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PAIR DIFFERENCE CORDIAL LABELING OF GRAPHS

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Abstract. Let $G = (V,E)$ be a (p,q) graph.

Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels.

Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph. In this paper we investigate the pair difference cordial labeling behavior of path, cycle, star, comb.

Keywords: path; cycle; complete graph; star; bistar; comb.

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1. INTRODUCTION

In this paper we consider only finite, undirected and simple graphs.The notion of diference cordial labeling of a graph was introduced and studied some properties of difference cordial labeling in [4] .The difference cordial labeling behavior of several graphs like path, cycle, star etc have been investigated in [4].In this paper we introduce the pair difference cordial labeling and investigate pair difference cordial labeling behavior of path,cycle,star,comb and bistar graph.

2. PRELIMINARIES

Definition 2.1. The ladder L_n is the product graph $P_n X K_2$ with $2n$ vertices and $3n-2$ edges.

Definition 2.2. The graph obtained by joining two disjoint cycles $u_1u_2, \cdots u_m u_1$ and $v_1v_2, \cdots v_n v_1$ with an edge u_1v_1 is called dumbbell graph and it is denoted by $Db(m,n)$.

3. PAIR DIFFERENCE CORDIAL LABELING

Definition 3.1. Let $G = (V,E)$ be a (p,q) graph.
Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels.
Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1.A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph.

Theorem 3.1. If G is a (p,q) pair difference cordial graph then

$$q \leq \begin{cases} 2p-3 & \text{if } p \text{ is even} \\ 2p-1 & \text{if } p \text{ is odd} \end{cases}$$

Proof. **Case 1.** p is even.

The maximum number of edges with the label 1 among the vertex labels $1,2,3,\cdots,\frac{p}{2}$ respectively is $\frac{p}{2}-1$. Also the maximum number of edges with the label 1 among the vertex labels $-1,-2,-3,\cdots,-\frac{p}{2}$ respectively is $\frac{p}{2}-1$. Therefore $\Delta_{f_1} \leq (\frac{p}{2}-1)+(\frac{p}{2}-1) = p-2$.That is $\Delta_{f_1} \leq p-2$, This implies $\Delta_{f_1^c} \geq q-p+2 \longrightarrow (1)$.

Type 1. $\Delta_{f_1^c} = \Delta_{f_1}+1$.

$$\begin{aligned} \text{By } (1), q-p+2 &\leq \Delta_{f_1^c}, \\ &\leq \Delta_{f_1}+1 \\ &\leq p-1. \text{This implies } q \leq 2p-3. \longrightarrow (2) \end{aligned}$$

Type 2. $\Delta_{f_1^c} = \Delta_{f_1}-1$.

$$\begin{aligned} \text{By } (1), q-(p-2) &\leq \Delta_{f_1^c}, \\ &\leq \Delta_{f_1}-1, \\ &\leq p-3. \text{This implies } q \leq 2p-5. \longrightarrow (3) \end{aligned}$$

Type 3. $\Delta_{f_1^c} = \Delta_{f_1}$.

$$\begin{aligned} \text{By } (1), q-(p-2) &\leq \Delta_{f_1^c}, \\ &\leq \Delta_{f_1}, \\ &\leq p-2. \end{aligned}$$

This implies $q \leq 2p-4 \longrightarrow (4)$.By (2),(3),(4),$q \leq 2p-3$.

Case 2. p is odd.

In this case,one vertex label is repeated.This vertex label contributes maximum two edges with label 1.Therefore, $\Delta_{f_1} \leq (\frac{p-1}{2}-1)+(\frac{p-1}{2}-1)+2 = p+1$.As in case (1), we get $q \leq 2p-1$. □

Theorem 3.2. The path P_n is pair difference cordial for all values of n except $n \neq 3$.

Proof. Let P_n be the path $u_1u_2\cdots u_n$.

Case. 1 n is odd.

There are two cases arises.

Subcase. 1 $n = 4t+1, t \in N \cup \{0\}$.

