Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 2792-2813
https://doi.org/10.28919/jmcs/5618
ISSN: 1927-5307

A NOVEL APPROACH FOR FRACTIONAL KAWAHARA AND MODIFIED KAWAHARA EQUATIONS USING ATANGANA-BALEANU DERIVATIVE OPERATOR

DNYANOBA B. DHAIGUDE, VIDYA N. BHADGAONKAR*

Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India

Abstract. In this manuscript, iterative laplace transform method is applied to obtain approximate solutions of the nonlinear time fractional Kawahara and modified Kawahara equations based on Atangana-Baleanu derivative operator. The noticeable features of the manuscript is to providing the existence and uniqueness conditions of solution for proposed technique and the graphical presentations of numerical solution of the concerned equations for various specific cases. The obtained approximate solutions are compared with the exact solutions to verify the applicability, efficiency and accuracy of the method.

Keywords: Kawahara equations; Atangana-Baleanu fractional differential operator; Laplace transform; existence and uniqueness; numerical simulations.

2010 AMS Subject Classification: Primary: 35A20, 35A22, 34A08, 35R11.

1. Introduction

Nonlinear wave phenomena play an essential role in various parts of mathematical physics and engineering such as dispersion, diffusion, reaction and convection. One such well-known nonlinear evolution equations is the fifth order Kawahara equation. These equation appeares in

*Corresponding author
E-mail address: bhadgaonkar.vidya@gmail.com
Received February 27, 2021

the study of shallow water waves having magneto-acoustic waves in a plasma, surface tension and capillary-gravity waves. Furthermore, these equations becomes the area of active research in recent times [28, 29, 30]. In 1972, Kawahara [32] proposed the kawahara equation first to describe solitary-wave propagation in media. Furthermore, the modified Kawahara equation has some useful applications in physics such as,capillary-gravity water waves, plasma waves, water waves with surface tension, etc. [31, 33, 34, 35, 36].

Fractional calculus allows differentiation and integration of arbitrary order and hence, becomes more popular in the past few decades in various fields of science and engineering, such as fluid mechanics, diffusive transport, electrical networks, electromagnetic theory, different branches of physics, biological sciences and groundwater problems , [1, 2, 3, 4, 5]. There are several useful applications of fractional calculus such as dissipation [19], modelling of processes such as anomalous diffusion [20, 21], control theory [22], relaxation [23], etc. Many mathematicians and researchers have tried to model several physical or biological processes using fractional differential equations. Solving these equations is turn out to be wide area of research and interest for researchers from various fields. Some of the recent analytical and numerical methods for solving linear and nonlinear fractional differential equations are the Adomian decomposition method ADM[6, 7, 8], Variational iteration method VIM [9, 10], Homotopy-perturbation method HPM [11], Homotopy analysis method [12], Finite difference method [13], monotone iterative method [15] and so on. In recent times, an iterative method was proposed by Daftardar-Gejji and Jafari [17, 18] which is known as new iterative method (NIM). This method is very useful and simple in fractional calculus for solving linear and nonlinear fractional partial differential equations.

These FDEs involves several fractional differential operators like Riemann-Liouville operator [24], Caputo operator [26], Hilfer operator [14], Caputo-Fabrizio operator [25, 16], etc. However these operators possesses a power law kernel, exponential kernel and has singularity. Hence these operators posseses some limitations in modelling physical problems. To overcome this difficulty, in recent times Atangana and Baleanu have proposed a reliable operator having nonlocal and nonsingular kernel in the form of Mittag-Leffler function known as Atanagana-Baleanu operator [27].

Motivated by above, in this paper we have applied iterative Laplace transform with to find approximate solutions of time fractional Kawahara and modified Kawahara equations having Atanagana-Baleanu operator. These equations are given below as follows:

$$\frac{{}^{ABC}\partial^{\kappa}u}{\partial t^{\kappa}}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}-\frac{\partial^5 u}{\partial x^5}=0, \qquad 0<\kappa\leq 1 \tag{1.1}$$

with initial condition

$$u(x,0)=\frac{105}{169}sech^4\left(\frac{x}{2\sqrt{13}}\right) \tag{1.2}$$

$$\frac{{}^{ABC}\partial^{\kappa}u}{\partial t^{\kappa}}+u^2\frac{\partial u}{\partial x}+m\frac{\partial^3 u}{\partial x^3}+n\frac{\partial^5 u}{\partial x^5}=0, \qquad 0<\kappa\leq 1 \tag{1.3}$$

where m, n are nonzero real constants and initial condition is.

$$u(x,0)=\frac{3m}{\sqrt{-10n}}sech^2(Kx),\ K=\frac{1}{2}\sqrt{\frac{-m}{5n}} \tag{1.4}$$

Equations (1.1) and (1.3) becomes the original Kawahara and modified Kawahara equations for $\kappa=1$ [32]

The rest of this paper is organized as follows. In Section 2, preliminaries of fractional calculus is presented. In Section 3 basic idea of iterative Laplace transform method is given. Section 4, presents existence and uniqueness criteria for obtained solutions of time fractional Kawahara and modified Kawahara equations. The numerical results and plots for the obtained solutions are presented in Section 5. Finally, we give our conclusions in Section 6.

2. Preliminaries

Definition 2.1. *The left and right sided Caputo fractional derivative for order $\kappa>0$ is defined as*

$${}_a^CD_t^{\kappa}f(t)=\frac{1}{\Gamma(n-\kappa)}\int_a^t(t-\zeta)^{n-\kappa-1}f^{(n)}(\zeta)\,d\zeta, \quad (left)$$

$${}_b^CD_t^{\kappa}f(t)=\frac{(-1)^n}{\Gamma(n-\kappa)}\int_t^b(t-\zeta)^{n-\kappa-1}f^{(n)}(\zeta)\,d\zeta \quad (right)$$

where $n-1<\kappa\leq n, n\in N, f\in C^{n-1}[0,t]$.

Definition 2.2. *The left and right Atangana-Baleanu fractional derivative for a given function for order in Caputo sense are defined as*

$$ {}_{a}^{ABC}D_{t}^{\kappa} f(t) = \frac{B(\kappa)}{1-\kappa} \int_{a}^{t} \frac{df(\zeta)}{d\zeta} E_{\kappa}\left[-\frac{\kappa}{1-\kappa}(t-\zeta)^{\kappa}\right] d\zeta, \quad (left) $$

$$ {}_{b}^{ABC}D_{t}^{\kappa} f(t) = -\frac{B(\kappa)}{1-\kappa} \int_{t}^{b} \frac{df(\zeta)}{d\zeta} E_{\kappa}\left[-\frac{\kappa}{1-\kappa}(t-\zeta)^{\kappa}\right] d\zeta, \quad (right) $$

where $B(\kappa) = (1-\kappa) + \frac{\kappa}{\Gamma(\kappa)}$ *is a normalization function and* $E_{\kappa}(\cdot)$ *is the Mittag-Leffler function.*

