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A FITTED NUMERICAL METHOD FOR A CLASS OF SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract: In this paper**,** a class of singularly perturbed differential-difference equations having boundary layer at one end is analysed to get its solution numerically by a fitted method. Such types of equations occur very frequently in various fields of applied mathematics and engineering such as fluid dynamics, quantum mechanics, optimal control, chemical reactor theory etc. The basic purpose of this study is to describe a numerical approach for the solution of singularly perturbed differential-difference equation based on deviating argument and interpolation. Thomas algorithm is used to solve the tri-diagonal system. Numerical examples are presented which demonstrate the applicability of this method.

Keywords: differential-difference equation; boundary layer; deviating argument; interpolation.
2010 AMS Subject Classification: 37M10.

1. INTRODUCTION

Due to the availability of supercomputing and cloud computing, now mathematicians are seriously concentrating on developing the robust numerical methods for solving most challenging problems like Boundary Layer Problems. In general, a region in which the solution of the problem changes rapidly is called Boundary Layer. In fact the solution changes rapidly to

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satisfy the given conditions in the problem. Any ordinary differential equation in which the highest order derivative is multiplied by a small positive parameter which is popularly known as singularly perturbation problem always exhibits the boundary layer phenomenon. Also any differential equation which contains at least one delay/advance parameters which is popularly called as delay/differential-difference equation also exhibits the boundary layer phenomenon. Solving these problems is very difficult due to the boundary layer phenomenon. These problems arise in the modelling of various practical phenomena in bioscience, engineering, control theory, such as in variational problems in control theory, in describing the human pupil-light reflex, in a variety of models for physiological processes or diseases and first exit time problems in the modelling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites. To solve these problems, perturbation methods such as Matched Asymptotic Expansions, WKB method are used extensively. These asymptotic expansions of solutions require skill, insight and experimentation. Further, the Matching Principle: matching of the coefficients of the inner and outer regions solution expansions is also a demanding process. Hence, researchers started developing numerical methods. If we use the existing numerical methods with the step size more than the perturbation parameters, for solving these problems we get oscillatory solutions due to the presence of the boundary layer. Existing numerical methods will produce good results only when we take step size less than the perturbation parameters. This is very costly and time-consuming process. Hence, the researchers are concentrating on developing robust numerical methods, which can work with a reasonable step size. In fact, these robust numerical methods should be independent of the parameters. The efficiency of such numerical method is determined by its accuracy, simplicity in computing the solution and its sensitivity to the parameters of the given problem. Lange and Miura [4-5], were first to publish a series of papers for solving these problems. Diddi Kumara Swami et al [6-7] have presented an accurate numerical method for singularly perturbed differential difference equations with mixed shifts. G. File and Y.N. Reddy [8] presented the numerical Integration of a class of singularly perturbed delay differential equations with small shift. Lakshmi Sirisha, et al[10-11] have presented a mixed finite difference method for singularly perturbed differential difference equations with mixed shifts. Adilaxmi and et al [13-15], have presented an initial value technique using exponentially fitted non-standard finite difference method for singularly perturbed differential - difference equations. M. K. Kadalbajoo and K. K. Sharma[16] have discussed the numerical analysis of boundary-value problems for singularly perturbed differential-difference equations with small shifts of mixed type. Kadalbajoo and Sharma [17-18], have given numerical treatment of boundary value problems for second order singularly perturbed delay differential equations. R. N. Rao and P. Chakravarthy [22] have presented a fitted Numerov method for singularly perturbed parabolic partial differential equation with a small negative shift arising in control theory. Reddy et all [23-25] have presented some simple methods for the solution of singularly perturbed differential-difference equations. For the more theory of perturbation problems, one may refer books: Bender and Orzag [2], Driver[19], El'sgol'ts and Norkin[12], Hale [9], Ali H. Nayfeh [1], RE Bellaman and Cook [20] O'Malley[21], Van Dyke [18].

In this paper**,** a class of singularly perturbed differential-difference equation having boundary layer at one end is analysed to get its solution numerically by fitted method. Such types of equations occur very frequently in various fields of applied mathematics and engineering such as fluid dynamics, quantum mechanics, optimal control, chemical reactor theory etc. The basic purpose of this study is to describe a numerical approach for the solution of singularly perturbed differential-difference equation based on deviating argument and interpolation. Thomas algorithm is used to solve the tri-diagonal system. Numerical examples are presented which demonstrate the applicability of this method.

2. DESCRIPTION OF THE FITTED METHOD

2.1 Type-I: Delay Differential Equation having boundary layer

Consider the delay differential equation of the form:

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1, \tag{1}$$

with boundary conditions

$$y(x) = \varphi(x), \quad -\delta \le x \le 0, \tag{2}$$

and

$$y(1) = \beta, \tag{3}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ is the small delay parameter, $a(x)$, $b(x)$and $f(x)$are sufficiently differentiable functions in $(0, 1)$.$\varphi(x)$ is also bounded continuous function on $[0, 1]$ and β is a finite constant.

From the Taylor's series expansion

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \tag{4}$$

Substituting Equation (4) into Equation (1), we get singularly perturbed ordinary differential equation:

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), 0 \le x \le 1 \tag{5}$$

with boundary conditions

$$y(0) = \alpha \tag{6}$$

$$y(1) = \beta \tag{7}$$

where $\varepsilon' = \varepsilon - a(x)\delta$, $A(x) = a(x)$, $B(x) = b(x)$ and α is a finite constant. Further it is established that, when $a(x) \ge M > 0$ in [0, 1], boundary layer will be at $x = 0$ and when $a(x) \le M < 0$ in [0, 1], boundary layer will be at $x = 1$, where M is some positive number. Since $0 < \delta \ll 1$, the transition from Equation (1) to Equation (5) is admitted. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [12]. Here we assume that $a(x) = a$ and $b(x) = b$ are constants.

2.2 Type-II: Differential-Difference Equation having boundary layer

Consider the differential-difference equation of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-\delta) + c(x)y(x) + d(x)y(x+\eta) = f(x), \tag{8}$$

$0 \le x \le 1$ with boundary conditions

$$y(x) = \varphi(x), \text{ on } -\delta \le x \le 0, \tag{9}$$

$$y(x) = \gamma(x), \text{ on } 1 \le x \le 1 + \eta, \tag{10}$$

with the constant coefficients (i.e., $a(x) = a, b(x) = b, c(x) = c$ and $d(x) = d$ are constants) and $f(x)$, $\varphi(x)$ and $\gamma(x)$ are smooth functions. $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ and $0 < \eta = O(\varepsilon)$ are the delay and advanced parameters respectively.