Assign the labels $1,2$ to the vertices u_1,u_2 respectively and assign the labels $-1,-2$ respectively to the vertices u_3,u_4.Next assign the labels $3,4$ respectively to the vertices u_5,u_6 and assign the labels $-3,-4$ to the vertices u_7,u_8 respectively.Proceeding like this untill we reach the vertex u_{n-1}.Finally assign the label -2 to the vertex u_n.Note that the vertices u_{n-4},u_{n-3} get the labels $\frac{n-3}{2},\frac{n-1}{2}$ respectively and the vertices u_{n-2},u_{n-1} receive the labels $-\frac{n-3}{2},-\frac{n-1}{2}$ respectively.
This vertex labeling gives the pair difference cordial labeling of path P_n,since $\Delta_{f_1} = \Delta_{f_1^c} = \frac{n-1}{2}$.

Subcase. 2 $n = 4t+3, t \in N$.

Assign the labels $1,2$ respectively to the vertices u_1,u_2 and assign the label $-1,-2$ to the vertices u_3,u_4 respectively.Next assign the labels $3,4$ respectively to the vertices u_5,u_6 and assign the labels $-3,-4$ to the vertices u_7,u_8 respectively.Proceeding like this untill we reached u_{n-3}.Assign the label $-\frac{n-3}{2}$ to the vertex u_n.Finally assign the labels $\frac{n-1}{2},-\frac{n-1}{2}$ respectively to the vertices u_{n-2},u_{n-1}.Note that the vertices u_{n-6},u_{n-5} received the labels $\frac{n-5}{2},\frac{n-3}{2}$ respectively and the vertices u_{n-4},u_{n-3} get the labels $-\frac{n-5}{2},-\frac{n-3}{2}$ respectively.
This vertex labeling gives the pair difference cordial labeling of path P_n,since $\Delta_{f_1} = \Delta_{f_1^c} = \frac{n-1}{2}$.

Subcase. 3 $n = 3$.

Suppose f is a pair difference cordial of P_3, then $\Delta_{f_1} = 0$ and $\Delta_{f_1^c} = 2$. This contradicts P_3 is not pair difference cordial.

Case. 2 n is even.

There are two cases arises.

Subcase. 1 $n = 4t, t \in N$.

Assign the labels $1,2$ to the vertices u_1, u_2 respectively and assign the labels $-1,-2$ to the vertices u_3, u_4 respectively.Next assign the labels $3,4$ to the vertices u_5, u_6 respectively and assign the labels $-3,-4$ respectively to the vertices u_7, u_8.Proceeding like this untill we reach the vertex u_n.Note that the vertices u_{n-3}, u_{n-2} respectively receive the labels $\frac{n-2}{2}, \frac{n}{2}$ and the vertices u_{n-1}, u_n get the labels $-\frac{n-2}{2}, -\frac{n}{2}$ respectively.

This vertex labeling gives a pair difference cordial labeling of the path P_n,since $\Delta_{f_1} = \frac{n}{2}, \Delta_{f_1^c} = \frac{n-2}{2}$.

Subcase. 2 $n = 4t+2, t \in N \cup \{0\}$.

Assign the labels $1,2$ respectively to the vertices u_1, u_2.Now assign the labels $-1,-2$ to the vertices u_3, u_4 respectively.Next assign the label $3,4$ respectively to the vertices u_5, u_6 and assign the label $-3,-4$ to the vertices u_7, u_8 respectively.Proceeding like this until we reach the vertex u_{n-2}.Finally assign the labels $\frac{n}{2}, -\frac{n}{2}$to the vertices u_{n-1}, u_n respectively.Note that the vertices u_{n-5}, u_{n-4} get the label $\frac{n-4}{2}, \frac{n-2}{2}$ respectively and the vertices u_{n-3}, u_{n-2} receive the labels $-\frac{n-4}{2}, -\frac{n-2}{2}$ respectively.

This vertex labeling gives the pair difference cordial labeling of path P_n,since $\Delta_{f_1} = \frac{n-2}{2}, \Delta_{f_1^c} = \frac{n}{2}$.