Definition 2.3. *Atangana-Baleanu fractional integral order* κ *is defined as*

$$ {}^{AB}I_{t}^{\kappa}(f(t)) = \frac{1-\kappa}{B(\kappa)} f(t) + \frac{\kappa}{B(\kappa)\Gamma(\kappa)} \int_{a}^{t} f(\zeta)(t-\zeta)^{\kappa-1} d\zeta, $$

if $f(t)$ *is a constant, integral will be resulted with zero.*

3. Fundamental Idea of Iterative Laplace Transform Method

In this section, we consider the arbitrary fractional order differential equation with Atangana Baleanu operator to demonstrate the fundamental solution procedure of the proposed algorithm by Daftardar-Gejji and Jafari [17]

$$ {}_{a}^{ABC}D_{t}^{\kappa} u(x,t) + \mathscr{R}u(x,t) + \mathscr{N}u(x,t) = f(x,t) \tag{3.1} $$

with initial condition

$$ u(x,0) = \psi(x,t) \tag{3.2} $$

Where $f(x,t)$ denotes source term, $\mathscr{R}$ and $\mathscr{N}$ are given linear and non-linear operator respectively. Applying Laplace transform on (3.1) we get

$$ \begin{aligned} & L\{u(x,t)\} - \frac{1}{s}u(x,0) + \frac{1}{B(\kappa)}\left(1-\kappa+\frac{\kappa}{s^{\kappa}}\right) \\ & \Big(L\{\mathscr{R}u(x,t)\} + L\{\mathscr{N}u(x,t)\} - L\{f(x,t)\}\Big) = 0. \end{aligned} \tag{3.3} $$

Rearranging terms we get

$$ L\{u(x,t)\} = \frac{1}{s}\psi(x,t) - \frac{1}{B(\kappa)}\left(1-\kappa+\frac{\kappa}{s^{\kappa}}\right)\Big(L\{\mathscr{R}u(x,t)\} + L\{\mathscr{N}u(x,t)\} - L\{f(x,t)\}\Big) \tag{3.4} $$

Next, we apply inverse laplace transform on (3.4) then we get

(3.5)
$$u(x,t) = \psi(x,t) + L^{-1}\left\{ \frac{-1}{B(\kappa)}\Big(1-\kappa+\frac{\kappa}{s^\kappa}\Big)\Big(L\{\mathscr{R}u(x,t)\} + L\{\mathscr{N}u(x,t)\} - L\{f(x,t)\}\Big)\right\}$$

Further, we apply iterative method [17]. We consider series solution as below given by,

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t), \tag{3.6}$$

since $\mathscr{R}$ is linear,

$$\mathscr{R}\left(\sum_{j=0}^{\infty} u_j(x,t)\right) = \sum_{j=0}^{\infty} \mathscr{R}\big(u_j(x,t)\big). \tag{3.7}$$

The nonlinear operator $\mathscr{N}$ is decomposed as

$$\mathscr{N}\left(\sum_{j=0}^{\infty} u_j(x,t)\right) = \mathscr{N}(u_0(x,t)) + \sum_{j=1}^{\infty}\left\{\mathscr{N}\left(\sum_{i=0}^{j} u_i(x,t)\right) - \mathscr{N}\left(\sum_{i=0}^{j-1} u_i(x,t)\right)\right\} \tag{3.8}$$

$$= \sum_{i=0}^{\infty} \mathscr{P}_i \tag{3.9}$$

where $\mathscr{P}_0 = \mathscr{N}(u_0(x,t))$ and $\mathscr{P}_i = \left\{\mathscr{N}\left(\sum_{j=0}^{i} u_j(x,t)\right) - \mathscr{N}\left(\sum_{j=0}^{i-1} u_j(x,t)\right)\right\}$, $i \geq 1$
In view of (3.6), (3.7) and (3.8), the equation (3.1) is equivalent to

$$\begin{aligned}\sum_{j=0}^{\infty} u_j(x,t) = \psi(x,t) + L^{-1}\Bigg\{ \frac{-1}{B(\kappa)}\Big(1-\kappa+\frac{\kappa}{s^\kappa}\Big)\Bigg(L\Big\{\sum_{j=0}^{\infty}\mathscr{R}\big(u_j(x,t)\big)\Big\} \\ + L\Big\{\mathscr{N}(u_0(x,t)) + \sum_{j=1}^{\infty}\left\{\mathscr{N}\left(\sum_{i=0}^{j} u_i(x,t)\right) - \mathscr{N}\left(\sum_{i=0}^{j-1} u_i(x,t)\right)\right\}\Big\} \\ - L\{f(x,t)\}\Bigg)\Bigg\}\end{aligned} \tag{3.10}$$

further, consider the recurrence relation as given below

$$u_0(x,t) = \psi(x,t)$$

$$u_1(x,t) = L^{-1}\left\{\frac{-1}{B(\kappa)}\left(1-\kappa+\frac{\kappa}{s^\kappa}\right)\left(L\{\mathscr{R}(u_0(x,t))\}+L\{\mathscr{N}(u_0(x,t))\}\right.\right\}$$

$$\vdots \tag{3.11}$$

$$u_{n+1}(x,t) = L^{-1}\left\{\frac{-1}{B(\kappa)}\left(1-\kappa+\frac{\kappa}{s^\kappa}\right)\left(L\left\{\mathscr{R}(u_n(x,t))\right\}\right.\right.$$

$$\left.\left.+L\left\{\mathscr{N}\left(\sum_{i=0}^{n}u_i(x,t)\right)-\mathscr{N}\left(\sum_{i=0}^{n-1}u_i(x,t)\right)\right\}\right\} \tag{3.12}$$

The n−term approximate solution is given by

$$u = u_0+u_1+u_2+\cdots+u_{n-1}. \tag{3.13}$$

4. EXISTENCE AND UNIQUENESS

4.1. Existence of solutions for the fractional Kawahara equation.

Here, we considered the fixed-point theorem to demonstrate the existence of the solution for the Kawahara equation. Let us consider the time fractional Kawahara equation (1.1) using Atangana-Baleanu fractional derivative operator as below

$$^{ABC}D^\kappa u(x,t) = \mathscr{G}(x,\ t,\ u(x,t)) = u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}-\frac{\partial^5 u}{\partial x^5} \tag{4.1}$$

where $^{ABC}D^\kappa$ represents the fractional operator of type Atangana-Baleanue-Caputo (ABC) having fractional order κ; where $0<\kappa<1$, subject to initial conditions $u_0(x,t) = u(x,0)$.

The equation (4.1) can be converted to the Volterra-type integral equation by using the ABC fractional integral which is written by referring definition 2.3 as follows:

$$u(x,t)-u(x,0) = \frac{1-\kappa}{B(\kappa)}\mathscr{G}(x,\ t,\ u)+\frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_a^t \mathscr{G}(x,\ \zeta,\ u)\,(t-\zeta)^{\kappa-1}\,d\zeta, \tag{4.2}$$