From Taylor's series expansion

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{11}$$

$$y(x+\eta) \approx y(x) + \eta y'(x) + \frac{\eta^2}{2} y''(x) \tag{12}$$

Substituting Equations (11)-(12) into Equation (8), we get singularly perturbed ordinary differential equation

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), 0 \le x \le 1 \tag{13}$$

with boundary conditions

$$y(0) = \alpha \tag{14}$$

$$y(1) = \beta \tag{15}$$

where

$$\varepsilon' = \varepsilon + b(x)\frac{\delta^2}{2} + d(x)\frac{\eta^2}{2}, \tag{16}$$

$$A(x) = a(x) - \delta b(x) + \eta d(x), \tag{17}$$

$$B(x) = b(x) + c(x) + d(x), \tag{18}$$

Since $0 < \delta \ll 1$ and $0 < \eta \ll 1$, the transition from Equation (8) to Equation (13) is admitted. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [12]. The behaviour of the boundary layer is given by the sign of $A(x)$ and $B(x)$. Further it is established that, if $B(x) \le 0$, $A(x) \ge M > 0$ in [0, 1] then Equation (8) has unique solution and a boundary layer at $x = 0$and if $B(x) \le 0$, $A(x) \le M < 0$ in [0, 1] then Equation (8) has unique solution and a boundary layer at $x = 1$, where M is a positive number.

2.3. Case(I): For Left-end boundary layer

Consider equation (5) or (13) with their boundary conditions

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), 0 \le x \le 1 \tag{19}$$

$$y(0) = \alpha \tag{20}$$

$$y(1) = \beta \tag{21}$$

From Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of the point x, we have

$$y\left(x - \sqrt{\varepsilon'}\right) \approx y(x) - \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x) \tag{22}$$

From equation (19) and (22), we have

$$y'(x) = p(x)y\left(x - \sqrt{\varepsilon'}\right) + q(x)y(x) + r(x) \tag{23}$$

where

$$p(x) = \frac{-2}{2\sqrt{\varepsilon'} + A(x)} \tag{24}$$

$$q(x) = \frac{2 - B(x)}{2\sqrt{\varepsilon'} + A(x)} \tag{25}$$

$$r(x) = \frac{f(x)}{2\sqrt{\varepsilon'} + A(x)} \tag{26}$$

The transition from equation (19) to (23) is valid, because of the condition that $\sqrt{\varepsilon'}$ is small. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [12].

Now, we divide the interval [0, 1] into n equal parts with constant mesh length $h = 1/n$.

Let $0 = x_0,\ x_1, \ldots, x_n = 1$ be the mesh points, then we have $x_i = ih,\ i = 0, 1, 2, \ldots, n$. From our earlier assumptions, $A(x)$ and $B(x)$ are constants. Therefore, $p(x)$ and $q(x)$ are constants. Equation (23) can be written as

$$y'(x) - qy(x) = py\left(x - \sqrt{\varepsilon'}\right) + r(x) \tag{27}$$

We take an integrating factor e^{-qx} to equation (27) and producing (as in B. J. McCartin [3])

$$\frac{d}{dx}\left[e^{-qx}y(x)\right] = e^{-qx}\left[py\left(x - \sqrt{\varepsilon'}\right) + r(x)\right] \tag{28}$$

On integrating equation (28) from x_i to x_{i+1}, we get

$$e^{-qx_{i+1}}y_{i+1} - e^{-qx_i}y_i = \int_{x_i}^{x_{i+1}} e^{-qx}\, py\left(x - \sqrt{\varepsilon'}\right)dx + \int_{x_i}^{x_{i+1}} e^{-qx}\, r(x)dx \tag{29}$$

Using the Hermite interpolation on $[x_i x_{i+1}]$ for $y(x-\sqrt{\varepsilon'})$ and $r(x)$ into the above equation, we get

$$y_{i+1} = e^{qh} y_i + p\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)} \{h_i(x-\sqrt{\varepsilon'}) * y(x_i-\sqrt{\varepsilon'}) + h_{i+1}(x-\sqrt{\varepsilon'}) * y(x_{i+1}-\sqrt{\varepsilon'})$$
$$+ \overline{h_i}(x-\sqrt{\varepsilon'}) * y'(x_i-\sqrt{\varepsilon'}) + \overline{h_{i+1}}(x-\sqrt{\varepsilon'}) * y'(x_{i+1}-\sqrt{\varepsilon'})\}dx$$
$$+ \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)} \{h_i(x) * r(x_i) + h_{i+1}(x) * r(x_{i+1}) + \overline{h_i}(x) * r'(x_i) + \overline{h_{i+1}}(x)$$
$$* r'(x_{i+1})\}dx \tag{30}$$

where h_i, h_{i+1}, $\overline{h_i}$ and $\overline{h_{i+1}}$ are given by Hermite interpolation

$$h_i = [(-2)x^3 + (3x_i + 3x_{i+1})x^2 + (-6x_i x_{i+1})x + 3x_i x_{i+1}^2 - x_{i+1}^3]/(-h^3)$$
$$h_{i+1} = [(-2)x^3 + (3x_i + 3x_{i+1})x^2 + (-6x_i x_{i+1})x + 3x_{i+1} x_i^2 - x_i^3]/(h^3)$$
$$\overline{h_i} = [x^3 + (-x_i - 2x_{i+1})x^2 + (2x_i x_{i+1} + x_{i+1}^2)x - x_i x_{i+1}^2]/(h^2)$$
$$\overline{h_{i+1}} = [x^3 + (-2x_i - x_{i+1})x^2 + (2x_i x_{i+1} + x_i^2)x - x_{i+1} x_i^2]/(h^2)$$