□

Remark. P_3 is difference cordial but not pair difference cordial [4].

Corollary 3.2.1. The cycle C_n is pair difference cordial if and only if $n > 3$.

Proof. Let C_n be the cycle $u_1 u_2 \cdots u_n u_1$. The function f in the theorem 3.3 is also a pair difference cordial labeling of the cycle C_n.

□

Theorem 3.3. The star $K_{1,n}$ is pair difference cordial if and only if $3 \leq n \leq 6$.

Proof. Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}, E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$.The graph $K_{1,n}$ has $n+1$ vertices and n edges.

Case 1. $3 \leq n \leq 6$.

Table 1 shows that the star $K_{1,n}, 3 \leq n \leq 6$ is pair difference cordial.

n	u	u_1	u_2	u_3	u_4	u_5	u_6
3	2	-1	1	-2			
4	2	-1	1	-2	2		
5	2	-1	1	-2	3	-3	
6	2	1	-1	1	-2	3	-3

TABLE 1

Case 2. $n \geq 6$.

Suppose f is a pair difference cordial labeling of $K_{1,n}$.Assume $f(u) = l$,To get the edge label 1, the only possibly is that the pendant vertices receive the label $l-1$ or $l+1$.

Subcase 1. n is odd.

In this case, $\Delta_{f_1} \leq 2$.This implies $\Delta_{f_1} - \Delta_{f_1^c} \geq n-4 > 1$, a contradiction.

Subcase 2. n is even.

In this case, we may use one vertex label as twice.This implies $\Delta_{f_1} \leq 3$. Therefore $\Delta_{f_1} - \Delta_{f_1^c} \geq n-6 > 1$, a contradiction.

□

Remark. The star $K_{1,6}$ is pair difference cordial but not difference cordial[4].

Corollary 3.3.1. The complete graph K_p is pair difference cordial if and only if $p \leq 2$.

Proof. **Case 1**. $p \leq 2$.

By theorem 3.3, K_1, K_2 is pair difference cordial.

Case 2. $3 \leq p \leq 5$.

The Table 2 shows that K_3, K_4, K_5 is not pair diffrence cordial.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
3	3	0
4	2	4
5	3	7

TABLE 2

Case 2. $p \geq 6$.

Suppose K_p is pair difference cordial.By theorem 3.2, $\binom{p}{2} \leq 2p+1$.This implies $\frac{p(p-1)}{2} \leq 2p+1$,a contradiction to $p \geq 6$. □

Theorem 3.4. The comb $Pn \odot K_1$ is a pair difference cordial for all values of n.

Proof. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$.

Define a map $f : V(P_n \odot K_1) \to \{\pm 1, \pm 2, \cdots, \pm n\}$ by

$f(u_i) = i, 1 \leq i \leq n$, and $f(v_i) = -i, 1 \leq i \leq n$.Then $\Delta_{f_1} = n-1$, $\Delta_{f_1^c} = n$. □

Theorem 3.5. $K_2 + mK_1$ is pair difference cordial if and only if $m = 2$.

Proof. Let $V(K_2+mK_1) = \{u, v, u_i : 1 \leq i \leq m\}$ and $E(K_2+mK_1) = \{uu_i, vu_i : 1 \leq i \leq m\} \cup \{uv\}$.

Case 1.$m = 2$.

Define $f(u) = -1, f(v) = 1$ and $f(u_1) = 2, f(u_2) = -2$,Then $\Delta_{f_1^c} = 3, \Delta_{f_1} = 2$.

Case 2.$m \geq 3$.

Suppose f is a pair difference cordial.Assume $f(u) = l_1$ and $f(v) = l_2$.To get the edge label 1, the only possibly is that the vertices with degree two receive the label $l_1 - 1$ or $l_1 + 1$ and $l_2 - 1$ or $l_2 + 1$.

Subcase 1.m is even.