Theorem 4.1. *The kernels $\mathscr{G}(x,\ t,\ u)$ given in Eq.* (4.1) *satisfy the Lipschitz condition and contraction if the following condition holds*

$$0 \le \left(\frac{\tau_1}{2}(\rho_1+\rho_2)+\tau_2+\tau_3\right) < 1.$$

Proof: We have $\mathscr{G}(x,\ t,\ u(x,t)) = u\dfrac{\partial u}{\partial x} + \dfrac{\partial^3 u}{\partial x^3} - \dfrac{\partial^5 u}{\partial x^5}$.
Consider u_1 and u_2 be two functions, then we obtain the following

$$\begin{aligned}
&\parallel \mathscr{G}(x,\ t,\ u_1(x,t)) - \mathscr{G}(x,\ t,\ u_2(x,t)) \parallel \\
&= \left\| u_1(x,t)\frac{\partial u_1(x,t)}{\partial x} - u_2(x,t)\frac{\partial u_2(x,t)}{\partial x} + \frac{\partial^3 u_1(x,t)}{\partial x^3} - \frac{\partial^3 u_2(x,t)}{\partial x^3} \right. \\
&\qquad \left. + \frac{\partial^5 u_1(x,t)}{\partial x^5} - \frac{\partial^5 u_2(x,t)}{\partial x^5} \right\| \\
&\le \frac{1}{2}\left\| \frac{\partial}{\partial x}[u_1^2(x,t) - u_2^2(x,t)] \right\| + \left\| \frac{\partial^3}{\partial x^3}[u_1(x,t) - u_2(x,t)] \right\| \\
&\qquad + \left\| \frac{\partial^5}{\partial x^5}[u_1(x,t) - u_2(x,t)] \right\| \\
&\le \frac{\tau_1}{2}\left\| [u_1^2(x,t) - u_2^2(x,t)] \right\| + \tau_2 \left\| [u_1(x,t) - u_2(x,t)] \right\| + \tau_3 \left\| [u_1(x,t) - u_2(x,t)] \right\| \\
&\le \Big(\frac{\tau_1}{2}(\rho_1 + \rho_2) + \tau_2 + \tau_3 \Big) \left\| [u_1(x,t) - u_2(x,t)] \right\| \\
&\le \lambda \left\| [u_1(x,t) - u_2(x,t)] \right\|.
\end{aligned}$$

where $\tau_1 = \dfrac{\partial}{\partial x}$, $\tau_2 = \dfrac{\partial^3}{\partial x^3}$ and $\tau_3 = \dfrac{\partial^5}{\partial x^5}$ are the differential operators. Since, u_1 and u_2 are bounded functions, we have $\|u_1\| \le \rho_1$ $\|u_2\| \le \rho_2$.
Also $\lambda = \Big(\dfrac{\tau_1}{2}(\rho_1 + \rho_2) + \tau_2 + \tau_3 \Big)$. Therefore, we have

$$\parallel \mathscr{G}(x,\ t,\ u_1(x,t)) - \mathscr{G}(x,\ t,\ u_2(x,t)) \parallel \le \lambda \left\| [u_1(x,t) - u_2(x,t)] \right\| \tag{4.3}$$

This shows that the Lipschitz condition is obtained for $\mathscr{G}$. Moreover, we see that if $0 \le \Big(\dfrac{\tau_1}{2}(\rho_1 + \rho_2) + \tau_2 + \tau_3 \Big) < 1$ then it implies the contraction.
The recursive form of equation (4.2) defined as follows

$$u_n(x,t) = \frac{1-\kappa}{B(\kappa)}\mathscr{G}(x,\ t,\ u_{n-1}) + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{G}(x,\ \zeta,\ u_{n-1})\,(t-\zeta)^{\kappa-1}\,d\zeta. \tag{4.4}$$

Next, we get the difference between the successive iterative terms in the form of following expression

$$\begin{aligned}\theta_n(x,t) &= u_n(x,t) - u_{n-1}(x,t) \\ &= \frac{1-\kappa}{B(\kappa)}\big(\mathscr{G}(x,\ t,\ u_{n-1}) - \mathscr{G}(x,\ t,\ u_{n-2})\big) \\ &\quad + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \big(\mathscr{G}(x,\ \zeta,\ u_{n-1}) - \mathscr{G}(x,\ \zeta,\ u_{n-2})\big)(t-\zeta)^{\kappa-1}\,d\zeta. \end{aligned} \tag{4.5}$$

Notice that

$$u_n(x,t) = \sum_{i=1}^{n} \theta_i(x,t) \tag{4.6}$$

By using equation (4.4) and then using norm on equation (4.5), we get

$$\|\theta_n(x,t)\| \leq \frac{1-\kappa}{B(\kappa)}\lambda\|\theta_{n-1}(x,t)\| + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\lambda\int_0^t \|\theta_{n-1}(x,\zeta)\|d\zeta. \tag{4.7}$$

This completes the proof of the theorem.

Theorem 4.2. *The solution for the* (4.1) *will exist and unique under the condition that we can find* t_0 *satisfying*

$$\frac{1-\kappa}{B(\kappa)}\lambda + \frac{\kappa t_0}{B(\kappa)\Gamma(\kappa)}\lambda < 1.$$

Proof: First, we consider bounded function $u(x,t)$ satisfying the Lipschitz condition. From equations (4.5) and (4.7) we get the following equation

$$\|\theta_n(x,t)\| \leq \|u_n(x,0)\|\left[\frac{1-\kappa}{B(\kappa)}\lambda + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\lambda\right]^n. \tag{4.8}$$

Hence, the solution is smooth, moreover existence is proved for the obtained solution. Next, we show that the equation (4.8) is the solution for the equation (4.1). For this, we consider

$$u(x,t) - u(x,0) = u_n(x,t) - \delta_n(x,t) \tag{4.9}$$

where $\delta_n(x,t)$ are reminder terms of series solution. Then, we must show that these terms approach to zero at infinity, that is, $\|\delta_\infty(x,t)\| \to 0$.

$$\begin{aligned}
\|\delta_n(x,t)\| &= \left\| \frac{1-\kappa}{B(\kappa)}\big(\mathscr{G}(x,\ t,\ u)-\mathscr{G}(x,\ t,\ u_{n-1})\big) \right. \\
&\quad \left. + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \big(\mathscr{G}(x,\ \zeta,\ u)-\mathscr{G}(x,\ \zeta,\ u_{n-1})\big)(t-\zeta)^{\kappa-1}\,d\zeta \right\| \\
&\le \frac{1-\kappa}{B(\kappa)}\big\|\big(\mathscr{G}(x,\ t,\ u)-\mathscr{G}(x,\ t,\ u_{n-1})\big)\big\| \\
&\quad + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \big\|\big(\mathscr{G}(x,\ \zeta,\ u)-\mathscr{G}(x,\ \zeta,\ u_{n-1})\big)\big\|(t-\zeta)^{\kappa-1}\,d\zeta \\
&\le \frac{1-\kappa}{B(\kappa)}\lambda\|u-u_{n-1}\| + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\lambda\|u-u_{n-1}\|t. \qquad (4.10)
\end{aligned}$$

Therefore, Continuing this way recursively at t_0 we get:

$$\|\delta_n(x,t)\| = \left(\frac{1-\kappa}{B(\kappa)} + \frac{\kappa t_0}{B(\kappa)\Gamma(\kappa)}\right)^{n+1}\lambda^{n+1}M. \qquad (4.11)$$

where $M = \|u-u_{n-1}\|$. After taking limit of both sides as n tends to infinity, we get $\|\delta_n(x,t)\| \to 0$.