To solve equation (30), We firstly solve integrals

$$\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)} h_i dx = \frac{1}{-h^3}\Big[(3x_i + 3x_{i+1})\Big\{x_{i+1}^2\left(-\frac{1}{q}\right) - (2x_{i+1})\left(\frac{1}{q^2}\right) + 2\left(-\frac{1}{q^3}\right) -$$
$$x_i^2\left(\frac{e^{qh}}{-q}\right) + 2x_i\left(\frac{e^{qh}}{q^2}\right) - 2\left(\frac{e^{qh}}{-q^3}\right)\Big\} - 6x_i x_{i+1}\Big\{x_{i+1}\left(-\frac{1}{q}\right) - \left(\frac{1}{q^2}\right) -$$
$$x_i\left(\frac{e^{qh}}{-q}\right) + \left(\frac{e^{qh}}{q^2}\right)\Big\} - 2\Big\{x_{i+1}^3\left(-\frac{1}{q}\right) - 3x_{i+1}^2\left(\frac{1}{q^2}\right) + 6x_{i+1}\left(-\frac{1}{q^3}\right) -$$
$$6\left(\frac{1}{q^4}\right) - x_i^3\left(\frac{e^{qh}}{-q}\right) + 3x_i^2\left(\frac{e^{qh}}{q^2}\right) - 6x_i\left(\frac{e^{qh}}{-q^3}\right) + 6\left(\frac{e^{qh}}{q^4}\right)\Big\} + (3x_i x_{i+1}^2 -$$
$$x_{i+1}^3)\left\{-\frac{1}{q} + \frac{e^{qh}}{q}\right\}\Big] = X(i)(\text{Say}) \tag{31}$$

$$\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)} h_{i+1} dx = \frac{1}{h^3}\Big[(3x_i + 3x_{i+1})\Big\{x_{i+1}^2\left(-\frac{1}{q}\right) - (2x_{i+1})\left(\frac{1}{q^2}\right) + 2\left(-\frac{1}{q^3}\right) -$$
$$x_i^2\left(\frac{e^{qh}}{-q}\right) + 2x_i\left(\frac{e^{qh}}{q^2}\right) - 2\left(\frac{e^{qh}}{-q^3}\right)\Big\} - 6x_i x_{i+1}\Big\{x_{i+1}\left(-\frac{1}{q}\right) - \left(\frac{1}{q^2}\right) -$$
$$x_i\left(\frac{e^{qh}}{-q}\right) + \left(\frac{e^{qh}}{q^2}\right)\Big\} - 2\Big\{x_{i+1}^3\left(-\frac{1}{q}\right) - 3x_{i+1}^2\left(\frac{1}{q^2}\right) + 6x_{i+1}\left(-\frac{1}{q^3}\right) -$$

$$6\left(\frac{1}{q^4}\right)-x_i^3\left(\frac{e^{qh}}{-q}\right)+3x_i^2\left(\frac{e^{qh}}{q^2}\right)-6x_i\left(\frac{e^{qh}}{-q^3}\right)+6\left(\frac{e^{qh}}{q^4}\right)\Big\}+(3x_{i+1}x_i^2-x_i^3)\left\{-\frac{1}{q}+\frac{e^{qh}}{q}\right\}\Big]=Y(i) \quad \text{(Say)} \tag{32}$$

$$\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\overline{h_i}\,dx=\frac{1}{h^2}\Big[(-x_i-2x_{i+1})\Big\{x_{i+1}^2\left(-\frac{1}{q}\right)-(2x_{i+1})\left(\frac{1}{q^2}\right)+2\left(-\frac{1}{q^3}\right)-x_i^2\left(\frac{e^{qh}}{-q}\right)+2x_i\left(\frac{e^{qh}}{q^2}\right)-2\left(\frac{e^{qh}}{-q^3}\right)\Big\}+(2x_ix_{i+1}+x_{i+1}^2)\Big\{x_{i+1}\left(-\frac{1}{q}\right)-\left(\frac{1}{q^2}\right)-x_i\left(\frac{e^{qh}}{-q}\right)+\left(\frac{e^{qh}}{q^2}\right)\Big\}+\Big\{x_{i+1}^3\left(-\frac{1}{q}\right)-3x_{i+1}^2\left(\frac{1}{q^2}\right)+6x_{i+1}\left(-\frac{1}{q^3}\right)-6\left(\frac{1}{q^4}\right)-x_i^3\left(\frac{e^{qh}}{-q}\right)+3x_i^2\left(\frac{e^{qh}}{q^2}\right)-6x_i\left(\frac{e^{qh}}{-q^3}\right)+6\left(\frac{e^{qh}}{q^4}\right)\Big\}-x_ix_{i+1}^2\left\{-\frac{1}{q}+\frac{e^{qh}}{q}\right\}\Big]=Z(i) \quad \text{(Say)} \tag{33}$$

$$\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\overline{h_{i+1}}\,dx=\frac{1}{h^2}\Big[(-x_{i+1}-2x_i)\Big\{x_{i+1}^2\left(-\frac{1}{q}\right)-(2x_{i+1})\left(\frac{1}{q^2}\right)+2\left(-\frac{1}{q^3}\right)-x_i^2\left(\frac{e^{qh}}{-q}\right)+2x_i\left(\frac{e^{qh}}{q^2}\right)-2\left(\frac{e^{qh}}{-q^3}\right)\Big\}+(2x_ix_{i+1}+x_i^2)\Big\{x_{i+1}\left(-\frac{1}{q}\right)-\left(\frac{1}{q^2}\right)-x_i\left(\frac{e^{qh}}{-q}\right)+\left(\frac{e^{qh}}{q^2}\right)\Big\}+\Big\{x_{i+1}^3\left(-\frac{1}{q}\right)-3x_{i+1}^2\left(\frac{1}{q^2}\right)+6x_{i+1}\left(-\frac{1}{q^3}\right)-6\left(\frac{1}{q^4}\right)-x_i^3\left(\frac{e^{qh}}{-q}\right)+3x_i^2\left(\frac{e^{qh}}{q^2}\right)-6x_i\left(\frac{e^{qh}}{-q^3}\right)+6\left(\frac{e^{qh}}{q^4}\right)\Big\}-x_{i+1}x_i^2\left\{-\frac{1}{q}+\frac{e^{qh}}{q}\right\}\Big]=W(i) \quad \text{(Say)} \tag{34}$$

After Substituting equations (31), (32), (33) and (34) in equation (30), we obtain

$$y_{i+1}=e^{qh}y_i+\{py(x_i-\sqrt{\varepsilon'})+r(x_i)\}X(i)+\{py(x_{i+1}-\sqrt{\varepsilon'})+r(x_{i+1})\}Y(i)+\{py'(x_i-\sqrt{\varepsilon'})+r'(x_i)\}Z(i)+\{py'(x_{i+1}-\sqrt{\varepsilon'})+r'(x_{i+1})\}W(i) \tag{35}$$

From finite difference approximation, we have

$$y(x_{i+1}-\sqrt{\varepsilon'})\approx\left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i+1}+\frac{\sqrt{\varepsilon'}}{h}y_i$$

$$y(x_i-\sqrt{\varepsilon'})\approx\left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_i+\frac{\sqrt{\varepsilon'}}{h}y_{i-1}$$

$$y'(x_i-\sqrt{\varepsilon'})\approx y_i'-\sqrt{\varepsilon'}y_i''\approx(y_i-y_{i-1})/h$$

$$y'(x_{i+1}-\sqrt{\varepsilon'})\approx y_{i+1}'-\sqrt{\varepsilon'}y_{i+1}''\approx(y_{i+1}-y_i)/h$$

Therefore equation (35) becomes

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1,2,\dots,n-1 \tag{36}$$

where,

$$E_i = -\frac{p\sqrt{\varepsilon'}}{h}X(i) + \frac{p}{h}Z(i)$$

$$F_i = e^{qh} + p\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)X(i) + \frac{p\sqrt{\varepsilon'}}{h}Y(i) + \frac{p}{h}Z(i) - \frac{p}{h}W(i)$$

$$G_i = 1 - P\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)Y(i) - \frac{p}{h}W(i)$$

$$H_i = r_i X(i) + r_{i+1}Y(i) + r'_i Z(i) + r'_{i+1} W(i)$$

This is a tridiagonal system of $n-1$ equations. We solve this tridiagonal system with given two boundary conditions by Thomas algorithm.