In this case $\Delta_{f_1} \leq 2$,$\Delta_{f_1^c} \geq 2m-1$.This implies $\Delta_{f_1^c} - \Delta_{f_1} \geq 2m-3 > 1$, a contradiction.

Subcase 2.m is odd.

In this case we may use one vertex label as twice. This implies $\Delta_{f_1} \leq 3$,$\Delta_{f_1^c} \geq 2m-2$. Teherefore $\Delta_{f_1^c} - \Delta_{f_1} \geq 2m-5 > 1$, a contradiction.

□

Theorem 3.6. Th bistar $B_{1,n}$ is pair difference cordial if and only if $2 \leq n \leq 6$.

Proof. Let $V(B_{1,n}) = \{u, v, u_1, v_i : 1 \leq i \leq n\}$ and $E(B_{1,n}) = \{uu_1, vv_i, uv : 1 \leq i \leq n\}$.

Case 1. $2 \leq n \leq 6$. Define $f(u) = 2, f(u_1) = 1, f(v) = -2$ and Table 3 shows that the bistar $B_{1,n}, 2 \leq n \leq 6$ is pair difference cordial.

n	u_1	u_2	u_3	u_4	u_5	u_6
2	-1	2				
3	-1	3	-3			
4	-1	-3	1	3		
5	-1	-3	-4	3	4	
6	-1	-3	-4	3	4	-1

TABLE 3

Case 2. $n \geq 7$.

Suppose $f(u) = l_1, f(v) = l_2$, then the maximum value of Δ_{f_1} is attained when $f(u_1) = l_1 - 1, f(v_i) = l_2 - 1, f(v_j) = l_2 + 1$ for some i and j. Therefore $\Delta_{f_1} \leq 1 + 2 = 3$ .That is $\Delta_{f_1} \leq 3$. This implies $\Delta_{f_1^c} \geq n + 2 - 3$. Therefore $\Delta_{f_1^c} \geq n - 1$. Hence $\Delta_{f_1^c} - \Delta_{f_1} \geq n - 1 - 3 > 1$, a contradiction. □

Theorem 3.7. The bistar $B_{m,n}, (m \geq 2, n \geq 2)$ is pair difference cordial if and only if $m + n \leq 9$.

Proof. Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m 1 \leq j \leq n\}$ and $E(B_{3,n}) = \{uu_i, vv_j, uv : 1 \leq i \leq n, 1 \leq j \leq n\}$.

There are two cases arises.

Case 1. $m+n \leq 9$.

There are two subcase arises.

Subcase 1. $n = m = 2$.

Define $f(u) = 1, f(v) = -1, f(u_1) = 2, f(u_2) = -3, f(v_1) = -2, f(v_2) = 3$. Here $\Delta_{f_1} = 2$ and $\Delta_{f_1^c} = 3$.

Subcase 2. $n > 2, m > 2$.

Define $f :\longrightarrow \{\pm 1, \pm 2, \cdots, \pm \frac{m+n}{2}\}$ by $f(u) = 2, f(v) = -2, f(u_1) = 1, f(u_2) = 3, f(v_1) = -1, f(v_2) = -3$. Next assign the remaining labels to the remaining vertices in any order.

Case 2. $m+n \geq 10$.

There are two subcase arises.

Subcase 1. $m+n$ is even.

Suppose $f(u) = l_1, f(v) = l_2$, then the maximum value of Δ_{f_1} is attained when $f(u_i) = l_1 - 1, f(u_j) = l_1 + 1$ for some i and j ,$f(v_i) = l_2 - 1, f(v_j) = l_2 + 1$ for some i and j. Therefore $\Delta_{f_1} \leq 2+2 = 4$ .This implies that $\Delta_{f_1^c} \geq m+n+1-4$.Therefore $\Delta_{f_1^c} \geq m+n-3$. Hence $\Delta_{f_1^c} - \Delta_{f_1} \geq m+n-7$,a contradiction.

Subcase 2. $m+n$ is odd.