Next, it is necessity to demonstrate uniqueness for the solution of the proposed problem. Suppose, $u^*(x,t)$ be the another solution, then we get

$$\begin{aligned}
u(x,t)-u^*(x,t) &= \frac{1-\kappa}{B(\kappa)}\big(\mathscr{G}(x,\ t,\ u)-\mathscr{G}(x,\ t,\ u^*)\big) \\
&\quad + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \big(\mathscr{G}(x,\ \zeta,\ u)-\mathscr{G}(x,\ \zeta,\ u^*)\big)(t-\zeta)^{\kappa-1}\,d\zeta \qquad (4.12)
\end{aligned}$$

On applying norms on both side of above equation we get

$$\begin{aligned}
\|u(x,t)-u^*(x,t)\| &= \left\| \frac{1-\kappa}{B(\kappa)}\big(\mathscr{G}(x,\ t,\ u)-\mathscr{G}(x,\ t,\ u^*)\big) \right. \\
&\quad \left. + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \big(\mathscr{G}(x,\ \zeta,\ u)-\mathscr{G}(x,\ \zeta,\ u^*)\big)(t-\zeta)^{\kappa-1}\,d\zeta. \right\| \\
&\le \frac{1-\kappa}{B(\kappa)}\lambda\big\|\big(u(x,\ t)-u^*(x,\ t)\big)\big\| \\
&\quad + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\lambda t\big\|\big(u(x,\ t)-u^*(x,\ t)\big)\big\|. \qquad (4.13)
\end{aligned}$$

After simplification we get

$$\left\| \left(u(x,\, t) - u^{*}(x,\, t)\right) \right\| \left(1 - \frac{1-\kappa}{B(\kappa)}\lambda + \frac{\kappa t}{B(\kappa)\Gamma(\kappa)}\lambda \right) \leq 0. \tag{4.14}$$

From the above inequility, we get that if

$$\left(1 - \frac{1-\kappa}{B(\kappa)}\lambda + \frac{\kappa t}{B(\kappa)\Gamma(\kappa)}\lambda \right) \geq 0. \tag{4.15}$$

then $\left(u(x,\, t) - u^{*}(x,\, t)\right) = 0$.

Therefore, (4.15) is required condition for uniqueness.

4.2. Existence of solutions for the modified time fractional Kawahara equation. Let us consider the modified time fractional Kawahara equation (1.3) using Atangana-Baleanu fractional derivative operator as below

$$^{ABC}D^{\kappa}u(x,t) = \mathscr{H}(x,\, t,\, u(x,t)) = u^{2}\frac{\partial u}{\partial x} + m\frac{\partial^{3} u}{\partial x^{3}} + n\frac{\partial^{5} u}{\partial x^{5}} \tag{4.16}$$

where $^{ABC}D^{\kappa}$ represents the fractional operator of type Atangana-Baleanu-Caputo (ABC) having fractional order κ; where $0 < \kappa < 1$, subject to initial condition $u_0(x,t) = u(x,0)$. The equation (4.16) can be converted to the Volterra-type integral equation by using the ABC fractional integral as follows:

$$u(x,t) - u(x,0) = \frac{1-\kappa}{B(\kappa)}\mathscr{H}(x,\, t,\, u) + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_{0}^{t}\mathscr{H}(x,\, \zeta,\, u)\,(t-\zeta)^{\kappa-1}\,d\zeta, \tag{4.17}$$

Theorem 4.3. *The kernel $\mathscr{H}(x,\, t,\, u)$ given in Eq.* (4.16) *satisfy the Lipschitz condition and contraction if the following condition holds*

$$0 \leq \left(\frac{\tau_4}{3}(\rho_3^2 + \rho_3\rho_4 + \rho_4^2) + \tau_5 + \tau_6 \right) < 1.$$

Proof: We have $\mathscr{H}(x,\, t,\, u(x,t)) = u^{2}\frac{\partial u}{\partial x} + m\frac{\partial^{3} u}{\partial x^{3}} + n\frac{\partial^{5} u}{\partial x^{5}}$.

Consider u_1 and u_2 be two functions, then we obtain the following

$$\| \mathscr{H}(x,\, t,\, u_1(x,t)) - \mathscr{H}(x,\, t,\, u_2(x,t)) \| =$$

$$
\begin{aligned}
&= \left\| u_1^2(x,t)\frac{\partial u_1(x,t)}{\partial x} - u_2^2(x,t)\frac{\partial u_2(x,t)}{\partial x} + m\frac{\partial^3 u_1(x,t)}{\partial x^3} - m\frac{\partial^3 u_2(x,t)}{\partial x^3} \right. \\
&\qquad \left. + n\frac{\partial^5 u_1(x,t)}{\partial x^5} - n\frac{\partial^5 u_2(x,t)}{\partial x^5} \right\| \\
&\le \frac{1}{3}\left\| \frac{\partial}{\partial x}[u_1^3(x,t) - u_2^3(x,t)] \right\| + m\left\| \frac{\partial^3}{\partial x^3}[u_1(x,t) - u_2(x,t)] \right\| \\
&\qquad + n\left\| \frac{\partial^5}{\partial x^5}[u_1(x,t) - u_2(x,t)] \right\| \\
&\le \frac{\tau_4}{3}\left\| [u_1^3(x,t) - u_2^3(x,t)] \right\| + m\tau_5\left\| [u_1(x,t) - u_2(x,t)] \right\| + n\tau_6\left\| [u_1(x,t) - u_2(x,t)] \right\| \\
&\le \left(\frac{\tau_4}{3}(\rho_3^2 + \rho_3\rho_4 + \rho_4^2) + \tau_5 + \tau_6 \right)\left\| [u_1(x,t) - u_2(x,t)] \right\| \\
&\le \lambda_1 \left\| [u_1(x,t) - u_2(x,t)] \right\|.
\end{aligned}
$$

where $\tau_4 = \frac{\partial}{\partial x}$, $\tau_5 = \frac{\partial^3}{\partial x^3}$ and $\tau_6 = \frac{\partial^5}{\partial x^5}$ are the differential operators. Since, u_1 and u_2 are bounded functions, we have $\|u_1\| \le \rho_3$, $\|u_2\| \le \rho_4$.
Also $\lambda_1 = \left(\frac{\tau_4}{3}(\rho_3^2 + \rho_3\rho_4 + \rho_4^2) + \tau_5 + \tau_6 \right)$. Therefore, we have

$$
\| \mathscr{H}(x,\ t,\ u_1(x,t)) - \mathscr{H}(x,\ t,\ u_2(x,t)) \| \le \lambda_1 \left\| [u_1(x,t) - u_2(x,t)] \right\| \tag{4.18}
$$

This shows that the Lipschitz condition is obtained for $\mathscr{H}$. Moreover, we see that if $0 \le \left(\frac{\tau_4}{3}(\rho_3^2 + \rho_3\rho_4 + \rho_4^2) + \tau_5 + \tau_6 \right) < 1$, then it implies the contraction.
The recursive form of equation (4.17) defined as follows

$$
u_n(x,t) = \frac{1-\kappa}{B(\kappa)}\mathscr{H}(x,\ t,\ u_{n-1}) + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \mathscr{H}(x,\ \zeta,\ u_{n-1})(t-\zeta)^{\kappa-1}\,d\zeta. \tag{4.19}
$$

Next, we get the difference between the successive iterative terms in the form of following expression

$$
\begin{aligned}
\vartheta_n(x,t) &= u_n(x,t) - u_{n-1}(x,t) \\
&= \frac{1-\kappa}{B(\kappa)}\big(\mathscr{H}(x,\ t,\ u_{n-1}) - \mathscr{H}(x,\ t,\ u_{n-2}) \big) \\
&\quad + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \big(\mathscr{H}(x,\ \zeta,\ u_{n-1}) - \mathscr{H}(x,\ \zeta,\ u_{n-2}) \big)(t-\zeta)^{\kappa-1}\,d\zeta.
\end{aligned} \tag{4.20}
$$

Notice that

$$u_n(x,t) = \sum_{i=1}^{n} \vartheta_i(x,t) \tag{4.21}$$

By using equation (4.19) and then using norm on equation (4.20), we get

$$\|\vartheta_n(x,t)\| \leq \frac{1-\kappa}{B(\kappa)}\lambda \|\vartheta_{n-1}(x,t)\| + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\lambda \int_0^t \|\vartheta_{n-1}(x,\zeta)\| d\zeta. \tag{4.22}$$

This completes the proof of the theorem.