2.4. Case (II): For Right-end boundary layer

Consider equation (5) or (13) with their boundary conditions

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x), 0 \le x \le 1 \tag{37}$$

$$y(0) = \alpha \tag{38}$$

$$y(1) = \beta \tag{39}$$

From Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of the point x, we have

$$y\left(x + \sqrt{\varepsilon'}\right) \approx y(x) + \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x) \tag{40}$$

From equation (37) and (40), we have

$$y'(x) = p(x)y\left(x + \sqrt{\varepsilon'}\right) + q(x)y(x) + r(x) \tag{41}$$

where

$$p(x) = \frac{-2}{-2\sqrt{\varepsilon'} + A(x)} \tag{42}$$

$$q(x) = \frac{2 - B(x)}{-2\sqrt{\varepsilon'} + A(x)} \tag{43}$$

$$r(x) = \frac{f(x)}{-2\sqrt{\varepsilon'} + A(x)} \tag{44}$$

The transition from equation (37) to (41) is valid, because of the condition that $\sqrt{\varepsilon'}$ is small. For more details on the validity of this transition, one can refer El'sgolt's and Norkin [8]. Now, we divide the interval [0, 1] into n equal parts with constant mesh length $h = 1/n$.

Let $0 = x_0,\ x_1, \dots, x_n = 1$ be the mesh points, then we have $x_i = ih$, $i = 0, 1, 2, \dots, n$. From our earlier assumptions, $A(x)$ and $B(x)$ are constants. Therefore, $p(x)$ and $q(x)$ are constants. Equation (41) can be written as

$$y'(x) - qy(x) = py\left(x + \sqrt{\varepsilon'}\right) + r(x) \tag{45}$$

We take an integrating factor e^{-qx} to equation (45) and producing (as in B. J. McCartin[3])

$$\frac{d}{dx}\left[e^{-qx} y(x)\right] = e^{-qx}\left[py\left(x + \sqrt{\varepsilon'}\right) + r(x)\right] \tag{46}$$

On integrating equation (46) from x_{i-1} to x_i, we get

$$e^{-qx_i} y_i - e^{-qx_{i-1}} y_{i-1} = \int_{x_{i-1}}^{x_i} e^{-qx}\, py\left(x + \sqrt{\varepsilon'}\right) dx + \int_{x_{i-1}}^{x_i} e^{-qx}\, r(x) dx \tag{47}$$

Using the Hermite interpolation on $[x_{i-1} x_i]$ for $y\left(x + \sqrt{\varepsilon'}\right)$ and $r(x)$ into the above equation, we get

$$\begin{aligned} y_i = e^{qh} y_{i-1} + p \int_{x_{i-1}}^{x_i} e^{q(x_i - x)} \Big\{ & h_{i-1}\left(x + \sqrt{\varepsilon'}\right) * y\left(x_{i-1} + \sqrt{\varepsilon'}\right) + h_i\left(x + \sqrt{\varepsilon'}\right) * y\left(x_i + \sqrt{\varepsilon'}\right) \\ & + \overline{h_{i-1}}\left(x + \sqrt{\varepsilon'}\right) * y'\left(x_{i-1} + \sqrt{\varepsilon'}\right) + \overline{h_i}\left(x + \sqrt{\varepsilon'}\right) * y'\left(x_i + \sqrt{\varepsilon'}\right) \Big\} dx \\ & + \int_{x_{i-1}}^{x_i} e^{q(x_i - x)} \Big\{ h_{i-1}(x) * r(x_{i-1}) + h_i(x) * r(x_i) + \overline{h_{i-1}}(x) * r'(x_{i-1}) + \overline{h_i}(x) \\ & * r'(x_i) \Big\} dx \end{aligned} \tag{48}$$

where h_{i-1}, h_i, $\overline{h_{i-1}}$ and $\overline{h_i}$ are given by Hermite interpolation as in case of left end boundary layer. In the similar way, we get

$$\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\, h_{i-1} dx = X(i) \tag{49}$$

$$\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\, h_i dx = Y(i) \tag{50}$$

$$\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\, \overline{h_{i-1}} dx = Z(i) \tag{51}$$

$$\int_{x_{i-1}}^{x_i} e^{q(x_i-x)}\, \overline{h_i} dx = W(i) \tag{52}$$

After Substituting equations (49), (50), (51) and (52) in equation (48) and using finite difference approximation, we obtain

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1,2, \dots, n-1 \tag{53}$$

where,

$$E_i = -e^{qh} - p\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) X(i) + \frac{p}{h} Z(i)$$

$$F_i = -1 + p\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right) Y(i) + \frac{p\sqrt{\varepsilon'}}{h} X(i) + \frac{p}{h} Z(i) - \frac{p}{h} W(i)$$

$$G_i = -\frac{p\sqrt{\varepsilon'}}{h} Y(i) - \frac{p}{h} W(i)$$

$$H_i = r_{i-1} X(i) + r_i Y(i) + \ r'_{i-1}\, Z(i) + r'_i\, W(i)$$

This is a tridiagonal system of $n-1$ equations. We solve this tridiagonal system with given two boundary conditions by Thomas algorithm.