When $m+n$ is odd, either m or n is odd.Hence one vertex label is repeated. Therefore $\Delta_{f_1} \leq 3+2$. That is $\Delta_{f_1} \leq 5$. This implies $\Delta_{f_1^c} \geq m+n-4$. Hence $\Delta_{f_1^c} - \Delta_{f_1} \geq m+n-9 > 1$, a contradiction.

Therefore $B_{m,n}, m+n \geq 10$ is not pair difference cordial.

□

Theorem 3.8. The laddar graph $P_2 \times P_n$ is pair difference cordial for all values of n.

Proof. Let $V(P_2 \times P_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_2 \times P_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\}$.

Case 1. $n = 2$.

Let $P_2 \times P_2 \cong C_4$, is pair difference cordial by theorem 3.3.

Case 2. $n \geq 3$.

First we assign the labels $-1, -2, -3, \cdots, -n$ to the vertices $u_1, u_2, u_3, \cdots, u_n$ respectively.Now consider the vertices $v_i, (1 \leq i \leq n)$.There are four cases arises.

Subcase 1. $n \equiv 0 \pmod 4$.

Assign the labels $1,2$ to the vertices v_1, v_2 respectively.Next assign the labels $3,5$ respectively to the vertices v_3, v_4 and assign the labels $4,6$ to the vertices v_5, v_6 respectively.Now assign the labels $7,9$ to the vertices v_7, v_8 respectively and assign the labels $8,10$ to the vertices v_9, v_{10} respectively.Proceeding like this until we reach v_n.Note that in this process the vertex v_n get the label $n-1$.

Subcase 2. $n \equiv 1 \pmod 4$.

As in Subcase 1, assign the labels to the vertices $v_i, (1 \leq i \leq n)$.Here the vertex v_n receive the label $n-1$.

Subcase 3. $n \equiv 2 \pmod 4$.

Assign the labels to the vertices $v_i, (1 \leq i \leq n)$ as in Subcase 1.In this case the vertex v_n get the label n.

Subcase 4. $n \equiv 3 \pmod 4$.

Similar to Subcase 1 assign the labels to the vertices $v_i, (1 \leq i \leq n)$.Note that the vertex v_n receive the label n.

The Table 4 given below establish that this vertex labeling f is a pair difference cordial of $P_n \times P_2$.

□

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-3}{2}$	$\frac{3n-1}{2}$
n is even	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

TABLE 4

Theorem 3.9. The dumbbell graph $Db(n,n)$ is pair difference cordial for all values n.

Proof. The vertex set and the edge set of $Db(n,n)$ is given in definition 2.2.

There are four cases arises.

Case 1. $n \equiv 0 \pmod 4$.

Assign the labels $1,2$ respectively to the vertices u_1,u_2 then assign the labels $4,3$ to the vertices u_3,u_4.Secondly assign the labels $5,6$ to the vertices u_5,u_6 then assign the labels $8,7$ to the vertices u_7,u_8.Proceeding like this until we reach the vertex u_n.Note that in this the vertex u_{n-1} get the label $n-1$.Next assign the label to the vertices $v_i, 1 \le i \le n$. Assign the labels $-1,-2$ to the vertices v_1,v_2 then assign the labels $-4,-3$ to the vertices v_3,v_4.Secondly assign the labels $-5,-6$ to the vertices v_5,v_6 then assign the labels $-8,-7$ to the vertices v_7,v_8.Proceeding like this until we reach the vertex v_n.Note that in this the vertex v_n receive the label $-n+1$.

Case 2. $n \equiv 1 \pmod 4$.

Assign the labels $1,2,3$ to the vertices u_1,u_2,u_3 then assign the labels $5,4$ to the vertices u_4,u_5.Secondly assign the labels $6,7$ to the vertices u_6,u_7 then assign the labels $9,8$ to the vertices u_8,u_9.Proceeding like this until we reach the vertex u_n.Note that in this the vertex u_n rceive the label $n-1$. As in case 1 assign the label to the vertices $v_i, 1 \le i \le n$.Note that in this the vertex v_{n-1},v_n get the label $-n+2,-n$.