Theorem 4.4. *The solution for the* (4.16) *will exist and unique under the condition that we can find t_0 satisfying*

$$\frac{1-\kappa}{B(\kappa)}\lambda_1 + \frac{\kappa t_0}{B(\kappa)\Gamma(\kappa)}\lambda_1 < 1.$$

Proof: First, we consider bounded function $u(x,t)$ satisfying the Lipschitz condition. From equations (4.20) and (4.22) we get the following equation

$$\|\vartheta_n(x,t)\| \leq \|u_n(x,0)\| \left[\frac{1-\kappa}{B(\kappa)}\lambda_1 + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\lambda_1\right]^n. \tag{4.23}$$

Hence, the solution is smooth, moreover existence is proved for the obtained solution. Next, we show that the equation (4.23) is the solution for the equation (4.16). For this, we consider

$$u(x,t) - u(x,0) = u_n(x,t) - \psi_n(x,t) \tag{4.24}$$

where $\psi_n(x,t)$ are reminder terms of series solution. Then, we must show that these terms approach to zero at infinity, that is, $\|\psi_\infty(x,t)\| \to 0$.

$$\begin{aligned} \|\psi_n(x,t)\| &= \Bigg\| \frac{1-\kappa}{B(\kappa)}\big(\mathscr{H}(x,\, t,\, u) - \mathscr{H}(x,\, t,\, u_{n-1})\big) \\ &\qquad + \frac{\kappa}{B(\kappa)\Gamma(\kappa)} \int_0^t \big(\mathscr{H}(x,\, \zeta,\, u) - \mathscr{H}(x,\, \zeta,\, u_{n-1})\big)(t-\zeta)^{\kappa-1} d\zeta \Bigg\| \\ &\leq \frac{1-\kappa}{B(\kappa)} \big\|\big(\mathscr{H}(x,\, t,\, u) - \mathscr{H}(x,\, t,\, u_{n-1})\big)\big\| \\ &\qquad + \frac{\kappa}{B(\kappa)\Gamma(\kappa)} \int_0^t \big\|\big(\mathscr{H}(x,\, \zeta,\, u) - \mathscr{H}(x,\, \zeta,\, u_{n-1})\big)\big\| (t-\zeta)^{\kappa-1} d\zeta \\ &\leq \frac{1-\kappa}{B(\kappa)}\lambda_1 \|u - u_{n-1}\| + \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\lambda_1 \|u - u_{n-1}\| t. \end{aligned} \tag{4.25}$$

Therefore, continuing this way recursively at t_0 we get:

$$\|\psi_n(x,t)\| = \left(\frac{1-\kappa}{B(\kappa)} + \frac{\kappa t_0}{B(\kappa)\Gamma(\kappa)}\right)^{n+1} \lambda_1^{n+1}\mathscr{M}. \tag{4.26}$$

where $\mathscr{M} = \|u-u_{n-1}\|$. After taking limit of both sides as n tends to infinity, we get $\|\psi_n(x,t)\| \to 0$.

Next, it is necessity to demonstrate uniqueness for the solution of the proposed problem. Suppose, $u^*(x,t)$ be the another solution, then we get

$$\begin{aligned} u(x,t)-u^*(x,t) &= \frac{1-\kappa}{B(\kappa)}\big(\mathscr{H}(x,\,t,\,u)-\mathscr{H}(x,\,t,\,u^*)\big) \\ &+ \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \big(\mathscr{H}(x,\,\zeta,\,u)-\mathscr{H}(x,\,\zeta,\,u^*)\big)(t-\zeta)^{\kappa-1}d\zeta. \end{aligned} \tag{4.27}$$

On applying norms on both side of above equation we get

$$\begin{aligned} \|u(x,t)-u^*(x,t)\| &= \Bigg\|\frac{1-\kappa}{B(\kappa)}\big(\mathscr{H}(x,\,t,\,u)-\mathscr{H}(x,\,t,\,u^*)\big) \\ &+ \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\int_0^t \big(\mathscr{H}(x,\,\zeta,\,u)-\mathscr{H}(x,\,\zeta,\,u^*)\big)(t-\zeta)^{\kappa-1}d\zeta.\Bigg\| \\ &\leq \frac{1-\kappa}{B(\kappa)}\lambda_1\big\|\big(u(x,\,t)-u^*(x,\,t)\big)\big\| \\ &+ \frac{\kappa}{B(\kappa)\Gamma(\kappa)}\lambda_1 t\big\|\big(u(x,\,t)-u^*(x,\,t)\big)\big\|. \end{aligned} \tag{4.28}$$

After simplification we get

$$\big\|\big(u(x,\,t)-u^*(x,\,t)\big)\big\|\left(1-\frac{1-\kappa}{B(\kappa)}\lambda_1+\frac{\kappa t}{B(\kappa)\Gamma(\kappa)}\lambda_1\right) \leq 0. \tag{4.29}$$

From the above inequility, we get that if

$$\left(1-\frac{1-\kappa}{B(\kappa)}\lambda_1+\frac{\kappa t}{B(\kappa)\Gamma(\kappa)}\lambda_1\right) \geq 0. \tag{4.30}$$

then $\big(u(x,\,t)-u^*(x,\,t)\big)=0$.

Therefore, (4.30) is required condition for uniqueness.

5. Numerical Application

In this section, we demonstrate the efficiency of iterative Laplace transform method by applying it on the time fractional Kawahara and modified Kawahara equations. Computations are done with the help of Mathematica software.

5.1. Approximate solution for time fractional Kawahara equation. Consider the time fractional Kawahara equation (1.1) with initial condition (1.2).