3. Numerical Experiments

In this section, six model examples are solved and the solutions are compared with the exact/available solutions. The exact solution of equation (8) is given by (with assumptions$f(x) = f$, $\varphi(x) = \varphi$ and $\gamma(x)= \gamma$ are constant)

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + f/c' \tag{54}$$

where

$$c' = b + c + d$$

$$m_1 = \left[-(a-\delta b+\eta d)+\sqrt{(a-\delta b+\eta d)^2-4\varepsilon c'}\right]/2\,\varepsilon$$

$$m_2 = \left[-(a-\delta b+\eta d)-\sqrt{(a-\delta b+\eta d)^2-4\varepsilon c'}\right]/2\,\varepsilon$$

$$c_1 = [-f+\gamma c'+e^{m_2}(f-\varphi c')]/[(e^{m_1}-e^{m_2})c']$$

$$c_2 = [f-\gamma c'+e^{m_1}(-f+\varphi c')]/[(e^{m_1}-e^{m_2})c']$$

Example 1. Consider the delay differential equation having left boundary layer:

$$\varepsilon y''(x)+y'(x-\delta)-y(x)=0,\ 0\le x\le 1;\text{with}\, y(0)=1 \text{ and } y(1)=1.$$

The exact solution is given by

$$y=((1-e^{m_2})e^{m_1x}+(e^{m_1}-1)e^{m_2x})/(e^{m_1}-e^{m_2})$$

where

$$m_1=\frac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}\text{and}\, m_2=\frac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$$

The computational results are shown in table-1 & 2, the layer behaviour in fig. 1 & 2 for different values of parameters.

Example 2. Consider the differential-differential equation having left boundary layer:

$$\varepsilon y''(x)+y'(x)-2y(x-\delta)-5y(x)+y(x+\eta)=0,\qquad 0\le x\le 1;$$

$$\text{with } y(0)=1 \text{ and } y(1)=1.$$

The exact solution is given by equation (54) and computational results are shown in table-3 & 4 and the layer behaviour in fig. 3 & 4 for different values of parameters.

Example 3. Consider the differential-differential equation having left boundary layer:

$$\varepsilon y''(x)+y'(x)-3y(x)+2y(x+\eta)=0,\ 0\le x\le 1;\text{ with } y(0)=1 \text{ and } y(1)=1.$$

The exact solution is given by equation (54) and computational results are shown in table-5 & 6, the layer behaviour in fig. 5 & 6 for different values of parameters.

Example 4. Now we consider the delay differential equation having right boundary layer:

$\varepsilon y''(x)-y'(x-\delta)-y(x)=0,\ 0\le x\le 1;$ with $y(0)=1$ and $y(1)=-1$.

The exact solution is given by

$$y = ((1 + e^{m_2})e^{m_1 x} - (e^{m_1} + 1)e^{m_2 x})/(e^{m_2} - e^{m_1})$$

where

$$m_1 = \frac{1 - \sqrt{1 + 4(\varepsilon + \delta)}}{2(\varepsilon + \delta)} \text{and} m_2 = \frac{1 + \sqrt{1 + 4(\varepsilon + \delta)}}{2(\varepsilon + \delta)}$$

The computational results are shown in table-7 & 8, the layer behaviour in fig. 7 & 8 for different values of parameters.

Example 5. Consider the differential-differential equation having right boundary layer:

$$\varepsilon y''(x) - y'(x) - 2y(x - \delta) + y(x) - 2y(x + \eta) = 0, \qquad 0 \le x \le 1;$$

$$\text{with } y(0) = 1 \text{ and } y(1) = -1.$$

The exact solution is given by equation (54) and computational results are shown in table-9 & 10, the layer behaviour in fig. 9 & 10 for different values of parameters.

Example 6. Consider the differential-differential equation having right boundary layer:

$$\varepsilon y''(x) - y'(x) + y(x) - 2y(x + \eta) = 0, \ 0 \le x \le 1; \text{with } y(0) = 1 \text{ and } y(1) = -1.$$

The exact solution is given by equation (54) and computational results are shown in table-11 & 12, the layer behaviour in fig. 11 & 12 for different values of parameters.

3. DISCUSSION AND CONCLUSIONS

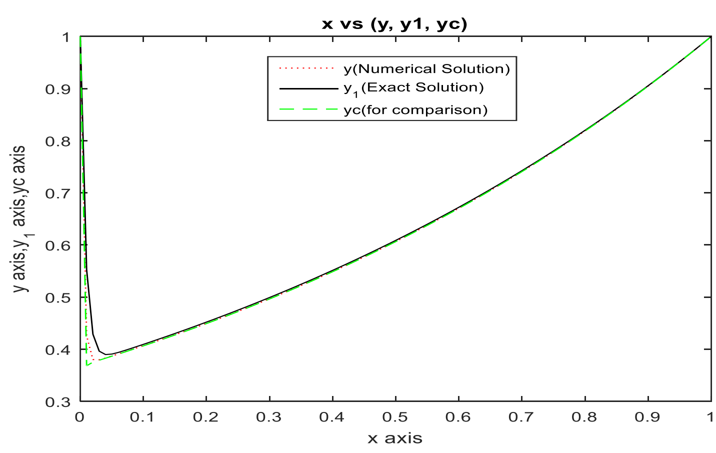
A fitted numerical scheme is presented to get solutions of singularly perturbed differential-difference equations having boundary layer at one end. Original Problem is converted into an asymptotically equivalent singularly perturbed differential equation by using Taylor's transformation. In this scheme deviating argument and Hermite interpolation concepts are used. This scheme is implemented on six standard model examples and found that the numerical solutions are in agreement with available or exact solution. Solutions obtained from this scheme and exact solutions are presented in their respective tables also layer behaviour for different values of the parameters. Scheme is simple and easy to implement on the class of singularly perturbed differential-difference equations having layer at one end.

Example 1: $h = 0.01, \varepsilon = 0.01$ and $\delta = 0.001$

Table-1.

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [8] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.38024295	0.42885272	0.37530590
0.04	0.38327294	0.38987298	0.38287105
0.06	0.39097449	0.39385789	0.39060602
0.08	0.39886493	0.40144299	0.39849726
0.1	0.40691486	0.40946339	0.40654792
0.2	0.44966807	0.45216849	0.44930761
0.3	0.49691321	0.49933011	0.49656465
0.4	0.54912224	0.55141074	0.54879207
0.5	0.60681671	0.60892343	0.60651264
0.6	0.67057294	0.67243474	0.67030411
0.7	0.74102783	0.74257036	0.74080501
0.8	0.81888518	0.82002118	0.81872102
0.9	0.90492274	0.90555021	0.90483204
1.0	1.00000000	1.00000000	1.00000000

Fig.1

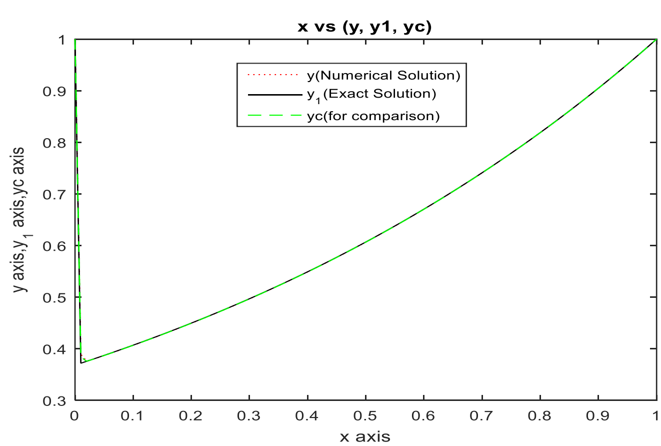


Example 1: $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.0001$

Table-2.