Case 3. $n \equiv 2 \pmod 4$.

As in case 1 assign the label to the vertices $u_i, 1 \le i \le n$.Note that in this the vertex u_{n-1},u_n receive the label $n-1,n$.Assign the label as in case 1 to the vertices $v_i, 1 \le i \le n$.Note that in this way the vertex v_{n-1},v_n get the label $-n+1,-n$.

Case 4. $n \equiv 3 \pmod 4$.

As in case 1 assign the label to the vertices $u_i, 1 \le i \le n$.Note that in this process the vertex u_{n-1},u_n receive the label $n-1,n$.Assign the label as in case 1 to the vertices $v_i, 1 \le i \le n$.Note that here the vertices v_{n-1},v_n get the label $-n,-n+1$.

The Table 5 given below establish that this vertex labeling f is a pair difference cordial of $Db(n,n)$.

□

Theorem 3.10. The dumbbell graph $Db(n+1,n)$ is pair difference cordial for all values n.

Proof. The vertex set and the edge set of $Db(n+1,n)$ is given in definition 2.2.

Case 1. $n \equiv 0 \pmod 4$.

Subcase 1. $n > 4$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$n \equiv 0 \pmod 4$	$n+1$	n
$n \equiv 1 \pmod 4$	n	$n+1$
$n \equiv 2 \pmod 4$	n	$n+1$
$n \equiv 3 \pmod 4$	$n+1$	n

TABLE 5

Assign the labels $1,2$ respectively to the vertices u_1,u_2 then assign the labels $4,3$ to the vertices u_3,u_4.Secondly assign the labels $5,6$ to the vertices u_5,u_6 then assign the labels $8,7$ to the vertices u_7,u_8.Proceeding like this until we reach the vertex u_n.Next assign the label 2 to the vertex u_{n+1}.Now we consider the vertices $v_i, 1 \leq i \leq n$. Assign the labels $-1,-2$ to the vertices v_1,v_2 then assign the labels $-4,-3$ to the vertices v_3,v_4.Secondly assign the labels $-5,-6$ to the vertices v_5,v_6 then assign the labels $-8,-7$ to the vertices v_7,v_8.Proceeding like this until we reach the vertex v_n.Note that in this the vertex v_n receive the label $-n+1$.

Subcase 2. $n = 4$.

As in case 1, assign the labels to the vertices $u_i, 1 \leq i \leq 4$ and $v_i, 1 \leq i \leq 4$.Finally assign the label 1 to the vertex u_5.

Case 2. $n \equiv 1 \pmod 4$.

Subcase 1. $n > 5$.

As in case 1, assign the labels to the vertices $u_i, 1 \leq i \leq n+1$.Next consider the vertices $v_i, 1 \leq i \leq n$. Assign the labels $-1,-2,-3$ to the vertices v_1,v_2,v_3 then assign the labels $-5,-4$ to the vertices v_4,v_5.Secondly assign the labels $-6,-7$ to the vertices v_6,v_7 then assign the labels $-8,-7$ to the vertices v_8,v_9.Proceeding like this until we reach the vertex v_n.Note that in this the vertex v_n receive the label $-n+1$.

Subcase 2. $n = 5$.

As in case 1, assign the labels to the vertices $u_i, 1 \leq i \leq 5$ and $v_i, 1 \leq i \leq 5$.Finally assign the label 1 to the vertex u_5.

Case 3. $n \equiv 2 \pmod 4$.

As in case 1, assign the labels to the vertices $u_i, 1 \leq i \leq n+1$ and $v_i, 1 \leq i \leq n$.

Case 4. $n \equiv 2 \pmod 4$.

Subcase 1. $n > 3$.

As in case 2, assign the labels to the vertices $u_i, 1 \leq i \leq n$ and $v_i, 1 \leq i \leq n$.Finally assign the label 1 to the vertex u_{n+1}.

Subcase 2. $n = 3$.

