.2 and theorem 2.3 we see that $\lambda K^*_{m,n}$ has a $\vec{P}_5$-factorization.

Case (II) 3n = 2m: Obviously, $\lambda K^*_{m,n}$ has a $\vec{P}_5$-factorization.

Case III($3m > 2n$ and $3n > 2m$): Let

$a = \frac{3n-2m}{5}$, $b = \frac{3m-2n}{5}$, $t = \frac{m+n}{5}$ and $r = \frac{5\lambda mn}{2(m+n)}$.

Then from condition (1)-(4) in theorem 2.1, a, b, t and r will be integers, with $0 < a < m$ and 0¡ b ¡ n. We get $m = 2a + 3b$, $n = 3a + 2b$, $r = 3(a + b)\lambda + z$, where $z = \frac{\lambda ab}{2(a+b)}$.

Here,

t= the number of copies of $\vec{P}_5$ in any factor,

r = the number of $\vec{P}_5$ - factors in the factorization,

a = the number of copies of $\vec{P}_5$ with its endpoints in Y in a particular $\vec{P}_5$ - factor(type M),

b = the number of copies of $\vec{P}_5$ with its endpoints in X in a particular $\vec{P}_5$ - factor(type W),

c = the total number of copies of $\vec{P}_5$ in the whole factorization.

Let gcd $(2a, 3b) = d$ and therefore $2a = dp$, $3b = dq$ for some p, q. Where gcd $(p, q) = 1$, then $z = \frac{\lambda dpq}{2(3p+2q)}$.

These equalities imply the following equalities:

$d = \frac{2(3p+2q)z}{\lambda pq}$, $m = \frac{2(p+q)(3p+2q)z}{\lambda pq}$,

$n = \frac{(3p+2q)(9p+4q)z}{3\lambda pq}$, $r = \frac{(p+q)(9p+4q)z}{pq}$,

$a = \frac{p(3p+2q)z}{\lambda pq}$ and $b = \frac{2q(3p+2q)z}{3\lambda pq}$.

Now we established the following lemma (2.2). The aim of this lemma is to discuss the details of spectrum of $\vec{P}_5$-factorization in different cases, one of which is further discussed in lemma (2.3).

Lemma 2.2:

Case (1): If gcd $(p, 4) = 1$ and gcd $(q, 9) = 1$, then there exist a positive integer s such that,

$m = 6(p+q)(3p+2q)s/\lambda$, $n = (9p+4q)(3p+2q)s/\lambda$,

$a = 3p(3p+2q)s/\lambda$, $b = 3q(4p+3q)s/\lambda$ and $r = 3(p+q)(9p+4q)s$.

Case (2): If gcd $(p, 4) = 1$ and gcd $(q, 9) = 3$. Let $q = 3q_1$, then there exist a positive integer s such that

$m = 6(p+3q_1)(p+2q_1)s/\lambda$, $n = 3(3p+4q_1)(p+2q_1)s/\lambda$,

$a = 3p(p+2q_1)s/\lambda, b = 6q_1(p+2q_1)s/\lambda$ and $r = 3(p+3q_1)(3p+4q_1)s$.

Case (3): If gcd $(p, 4) = 1$ and gcd $(q, 9) = 9$. Let $q = 9q_2$, then there exist a positive integer s such that

$m = 2(p+9q_2)(p+6q_2)s/\lambda$, $n = 3(p+4q_2)(p+6q_2)s/\lambda$,

$a = p(p+6q_2)s/\lambda$, $b = 6q_2(p+6q_2)s/\lambda$ and $r = 18(p+4q_2)(p+9q_2)s$.

Case (4): If gcd $(p, 4) = 2$ and gcd $(q, 9) = 1$. Let $p = 2p_1$, then there exist a positive integer s such that

$m = 6(2p_1+q)(3p_1+q)s/\lambda$, $n = 2(9p_1+2q)(3p_1+q)s/\lambda$,

$a = 6p_1(3p_1+q)s/\lambda$, $b = 2q(3p_1+q)s/\lambda$ and $r = 3(2p_1+q)(9p_1+2q)s$.

Case (5): If gcd $(p, 4) = 2$ and gcd $(q, 9) = 3$. Let $p = 2p_1$, $q = 3q_1$, then there exist a positive integer s such that

$m = 6(2p_1+3q_1)(p_1+q_1)s/\lambda$, $n = 6(3p_1+2q_1)(p_1+q_1)s/\lambda$,

$a = 6p_1(p_1+q_1)s/\lambda$, $b = 6q_1(p_1+q_1)s/\lambda$ and $r = 18(2p_1+3q_1)(3p_1+2q_1)s$.