The exact solution to (1.1) is given in [33] as

$$u(x,t) = \frac{105}{169} sech^4\left(\frac{1}{2\sqrt{13}}\left(x - \frac{36t}{169}\right)\right) \tag{5.1}$$

The initial condition (1.2) is rewritten as

$$u(x,0) = \frac{1680}{169} \frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}} + 1\right)^4}$$

Applying laplace transform on both side of (1.1) we get

$$L\{u(x,t)\} - \frac{1}{s}u(x,0) + \frac{1}{B(\kappa)}\Big(1 - \kappa + \frac{\kappa}{s^\kappa}\Big)L\Big\{u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5}\Big\} = 0. \tag{5.2}$$

Rearranging terms we get

$$L\{u(x,t)\} = \frac{1}{s}\left(\frac{1680}{169} \frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}} + 1\right)^4}\right) - \frac{1}{B(\kappa)}\Big(1 - \kappa + \frac{\kappa}{s^\kappa}\Big)L\Big\{u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5}\Big\} \tag{5.3}$$

Further, the inverse Laplace transform on (5.3), yields

$$u(x,t) = \frac{1680}{169} \frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}} + 1\right)^4} - L^{-1}\left\{\frac{1}{B(\kappa)}\Big(1 - \kappa + \frac{\kappa}{s^\kappa}\Big)L\Big\{u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5}\Big\}\right\} \tag{5.4}$$

The series solution obtained by the method is given by,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{5.5}$$

The nonlinear term $u\dfrac{\partial u}{\partial x}$ is written as $u_n\dfrac{\partial u_n}{\partial x} = \sum_{n=0}^{\infty}\mathbb{P}_n$; whereas $\mathbb{P}_n$ is further decomposed as follows

$$\mathbb{P}_n = \sum_{i=0}^{n} u_i.\frac{\partial}{\partial x}\Big(\sum_{i=0}^{n} u_i\Big) - \sum_{i=0}^{n-1} u_i.\frac{\partial}{\partial x}\Big(\sum_{i=0}^{n-1} u_i\Big)$$

by using $u_0(x,\ t) = \frac{1680}{169}\frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}}+1\right)^4}$, we get the recursive formula as follows

$$u_n(x,t) = u_0(x,\ t) - L^{-1}\left\{\frac{1}{B(\kappa)}\left(1-\kappa+\frac{\kappa}{s^\kappa}\right)L\left\{u_n\frac{\partial u_n}{\partial x}+\frac{\partial^3 u_n}{\partial x^3}-\frac{\partial^5 u_n}{\partial x^5}\right\}\right\} \tag{5.6}$$

The n−term approximate solution is given by

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots+u_{n-1}(x,t). \tag{5.7}$$

Therefore, using (5.6) the first three terms of approximate solution of (1.1) are obtained as follows

$$u_0 = \frac{1680}{169}\frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}}+1\right)^4}$$

$$u_1 = \frac{7560\tanh\left(\frac{x}{2\sqrt{13}}\right)\operatorname{sech}^4\left(\frac{x}{2\sqrt{13}}\right)\left(-\kappa\Gamma(\kappa)+\Gamma(\kappa)+t^\kappa\right)}{28561\sqrt{13}(\kappa+\kappa(-\Gamma(\kappa))+\Gamma(\kappa))}$$

$$\begin{aligned} u_2 = &\frac{174182400\kappa^2 e^{\frac{9x}{2\sqrt{13}}}\Gamma(\kappa)^2 t^\kappa\cosh\left(\frac{x}{2\sqrt{13}}\right) - 174182400\kappa e^{\frac{9x}{2\sqrt{13}}}\Gamma(\kappa)^2 t^\kappa\cosh\left(\frac{x}{2\sqrt{13}}\right)}{62748517\left(e^{\frac{x}{\sqrt{13}}}+1\right)^9(\kappa+\kappa(-\Gamma(\kappa))+\Gamma(\kappa))^2\Gamma(\kappa+1)} \\ &+\frac{34836480\kappa e^{\frac{9x}{2\sqrt{13}}}\Gamma(\kappa)^2 t^\kappa\cosh\left(\frac{5x}{2\sqrt{13}}\right) - 34836480\kappa^2 e^{\frac{9x}{2\sqrt{13}}}\Gamma(\kappa)^2 t^\kappa\cosh\left(\frac{5x}{2\sqrt{13}}\right)}{62748517\left(e^{\frac{x}{\sqrt{13}}}+1\right)^9(\kappa+\kappa(-\Gamma(\kappa))+\Gamma(\kappa))^2\Gamma(\kappa+1)} \\ &-\frac{87091200\kappa e^{\frac{9x}{2\sqrt{13}}}\Gamma(\kappa)^3\cosh\left(\frac{x}{2\sqrt{13}}\right)}{62748517\left(e^{\frac{x}{\sqrt{13}}}+1\right)^9(\kappa+\kappa(-\Gamma(\kappa))+\Gamma(\kappa))^2\Gamma(\kappa+1)}+\cdots \end{aligned}$$

Continuing in the same way, remaining terms of the iteration formula (5.6) can be calculated.

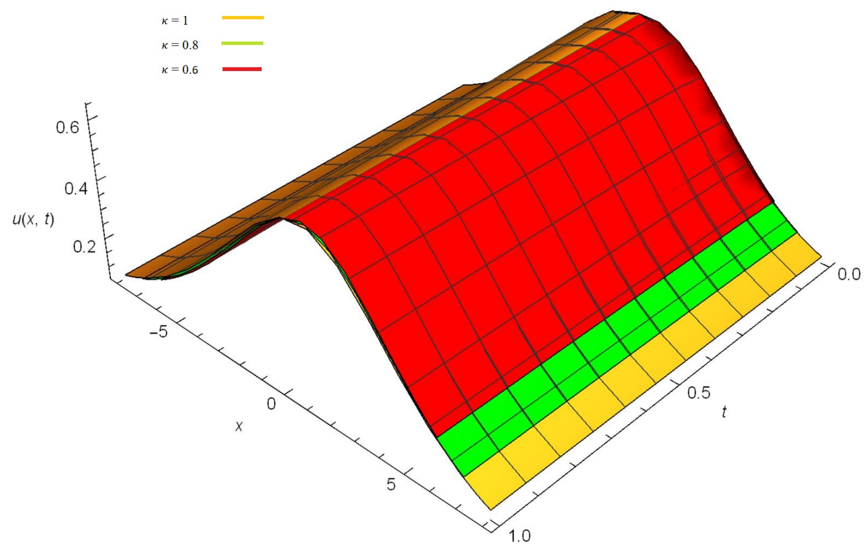

FIGURE 1. Approx. soln of Eq. (1.1), for $\kappa = 1,\ 0.8,\ 0.6$

t	x	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$	Absolute error $\lvert u_{\text{exact}} - u_{\text{apprx}} \rvert$ for $\kappa = 1$
0.02	-5	0.225399	0.237974	0.248996	0.00043072
	0	0.620115	0.620928	0.621263	1.76859×10^{-13}
	5	0.284071	0.270963	0.259721	0.00043072
0.06	-5	0.222523	0.235295	0.247044	0.00129216
	0	0.619872	0.620802	0.621234	1.43069×10^{-11}
	5	0.287106	0.273724	0.261693	0.00129216
0.1	-5	0.220546	0.233153	0.245231	0.00215361
	0	0.619694	0.620692	0.621201	1.10391×10^{-10}
	5	0.289199	0.275939	0.263527	0.00215361

TABLE 1. The numerical results for various values of κ and comparison of absolute error between the exact solution with three term approximations obtained by ILTM of (1.1) for $\kappa = 1$

5.2. Approximate solution for modified time fractional Kawahara equation

Next we consider the modified time fractional Kawahara equation (1.3) with initial condition (1.4).