Assign the labels $-1, -2, -3$ to the vertices v_1, v_2, v_3.Now assign the labels $1, 2, 3$ to the vertices u_1, u_2, u_3.Finally assign the label 1 to the vertex u_4.

□

Theorem 3.11. The dumbbell graph $Db(m,n)$ is pair difference cordial for all values $m > n+1$.

Proof. Take the vertex set and edge set in definition 2.2.

There are four cases arises.

Case 1. $n \equiv 0 \pmod 4$.

Assign the labels $-1, -2$ respectively to the vertices v_1, v_2 and assign the labels $-4, -3$ to the vertices v_3, v_4 respectively.Secondly assign the labels $-5, -6$ to the vertices v_5, v_6 respectively.Next assign the labels $-8, -7$ to the vertices v_7, v_8 respectively.Proceeding like this until we reach the vertex v_n.Note that in this the vertex v_n receive the label $-n+1$.Next cosider the vertices $u_i, 1 \leq i \leq m$.

Assign the labels $1, 2$ to the vertices u_1, u_2 respectively and assign the labels $4, 3$ respectively to the vertices u_3, u_4.Now assign the labels $5, 6$ to the vertices u_5, u_6 respectively and assign the labels $8, 7$ respectively to the vertices u_7, u_8.Proceeding like this until we reach the vertex u_n.Finally consider the remaining $m-n$ vertices.There are four cases arises.

Subcase 1. $m \equiv 0 \pmod 4$.

Assign the labels $n+1, n+2$ to the vertices u_{n+1}, u_{n+2} respectively and assign the labels $-n-1, -n-2$ respectively to the vertices u_{n+3}, u_{n+4}.Secondly assign the labels $n+3, n+4$ to the vertices u_{n+5}, u_{n+6} respectively.Next assign the labels $-n-3, -n-4$ respectively to the vertices u_{n+7}, u_{n+8}.Proceeding like this until we reach the vertex u_m.

Subcase 2. $m \equiv 1 \pmod 4$.

As in subcase 1 assign the labels to the vertices $u_i, 1 \leq i \leq m-1$ and assign the label $m-1$ to the vertex u_m.

Subcase 3. $m \equiv 2 \pmod 4$.

Assign the labels as in subcase 1 to the vertices $u_i, 1 \le i \le m-1$.Next assign the label $\frac{m+n}{2}$ to the vertex u_m.

Subcase 4. $m \equiv 3 \pmod 4$.

Assign the labels as in subcase 1 to the vertices $u_i, 1 \le i \le m-3$ and lastly assign the labels $-\frac{m+n}{2}, \frac{m+n}{2}, 2$ respectively to the vertices u_{m-2}, u_{m-1}, u_m.

The Table 6 given below establish that this vertex labeling f is a pair difference cordial of $Db(m,n)$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$m \equiv 0 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n+2}{2}$
$m \equiv 1 \pmod 4$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$
$m \equiv 2 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n+2}{2}$
$m \equiv 3 \pmod 4$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$

TABLE 6

Case 2. $n \equiv 1 \pmod 4$.

Assign the labels as in case 1 to the vertices $v_i, (1 \le i \le n)$.Here note that the vertex v_n receive the label $-n+1$.

Next consider the remaining $m-n$ vertices.There are four cases arises.

Subcase 1. $m \equiv 0 \pmod 4$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, (1 \le i \le m-3)$ and assign the labels $\frac{m+n-1}{2}, \frac{m+n-1}{2}, 2$ to the vertices u_{m-2}, u_{m-1}, u_m respectively.

Subcase 2. $m \equiv 1 \pmod 4$.

As in case 1, assign the labels to the vertices $u_i, 1 \le i \le m$.

Subcase 3. $m \equiv 2 \pmod 4$.

Assign the labels as in subcase 1 to the vertices $u_i, 1 \le i \le m-1$.Next assign the label 2 to the vertex u_m.

Subcase 4. $m \equiv 3 \pmod 4$.