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [8] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.37590123	0.37560498	0.37562175
0.04	0.38300304	0.38318659	0.38296304
0.06	0.39073751	0.39092122	0.39069782
0.08	0.39862855	0.39881199	0.39858892
0.1	0.40667895	0.40686202	0.40663940
0.2	0.44943632	0.44961616	0.44939748
0.3	0.49668912	0.49686302	0.49665156
0.4	0.54890998	0.54907470	0.54887440
0.5	0.60662123	0.60677293	0.60658846
0.6	0.67040012	0.67053424	0.67037115
0.7	0.74088459	0.74099575	0.74086058
0.8	0.81877965	0.81886155	0.81876196
0.9	0.90486444	0.90490969	0.90485466
1.0	1.00000000	1.00000000	1.00000000

Fig. 2.

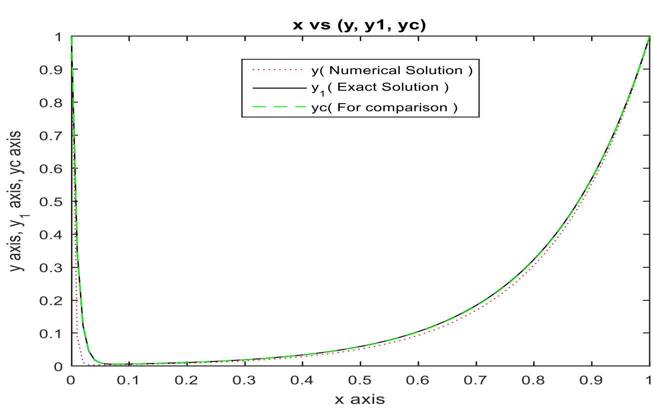


Example 2: $h = 0.01, \varepsilon = 0.01, \delta = 0.001$ and $\eta = 0.005$

Table-3.

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [11] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.01060194	0.12303611	0.12573638
0.04	0.00344628	0.01867136	0.01924308
0.06	0.00381518	0.00667501	0.00668484
0.08	0.00429460	0.00577240	0.00566896
0.1	0.00483483	0.00625879	0.00612688
0.2	0.00874323	0.01096036	0.01074933
0.3	0.01581114	0.01926844	0.01894343
0.4	0.02859264	0.03387416	0.03338381
0.5	0.05170651	0.05955119	0.05883196
0.6	0.09350529	0.10469173	0.10367897
0.7	0.16909358	0.18404935	0.18271239
0.8	0.30578634	0.32356102	0.32199219
0.9	0.55297951	0.56882424	0.56744355
1.0	1.00000000	1.00000000	1.00000000

Fig.3.

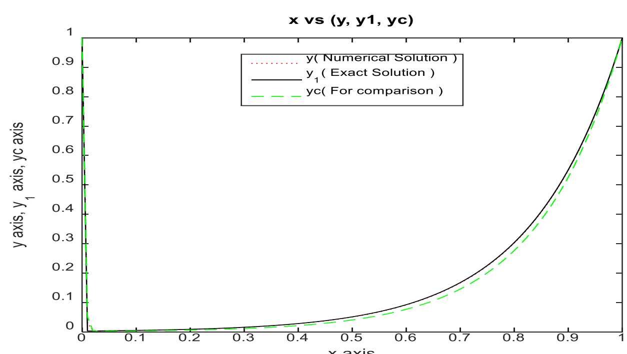


Example 2: $h = 0.01, \varepsilon = 0.001, \delta = 0.0001$ and $\eta = 0.0005$

Table-4.

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [11] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.00365332	0.00290566	0.00427314
0.04	0.00319817	0.00327352	0.00209563
0.06	0.00360408	0.00368796	0.00237638
0.08	0.00406235	0.00415486	0.00270237
0.1	0.00457889	0.00468088	0.00307310
0.2	0.00833059	0.00849532	0.00584433
0.3	0.01515624	0.01541816	0.01111458
0.4	0.02757446	0.02798241	0.02113738
0.5	0.05016751	0.05078525	0.04019845
0.6	0.09127210	0.09217010	0.07644823
0.7	0.16605560	0.16727944	0.14538701
0.8	0.30211273	0.30359530	0.27649274
0.9	0.54964782	0.55099482	0.52582577
1.0	1.00000000	1.00000000	1.00000000

Fig. 4.

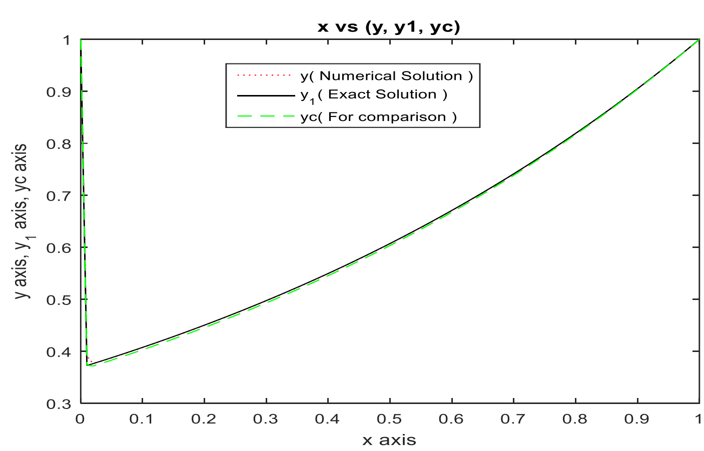


Example 3: $h = 0.01, \varepsilon = 0.001, \delta = 0.001$ and $\eta = 0.001$

Table-5.

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [11] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.37669432	0.37641209	0.37180442
0.04	0.38374303	0.38399316	0.37934369
0.06	0.39147662	0.39172692	0.38708203
0.08	0.39936653	0.39961644	0.39497823
0.1	0.40741545	0.40766486	0.40303550
0.2	0.45015975	0.45040470	0.44585542
0.3	0.49738860	0.49762541	0.49322467
0.4	0.54957251	0.54979677	0.54562660
0.5	0.60723132	0.60743781	0.60359590
0.6	0.67093945	0.67112197	0.66772406
0.7	0.74133157	0.74148282	0.73866542
0.8	0.81910894	0.81922034	0.81714384
0.9	0.90504637	0.90510791	0.90396009
1.0	1.00000000	1.00000000	1.00000000

Fig.5.