The exact solution for the classical modified Kawahara equation is given by [33]

$$u(x,t) = \frac{3p}{\sqrt{-10q}} sech^2[K(x-ct)],\ c = \frac{25q-4p^2}{25q}. \tag{5.8}$$

Applying laplace transform on both side of (1.3) we get,

$$L\{u(x,t)\} - \frac{1}{s}u(x,0) + \frac{1}{B(\kappa)}\Big(1-\kappa+\frac{\kappa}{s^\kappa}\Big)L\Big\{u^2\frac{\partial u}{\partial x} + m\frac{\partial^3 u}{\partial x^3} + n\frac{\partial^5 u}{\partial x^5}\Big\} = 0. \tag{5.9}$$

Rearranging terms we get,

(5.10)

$$L\{u(x,t)\} = \frac{1}{s}\Big(\frac{3m}{\sqrt{-10n}} sech^2(Kx)\Big) - \frac{1}{B(\kappa)}\Big(1-\kappa+\frac{\kappa}{s^\kappa}\Big)L\Big\{u^2\frac{\partial u}{\partial x} + m\frac{\partial^3 u}{\partial x^3} - n\frac{\partial^5 u}{\partial x^5}\Big\}$$

Further, the inverse Laplace transform on (5.10), yields

$$u(x,t) = \frac{3m}{\sqrt{-10n}} sech^2(Kx) - L^{-1}\Big\{\frac{1}{B(\kappa)}\Big(1-\kappa+\frac{\kappa}{s^\kappa}\Big)L\Big\{u^2\frac{\partial u}{\partial x} + m\frac{\partial^3 u}{\partial x^3} + n\frac{\partial^5 u}{\partial x^5}\Big\}\Big\} \tag{5.11}$$

The series solution obtained by the method is given by,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{5.12}$$

The nonlinear term $u^2\frac{\partial u}{\partial x}$ is written as $u_n^2\frac{\partial u_n}{\partial x} = \sum_{n=0}^{\infty}\mathbb{J}_n$; whereas $\mathbb{J}_n$ is further decomposed as follows

$$\mathbb{J}_n = \sum_{i=0}^{n} u_i^2 . \frac{\partial}{\partial x}\Big(\sum_{i=0}^{n} u_i\Big) - \sum_{i=0}^{n-1} u_i^2 . \frac{\partial}{\partial x}\Big(\sum_{i=0}^{n-1} u_i\Big)$$

by using $u_0(x,\ t) = \frac{3m}{\sqrt{-10n}} sech^2(Kx)$, we get the recursive formula as follows

$$u_n(x,t) = u_0(x,\ t) + L^{-1}\Big\{\frac{-1}{B(\kappa)}\Big(1-\kappa+\frac{\kappa}{s^\kappa}\Big)L\Big\{u_n^2\frac{\partial u_n}{\partial x} + m\frac{\partial^3 u_n}{\partial x^3} + n\frac{\partial^5 u_n}{\partial x^5}\Big\}\Big\} \tag{5.13}$$

The n−term approximate solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots + u_{n-1}(x,t). \tag{5.14}$$

$$u_0 = \frac{6\sqrt{\frac{2}{5}}me^{\frac{x\sqrt{-\frac{m}{n}}}{\sqrt{5}}}}{\sqrt{-n}\left(e^{\frac{x\sqrt{-\frac{m}{n}}}{\sqrt{5}}}+1\right)^2}$$

$$u_1 = -\frac{24\sqrt{2}m^3\Gamma(\kappa)\sqrt{-\frac{m}{n}}e^{\frac{x\sqrt{-\frac{m}{n}}}{\sqrt{5}}}\left(e^{\frac{x\sqrt{-\frac{m}{n}}}{\sqrt{5}}}-1\right)(\kappa t^{\kappa}-(\kappa-1)\Gamma(\kappa+1))}{125(-n)^{3/2}((\kappa-1)\Gamma(\kappa)-\kappa)\Gamma(\kappa+1)\left(e^{\frac{x\sqrt{-\frac{m}{n}}}{\sqrt{5}}}+1\right)^3}$$

$$u_2 = \frac{1}{-\kappa+\frac{\kappa}{\Gamma(\kappa)}+1}\left(\frac{3.654931423025195\text{*^-28}e^{0.183848x}-1.0964794269075588\text{*^-27}e^{0.212132x}}{(e^{0.0141421x}+1)^{22}\left(-\kappa+\frac{\kappa}{\Gamma(\kappa)}+1\right)^3}\right.$$
$$+\frac{1.0964794269075587\text{*^-27}e^{0.240416x}-3.654931423025195\text{*^-28}e^{0.268701x}}{(e^{0.0141421x}+1)^{22}\left(-\kappa+\frac{\kappa}{\Gamma(\kappa)}+1\right)^3}$$
$$-\frac{7.30986284605039\text{*^-28}e^{0.169706x}+4.385917707630235\text{*^-27}e^{0.19799x}}{(e^{0.0141421x}+1)^{21}\left(-\kappa+\frac{\kappa}{\Gamma(\kappa)}+1\right)^3}$$
$$-\frac{6.578876561445353\text{*^-27}e^{0.226274x}}{(e^{0.0141421x}+1)^{21}\left(-\kappa+\frac{\kappa}{\Gamma(\kappa)}+1\right)^3}+\cdots.$$

t	x	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$	Absolute error $\lvert u_{\text{exact}} - u_{\text{apprx}}\rvert$ for $\kappa = 1$
0.1	0	0.00094868329	0.00094868329	0.00094868329	4.74342×10^{-10}
	5	0.00094749843	0.00094749843	0.00094749843	4.68832×10^{-8}
	10	0.00094394464	0.00094394464	0.00094394464	9.37736×10^{-8}
	15	0.00093809014	0.00093809014	0.00093809014	1.39735×10^{-7}
	20	0.00092999597	0.00092999597	0.00092999597	1.84324×10^{-7}

TABLE 2. The numerical results for various values of κ and comparison of absolute error between the exact solution with three term approximations obtained by ILTM of (1.3) for $\kappa = 1$

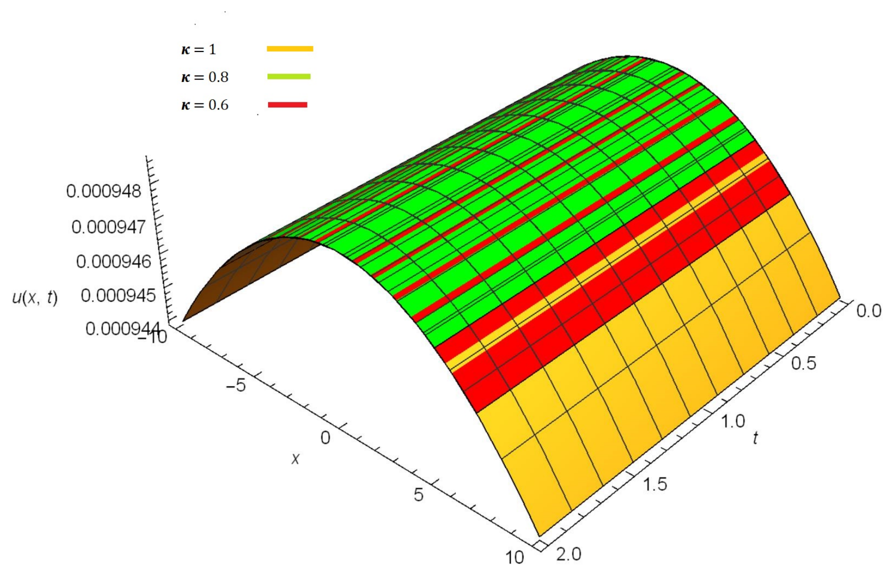

FIGURE 2. Approx. soln of Eq. (1.3), for $\kappa = 1, 0.8, 0.6$

In tables 1 and 2 the computational results are obtained to get approximate solution of equations (1.1) and (1.3) respectively with $m = 0.001$ and $q = -1.$ in Eq.(1.3). This technique provides accurate numerical solutions even if lower order approximations are used. The accuracy for the time fractional Kawahara and modified kawahara equation is demonstrated for the absolute errors of Eq. (1.1) and (1.3) respectively with their exact solutions.