Case (6): If gcd $(p, 4) = 2$, and $gcd(q, 9) = 9$. Let $p = 2p_1$, $q = 9q_2$, then there exist a positive integer s such that

$m = 2(2p_1+9q_2)(p_1+3q_2)s/\lambda$, $n = 6(p_1+2q_2)(p_1+3q_2)s/\lambda$,

$a = 2p_1(p_1+3q_2)s/\lambda$, $b = 6q_2(p_1+3q_2)s/\lambda$ and $r = 3(2p_1+9q_2)(p_1+3q_2)s$.

Case (7): If gcd $(p, 4) = 4$ and $gcd(q, 9) = 1$. Let $p = 4p_2$, then there exist a positive integer s such that

$m = 3(4p_2+q)(6p_2+q)s/\lambda$, $n = 2(9p_2+q)(6p_2+q)s/\lambda$,

$a = 6p_2(6p_2+q)s/lambda$, $b = q(6p_2+q)s/\lambda$ and $r = 12(4p_2+q)(9p_2+q)s$.

Case (8): If gcd $(p, 4) = 4$, gcd $(q, 9) = 3$. Let $p = 4p_2$, $q = 3q_1$, then there exist a positive integer s such that

$m = 3(4p_2 + 3q_1)(2p_2 + q_1)s/\lambda$, $n = 6(3p_2 + q_1)(2p_2 + q_1)s/\lambda$,

$a = 6p_2(2p_2 + q_1)s/\lambda$, $b = 3q_1(2p_2 + q_1)s/\lambda$ and $r = 3(4p_2 + 3q_1)(3p_2 + q_1)s$.

Case (9): If gcd $(p, 4) = 4$ and gcd $(q, 9) = 9$. Let $p = 4p_2$, $q = 9q_2$, then there exist a positive integer s such that

$m = (4p_2 + 9q_2)(2p_2 + 3q_2)s/\lambda$, $n = 6(p_2 + q_2)(2p_2 + 3q_2)s/\lambda$,

$a = 2p_2(2p_2 + 3q_2)s/\lambda$, $b = 3q_2(2p_2 + 3q_2)s/\lambda$ and $r = 3(4p_2 + 9q_2)(p_2 + q_2)s$.

Proof: Here we are giving the proof of case (1). If gcd $(p, q) = 1$, gcd $(p, 4) = 1$ and gcd $(q, 9) = 1$, then gcd $(9p + 4q, 3) = gcd(3p + 2q, 3) = 1$ and gcd $(9p, 4) = gcd(3p, 2) = gcd(9p + 4q, 2) = 1$ hold. Hence, gcd(9p + 4q, pq) = gcd(3p + 2q, pq) = 1 (lemma 2.1). Since $n = \frac{(9p+4q)(3p+2q)z}{3\lambda pq}$ is an integer, hence we observe that $\frac{z}{3\lambda pq}$ (call it s) will be an integer. Then the equalities in (1) hold.

The proofs of other equalities of lemma (2.2) in different cases are similar to (1). Now we will establish the value of m and n for $\overrightarrow{P}_5$-factorization. We observe that cases (1) and (9), (2) and (8), (3) and (7), and (4) and (6) are symmetrical. Therefore, we are giving the direct construction of case (1) only, others will be similar. Here we are taking $s = 1$.

Lemma 2.3: For any positive integer p and q let $m = 6(p + q)(3p + 2q)/\lambda$, and $n = (9p + 4q)(3p + 2q)/\lambda$. Then $\lambda K^*_{m,n}$ has $\overrightarrow{P}_5$ -factorization.

Proof: Let $a = 3p(3p + 2q)/\lambda$, and $b = 2q(3p + 2q)/\lambda$. It implies $r = 3(p+q)(9p+4q) = r_1.r_2$(say). With $r_1 = 3(p + q)$ and $r_2 = (9p + 4q)$. Also $m_0 = m/r_1 = 2(3p + 2q)/\lambda$ and $n_0 = n/r_2 = (3p + 2q)/\lambda$.

Let X, Y be two partite sets of $\lambda K^*_{m,n}$ such that

$X = \{x_{i,j}; 1 \leq i \leq r_1, 1 \leq j \leq m_0\}$,

$Y = \{y_{i,j}; 1 \leq i \leq r_2, 1 \leq j \leq n_0\}$.

Where first subscript of $x_{i,j}$'s and $y_{i,j}$'s taken addition modulo r_1 and r_2. The second subscript of $x_{i,j}$'s and $y_{i,j}$'s taken addition modulo m_0 and n_0.