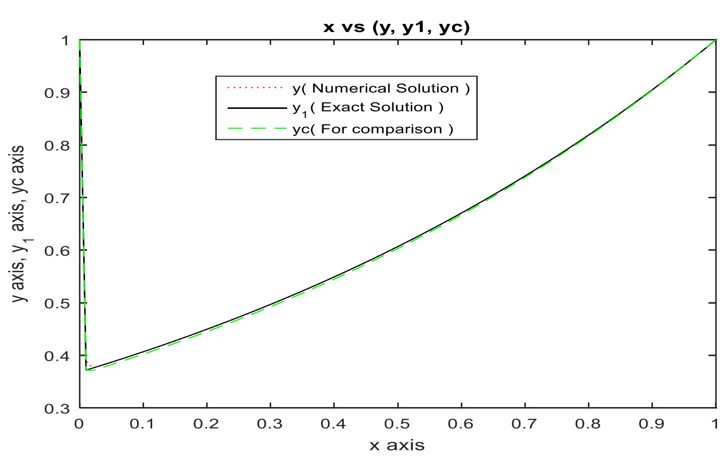


Example 3: $h = 0.01, \varepsilon = 0.001, \delta = 0.001$ and $\eta = 0.0001$

Table-6.

x	Numerical Solution $y(x)$	Exact Solution $y_1(x)$	Result by [11] $y_c(x)$
0.0	1.00000000	1.00000000	1.00000000
0.02	0.37603526	0.37575175	0.37114042
0.04	0.38308252	0.38333326	0.37867982
0.06	0.39081682	0.39106774	0.38641872
0.08	0.39870774	0.39895828	0.39431577
0.1	0.40675798	0.40700803	0.40237422
0.2	0.44951396	0.44975958	0.44520510
0.3	0.49676420	0.49700169	0.49259513
0.4	0.54898109	0.54920605	0.54502961
0.5	0.60668672	0.60689389	0.60304550
0.6	0.67045802	0.67064117	0.66723691
0.7	0.74093258	0.74108438	0.73826121
0.8	0.81881501	0.81892684	0.81684571
0.9	0.90488397	0.90494576	0.90379517
1.0	1.00000000	1.00000000	1.00000000

Fig. 6.

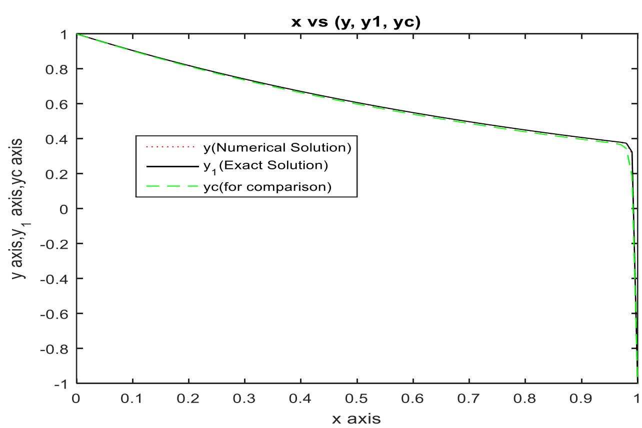


Example 4: $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.001$

Table-7.

x	Present Solution	Exact Solution	Result by [8]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90487768	0.90510729	0.90254597
0.2	0.81880362	0.81921921	0.81458924
0.3	0.74091713	0.74148128	0.73520424
0.4	0.67043938	0.67112011	0.66355563
0.5	0.60666563	0.60743571	0.59888947
0.6	0.54895819	0.54979449	0.54052528
0.7	0.49674002	0.49762300	0.48784892
0.8	0.44948896	0.45040221	0.44030608
0.9	0.40673253	0.40766232	0.39739648
0.92	0.39868224	0.39961390	0.38932991
0.94	0.39079127	0.39172437	0.38142112
0.96	0.38305253	0.38398849	0.37331288
0.98	0.37314318	0.37470173	0.34361581
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 7.

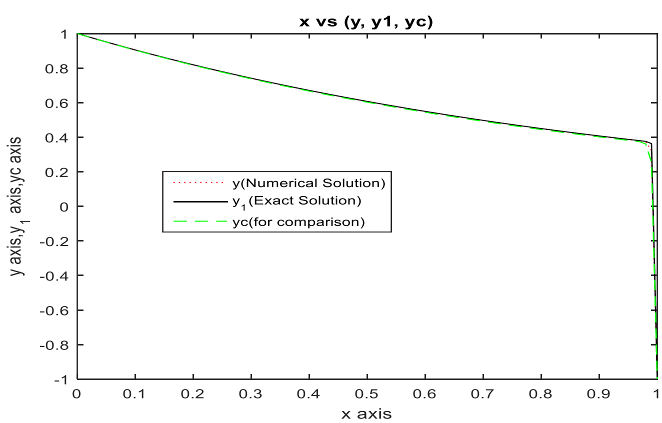


Example 4: $h = 0.01, \varepsilon = 0.001$ and $\delta = 0.0005$

Table-8.

x	Present Solution	Exact Solution	Result by [8]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90487229	0.90501768	0.90361600
0.2	0.81879387	0.81905700	0.81652188
0.3	0.74090389	0.74126107	0.73782223
0.4	0.67042340	0.67085438	0.66670798
0.5	0.60664756	0.60713507	0.60244800
0.6	0.54893857	0.54946798	0.54438165
0.7	0.49671931	0.49727823	0.49191197
0.8	0.44946754	0.45004559	0.44449953
0.9	0.40671073	0.40729922	0.40165688
0.92	0.39866039	0.39925006	0.39359720
0.94	0.39076939	0.39135997	0.38569882
0.96	0.38303236	0.38362581	0.37789776
0.98	0.37370459	0.37598358	0.36118295
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 8.

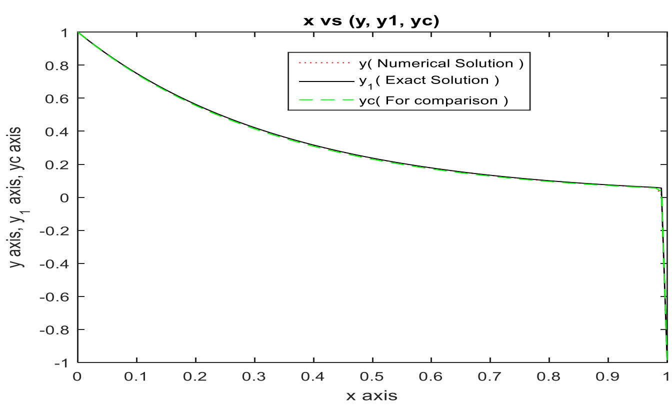


Example 5: $h = 0.01, \varepsilon = 0.002, \delta = 0.001$ and $\eta = 0.02$

Table-9.