Also fig. 1 shows surfaces for approximate solution of Eq. (1.1) and exact solution of classical Kawahara equation and fig. 2 shows surfaces for approximate solution of Eq. (1.3) and exact solution of classical modified Kawahara equation. Also, from these surfaces it can be observed that approximate solution and exact solution are of both the equations are very near to each other.

6. CONCLUSIONS

In this study, we have obtained the approximate solutions of time fractional Kawahara and modified Kawahara equations based on Atangana-Baleanue fractional derivative operator by using Laplace transform with iterative method. It is seen that the solutions obtained converges very rapidly to the exact solutions in only second order approximations i.e. approximate solutions are very near to the exact solutions. We can conclude from the numerical results that the present technique is straightforward, efficient and provides very high accuracy. This is very simple, reliable and powerful technique for finding approximate solutions of many fractional physical models arising in science and engineering.

ACKNOWLEDGEMENT

The authors are thankful to the referees for their valuable comments.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[2] R. Hilfer, Applications of fractional calculus in physics, World Scientific, Singapore, 2000.

[3] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[4] R. Herrmann. Fractional calculus: an introduction for physicists, 2nd ed. World Scientific, Singapore, 2014.

[5] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, 1994.

[6] G. Adomian, R. Rach, Modified Adomian Polynomials, Math. Computer Model. 24 (1996), 39–46.

[7] D.B. Dhaigude, G.A. Birajdar and V.R. Nikam, Adomian Decomposition Method for Fractional Benjamin-Bona-Mahony-Burgers Equations, Int. J. Appl. Math. Mech. 8 (2012), 42–51.

[8] D.B. Dhaigude, G.A. Birajdar, Numerical Solution of System of Fractional Partial Differential Equations by Discrete Adomian Decomposition Method, J. Frac. Cal. Appl. 12 (2012), 1–11.

[9] J.H. He, Variational iteration method – a kind of non-linear analytical technique: some examples, Int. J. Nonlinear Mech. 34(4) (1999), 699-708.

[10] B.R. Sontakke, A.S. Shelke, A.S. Shaikh, Solution of non-linear fractional differential equations by variational iteration method and applications, Far East J. Math. Sci. 110(1) (2019), 113-129.

[11] J.-H. He, Homotopy perturbation technique, Computer Meth. Appl. Mech. Eng. 178 (1999), 257–262.

[12] S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A. 360 (2006), 109-113,

[13] M.Yu. Kokurin, S.I. Piskarev, M. Spreafico, Finite-Difference Methods for Fractional Differential Equations of Order 1/2, J. Math. Sci. 230 (2018), 950–960.

[14] K.M. Furati, M.D. Kassim, N.e-. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, Computers Math. Appl. 64 (2012), 1616–1626.

[15] D.B. Dhaigude, S.B. Kiwne, R.M. Dhaigude, Monotone iterative scheme for weakly coupled system of finite difference reaction-diffusion equations, Commun. Appl. Anal. 12(2) (2008), 161-173.

[16] D.B. Dhaigude, V.N. Bhadgaonkar, An analytical study of reaction diffusion, (3 + 1)-dimensional diffusion equations using Caputo Fabrizio fractional differential operator, J. Math. Comput. Sci. 11 (2021), 183-202

[17] V. Daftardar-Gejji, H. Jafari, An iterative method for solving non linear functional equations, J. Math. Anal. Appl. 316 (2006), 753-763.

[18] V. Daftardar-Gejji, S. Bhalekar, Solving fractional diffusion-wave equations using a new iterative method, Frac. Calc. Appl. Anal. 11(2) (2008), 193-202.

[19] L. Gaul, P. Klein, S. Kemple, Damping description involving fractional operators, Mech. Syst. Signal Process. 5 (1991), 81–88.

[20] R. Figueiredo Camargo, E. Capelas de Oliveira, J. Vaz, On anomalous diffusion and the fractional generalized Langevin equation for a harmonic oscillator, J. Math. Phys. 50 (2009), 123518.

[21] T.A.M. Langlands, B.I. Henry, S.L. Wearne, Fractional cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions, J. Math. Biol. 59 (2009), 761–808.

[22] J.T. Machado, Fractional Calculus: Application in Modeling and Control, in: C. Constanda, B.E.J. Bodmann, H.F. de C. Velho (Eds.), Integral Methods in Science and Engineering, Springer, New York, 2013, pp. 279–295.

[23] E.C. de Oliveira, F. Mainardi, J. Vaz, Fractional models of anomalous relaxation based on the Kilbas and Saigo function, Meccanica. 49 (2014), 2049–2060.

[24] B.R. Sontakke, A. Shaikh, K.S. Nisar, Approximate Solutions of a Generalized Hirota-Satsuma Coupled KdV and a Coupled mKdV Systems with Time Fractional Derivatives, Malays. J. Math. Sci. 12(2) (2018), 175-196.

[25] A. Shaikh, A. Tassaddiq, K. S. Nisar and D. Baleanu, Analysis of differential equations involving Caputo–Fabrizio fractional operator and its applications to reaction–diffusion equations, Adv. Differ. Equ. 2019 (2019), 78.

[26] B.R. Sontakke, A.S. Shaikh, Properties of caputo operator and its applications to linear fractional differential equations, Int. J. Eng. Res. Appl. 5(5) (2015), 22-27.

[27] A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, Therm. Sci. 20(2) (2016), 763-769.

[28] N.A. Kudryashov, A note on new exact solutions for the Kawahara equation using Exp. function method, J. Comput. Appl. Math. 234 (2012), 3511-3512.

[29] A. Biswas, Solitary wave solution for the generalized Kawahara equation, Appl. Math. Lett. 22(2) (2009), 208-210.

[30] M.N. Rasoulizadeh, J. Rashidinia, Numerical solution for the Kawahara equation using local RBF-FD meshless method, J. King Saud Univ.-Sci. 32 (2020). 2277–2283.

[31] J. Lu, Analytical approach to Kawahara equation using Variational Iteration Method and Homotopy Perturbation Method, Topol. Meth. Nonlinear Anal. 31 (2008), 287-293.

[32] T. Kawahara, Oscillatory solitary waves in dispersive media, J. Phys. Soc. Japan. 33 (1972), 260-264.

[33] Sirendaoreji, New exact traveling wave solutions for the Kawahara and modified Kawahara equations, Chaos Solitons Fractals, 19 (2004), 147-150.

[34] A. M.Wazwaz, New solitary wave solutions to the modified Kawahara equation, Phys. Lett. A. 8 (2007), 588-592.

[35] L. Jin, Application of variational iteration method and homotopy perturbation method to the modified Kawahara equation, Math. Computer Model. 49 (2009), 573-578

[36] M. Kurulay, Approximate analytic solutions of the modified Kawahara equation with homotopy analysis method, Adv. Differ. Equ. 2012 (2012), 178.