Assign the labels as in subcase 1 to the vertices $u_i, 1 \le i \le m-2$.Finally assign the labels $\frac{m+n}{2}, -\frac{m+n}{2}$ respectively to the vertices u_{m-1}, u_m.

The Table 7 given below establish that this vertex labeling f is a pair difference cordial of $Db(m,n)$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$m \equiv 0 \pmod 4$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$
$m \equiv 1 \pmod 4$	$\frac{m+n+2}{2}$	$\frac{m+n}{2}$
$m \equiv 2 \pmod 4$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$
$m \equiv 3 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n+2}{2}$

TABLE 7

Case 3. $n \equiv 2 \pmod 4$.

Assign the labels as in case 1 to the vertices $v_i, (1 \le i \le n)$.Here note that the vertex v_n received the label $-n$.

Finally we consider the remaining $m-n$ vertices.There are four cases arises.

Subcase 1. $m \equiv 0 \pmod 4$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, (1 \le i \le m-2)$ and assign the labels $-\frac{m+n}{2}, \frac{m+n}{2}$ respectively to the vertices u_{m-1}, u_m.

Subcase 2. $m \equiv 1 \pmod 4$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, (1 \le i \le m-3)$ and assign the labels $\frac{m+n-1}{2}, -\frac{m+n-1}{2}, 2$ to the vertices u_{m-2}, u_{m-1}, u_m respectively.

Subcase 3. $m \equiv 2 \pmod 4$.

Assign the label as in subcase 1 to the vertices $u_i, (1 \le i \le m)$.

Subcase 4. $m \equiv 3 \pmod 4$.

Assign the label as in subcase 1 to the vertices $u_i, (1 \le i \le m-2)$ and assign the label $-\frac{m+n}{2}, \frac{m+n}{2}$ respectively to the vertices u_{m-1}, u_m.

The Table 8 given below establish that this vertex labeling f is a pair difference cordial of $Db(m,n)$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$m \equiv 0 \pmod 4$	$\frac{m+n}{2}$	$\frac{m+n+2}{2}$
$m \equiv 1 \pmod 4$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$
$m \equiv 2 \pmod 4$	$\frac{m+n+2}{2}$	$\frac{m+n}{2}$
$m \equiv 3 \pmod 4$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$

TABLE 8

Case 4. $n \equiv 3 \pmod 4$.

Assign the labels as in case 1 to the vertices $v_i, 1 \leq i \leq n$ and $u_i, 1 \leq i \leq n$.Here note that the vertex v_n received the label $-n$.

We now consider the remaining $m-n$ vertices.There are four cases arises.

Subcase 1. $m \equiv 0 \pmod 4$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, n+1 \leq i \leq m-1$ and assign the labels 2 to the vertex u_m.

Subcase 2. $m \equiv 1 \pmod 4$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, n+1 \leq i \leq m-2$ and assign the labels $\frac{m+n-1}{2}, -\frac{m+n-1}{2}$ to the vertices u_{m-1}, u_m respectively.

Subcase 3. $m \equiv 2 \pmod 4$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, n+1 \leq i \leq m-3$.Finally assign the labels $\frac{m+n-1}{2}, -\frac{m+n-1}{2}, \frac{m+n-1}{2}$ respectively to the vertices u_{m-2}, u_{m-1}, u_m.

Subcase 4. $m \equiv 3 \pmod 4$.

Assign the label as in subcase 1 to the vertices $u_i, 1 \leq i \leq m$.

The Table 9 given below establish that this vertex labeling f is a pair difference cordial of $Db(m,n)$.

□

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$m \equiv 0 \pmod 4$	$\frac{m+n+1}{2}$	$\frac{m+n+2+1}{2}$
$m \equiv 1 \pmod 4$	$\frac{m+n+2}{2}$	$\frac{m+n}{2}$
$m \equiv 2 \pmod 4$	$\frac{m+n+1}{2}$	$\frac{m+n+1}{2}$
$m \equiv 3 \pmod 4$	$\frac{m+n+2}{2}$	$\frac{m+n}{2}$

TABLE 9

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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