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.74923676	0.74995512	0.74540014
0.2	0.56135572	0.56243268	0.55562138
0.3	0.42058834	0.42179927	0.41416026
0.4	0.31512024	0.31633052	0.30871512
0.5	0.23609967	0.23723370	0.23011629
0.6	0.17689455	0.17791462	0.17152872
0.7	0.13253590	0.13342798	0.12785753
0.8	0.09930077	0.10006500	0.09530502
0.9	0.07439978	0.07504426	0.07104038
0.92	0.07022560	0.07084750	0.06698586
0.94	0.06628561	0.06688544	0.06316275
0.96	0.06256503	0.06314495	0.05955775
0.98	0.05773454	0.05961135	0.05585811
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 9.

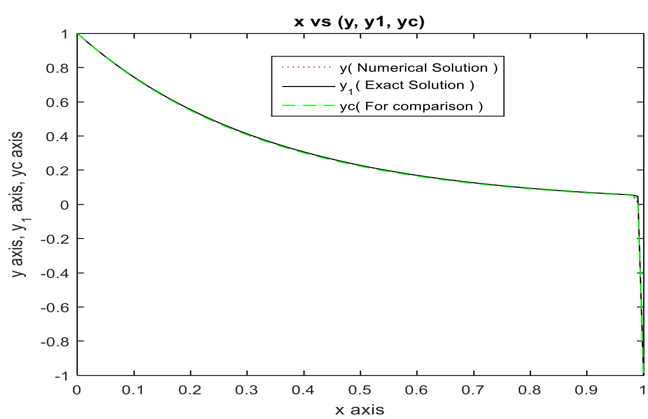


Example 5: $h = 0.01, \varepsilon = 0.002, \delta = 0.0001$ and $\eta = 0.005$

Table-10.

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.74326245	0.74424718	0.74075573
0.2	0.55243907	0.55390386	0.54871906
0.3	0.41060722	0.41224139	0.40646679
0.4	0.30518893	0.30680949	0.30109261
0.5	0.22683547	0.22834210	0.22303607
0.6	0.16859829	0.16994296	0.16521525
0.7	0.12531278	0.12647957	0.12238414
0.8	0.09314028	0.09413206	0.09065676
0.9	0.06922767	0.07005752	0.06715451
0.92	0.06523913	0.06603865	0.06324267
0.94	0.06148038	0.06225033	0.05955870
0.96	0.05793541	0.05867932	0.05608921
0.98	0.05288638	0.05527724	0.05247124
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 10.

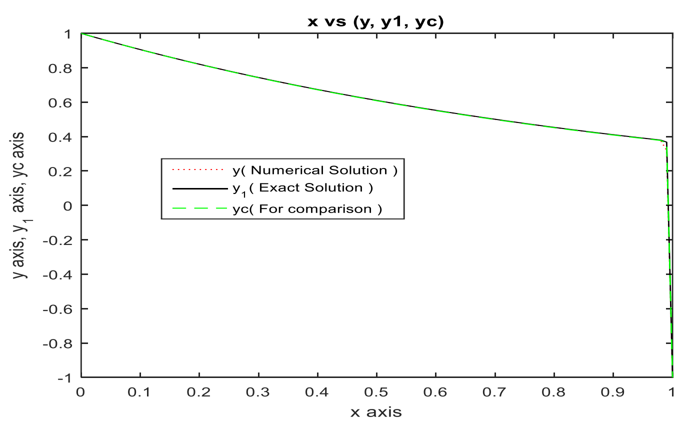


Example 6: $h = 0.01, \varepsilon = 0.002, \delta = 0.001$ and $\eta = 0.005$

Table-11.

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90577262	0.90590670	0.90542830
0.2	0.82042404	0.82066696	0.81980040
0.3	0.74311763	0.74344771	0.74227049
0.4	0.67309560	0.67349426	0.67207271
0.5	0.60967157	0.61012297	0.60851365
0.6	0.55222381	0.55271449	0.55096548
0.7	0.50018921	0.50070777	0.49885974
0.8	0.45305769	0.45359453	0.45168172
0.9	0.41036725	0.41091432	0.40896541
0.92	0.40232455	0.40287283	0.40091970
0.94	0.39443947	0.39498870	0.39303227
0.96	0.38670520	0.38725886	0.38530000
0.98	0.37686475	0.37963166	0.37757875
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 11.

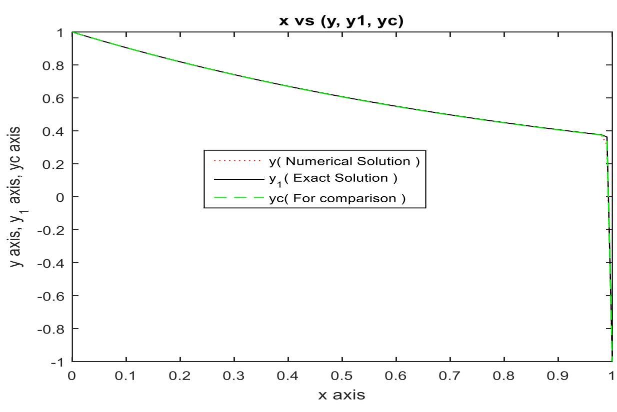


Example 6: $h = 0.01, \varepsilon = 0.002, \delta = 0.0003$ and $\eta = 0.0006$

Table-12.

x	Present Solution	Exact Solution	Result by [11]
0.0	1.00000000	1.00000000	1.00000000
0.1	0.90498600	0.90512548	0.90465207
0.2	0.81899966	0.81925214	0.81839538
0.3	0.74118322	0.74152599	0.74036308
0.4	0.67076044	0.67117407	0.66977100
0.5	0.60702881	0.60749675	0.60590973
0.6	0.54935257	0.54986079	0.54813749
0.7	0.49715639	0.49769301	0.49587372
0.8	0.44991957	0.45047463	0.44859319
0.9	0.40717091	0.40773607	0.40582076
0.92	0.39912150	0.39968779	0.39776868
0.94	0.39123120	0.39179839	0.38987636
0.96	0.38349295	0.38406471	0.38214061
0.98	0.37358889	0.37642359	0.37439988
1.0	-1.00000000	-1.00000000	-1.00000000

Fig. 12.



CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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