For constructing a $\overrightarrow{P}_5$ -factor of $\lambda K^*_{m,n}$, we need $t = (m + n)/5 = (3p + 2q)^2/\lambda$ number of vertex disjoint copies of $\overrightarrow{P}_5$. In these copies, there are $a = 3p(3p + 2q)/\lambda$ number of type M, $\overrightarrow{P}_5$-factor and $b = 2q(3p + 2q)/\lambda$ number of type W $\overrightarrow{P}_5$-factor. Here type M denotes

$\overrightarrow{P}_5$-factor with its end points in Y, and type W denotes $\overrightarrow{P}_5$-factor with its end points in X.

Type M copies of $\overrightarrow{P}_5$;

Now for each $1 \leq i \leq 3p$, let

$E_i = \left\{x_{i+1,j+(3p+2q)u}, y_{3(i-1)+u+v+1,j+(i-1)+u} : 1 \leq j \leq (3p+2q)/\lambda, 0 \leq u, v \leq 1\right\}$.

Type W copies of $\overrightarrow{P}_5$;

Again for each $1 \leq i \leq q$, let

$E_{3p+i} = \{x_{3p+3(i-1)+u+v,j+(3p+2q)w}, y_{9p+4(i-1)+2w+u,j+3p+2(i-1)+u+v+w-1}$

$: 1 \leq j \leq (3p+2q)/\lambda, 1 \leq u \leq 2, 0 \leq v, w \leq 1\}$.

Let $\overrightarrow{F} = U_{1 \leq i \leq 3p+2q} E_i$ gives the total $\overrightarrow{P}_5$-factors. Obviously $\overrightarrow{F}$ contains $t = (m+n)/5 = (3p+2q)^2/\lambda = (3p+2q)n_0$ vertex disjoint and edge disjoint $\overrightarrow{P}_5$ components and span $\lambda K^*_{m,n}$. Define a bijection $\sigma : X \cup Y \overrightarrow{onto} X \cup Y$ in such a way that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$. Where $i \in (1, 2, ..., r_1)$ and each $j \in (1, 2, ..., r_2)$, let

$\overrightarrow{F}_{i,j} = \left\{\sigma^i(x)\sigma^j(y) : x \in X, y \in Y, xy \in \overrightarrow{F}\right\}$.

It is easy to show that the digraph, $\overrightarrow{F}_{i,j}$ $\{1 \leq i \leq r_1, 1 \leq j \leq r_2\}$,are edge disjoint $\overrightarrow{P}_5$ -factor of $\lambda K^*_{m,n}$ and its union is also $\lambda K^*_{m,n}$.

Thus $\left\{\overrightarrow{F}_{i,j} : 1 \leq i \leq r_1, 1 \leq j \leq r_2\right\}$ is a $\overrightarrow{P}_5$ -factorization of $\lambda K^*_{m,n}$.

This is the proof of lemma(2.3), similarly we give the proof of other cases in lemma (2.2).

Proof: By using theorem 2.2 and theorem (2.3) with lemma 2.3, it can be seen that when the parameters m and n satisfy condition (1)-(4) in theorem 2.1, the symmetric complete bipartite multi - digraph $\lambda K^*_{m,n}$ has $\overrightarrow{P}_5$ -factorization. This completes the proof of theorem 2.1.

References

[1] Wang H: P_{2p}-factorization of a complete bipartite graph, *Discrete math,* **120** (1993), 307 - 308.

[2] Beiling Du: P_{2k}-factorization of a complete bipartite multi graph. *Australasian Journal Of Combinatorrics,***21** (2000), 197 - 199.

[3] Ushio K: G-designs and related designs, *Discrete maths* **116** (1993), 299-311.

[4] Ushio K: P_3- factorization of complete bipartite graphs, *Discrete math,* **72** (1988), 361-366.

[5] Wang J and Du B, P_5 - factorization of complete bipartite graphs, *Discrete math,* **308** (2008), 1665-1673.

[6] Wang J: P_7 - factorization of complete bipartite graphs, *Australasian Journal Of Combinatorics,* **33** (2005), 129-137.

[7] Raput U S and Bal Govind Shukla :P_9 -factorization of complete bipartite graphs, *Applied Mathematical Sciences,***5**(2011) 921 - 928.

[8] Wang j and Du B; P_3-factorization of complete bipartite multigraphs and symmetric Complete bipartite multi - digraphs, *Utilitas Math,* **63** (2003), 213 -228.

[9] F. Harary, *Graph theory,*Adison Wesley, Massachusetts, 1972.