Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 2874-2892
https://doi.org/10.28919/jmcs/5624
ISSN: 1927-5307

A PICARD S^* ITERATIVE ALGORITHM FOR APPROXIMATING FIXED POINTS OF GENERALIZED α-NONEXPANSIVE MAPPINGS

PINKI LAMBA, ANJU PANWAR*

Department of Mathematics, Maharshi Dayanand University, Rohtak-124001, India

Abstract. In this article, we introduce a new iterative algorithm called Picard-S^* iterative algorithm. We show that the Picard-S^* iterative algorithm can be used to approximate fixed points of generalized α-nonexpansive mappings. We discuss the convergence results of generalized α-nonexpansive mappings in the framework of CAT(0) spaces using Picard S^* iteration process and demiclosed principle for the aforementioned class of mappings in CAT(0) spaces. This is the nonlinear version of some known results that have been demonstrated in Banach spaces. Some useful examples are obtained to illustrate facts. Some known iteration processes are also compared using numerical calculations. Our results broaden and improve the corresponding recent results announced by many authors.

Keywords: fixed point; convergence theorem; generalized α-nonexpansive mapping; Picard S^* iterative algorithm; CAT(0) space.

2010 AMS Subject Classification: 47H10, 54H25, 54E50.

1. Introduction

Let E be CAT(0) space and K a nonempty closed convex subset of E. It is well known that a mapping $T : K \to K$ is said to be nonexpansive whenever $d(Tx,Ty) \leq d(x,y)$ for all $x, y \in K$. It is called quasi-nonexpansive mapping if $F(T) \neq \phi$ and $d(Tx, p) \leq d(x, p)$ for all $x \in K$ and

*Corresponding author
E-mail address: anjupanwar15@gmail.com
Received March 01, 2021

$p \in F(T)$, where $F(T)$ is the set of fixed points of T, i.e., $F(T) = \{x \in K : Tx = x\}$. A number of extensions and generalization of nonexpansive mappings have been considered by many mathematicians, in recent years. In 2008, Suzuki[16] introduced the concept of generalized nonexpansive mappings and obtained some existence and convergence theorems for such mappings. A mapping $T : K \to K$ is said to satisfy condition(C) if $\frac{1}{2}d(x,Tx) \leq d(x,y)$ implies $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in K$.

Aoyama and Kohsaka [7] introduced the class of α-nonexpansive mappings in Banach spaces. A mapping $T : K \to K$ is said to be α-nonexpansive if there exists an $\alpha \in [0,1)$ such that for all $x,y \in K$,

$$d(Tx,Ty)^2 \leq \alpha d(Tx,y)^2 + \alpha d(x,Ty)^2 + (1-2\alpha)d(x,y)^2.$$

It is interesting to note that nonexpansive mappings are continuous on their domains but Suzuki-type generalized nonexpansive mapping and α-nonexpansive mapping do not have to be continuous. Therefore, these figures are more important from a theoretical and application perspective. Pant and Shukla[11] launched a new class of mappings that includes both the Suzuki-type generalized nonexpansive mapping and α-nonexpansive mappings. A mapping $T : K \to K$ is said to be generalized α-nonexpansive if there exist $\alpha \in [0,1)$ such that for all $x,y \in K$, $\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq \alpha d(Tx,y) + \alpha d(Ty,x) + (1-2\alpha)d(x,y)$.

By time, many iteration processes have been developed and it is impossible to cover them all. There exist some iteration processes that are often used to approximate fixed points of nonexpansive mappings.

The one-step Mann[22] iteration process for approximating fixed points follows.

$x_1 \in K$

$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$, $n \in \mathbb{N}$,

where $\{\alpha_n\}$ is a sequence in (0,1).

In 1974, Ishikawa [14] generalized Mann iteration process for lipschitzian pseudo-contractive maps from one step to two steps defined as:

$x_1 \in K$

$y_n = (1-\alpha_n)x_n + \alpha_n Tx_n$

$x_{n+1} = (1-\beta_n)x_n + \beta_n Ty_n$, $n \in \mathbb{N}$, where $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in (0,1).

In 2007, Agarwal et al.[10] introduced the following iteration process, known as S iteration process for nearly asymptotically nonexpansive mappings.

$x_1 \in K$

$x_{n+1} = (1-\beta_n)Tx_n + \beta_n Ty_n$

$y_n = (1-\alpha_n)x_n + \alpha_n Tx_n,\ n \in \mathbb{N},$

where $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in (0,1).

In 2013, Karahan and Ozdemir[6] used the following S^* iteration procedure to approximate the fixed point of nonexpansive mappings.

$x_1 \in K$

$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n$

$y_n = (1-\beta_n)Tx_n + \beta_n Tz_n$

$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n,\ n \in \mathbb{N},$

where $\{\gamma_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in (0,1).

Very recently, in 2019, Panwar and Reena introduced a hybrid iterative scheme named Picard Noor-type hybrid iterative scheme to approximate the fixed points of a multivalued ρ-quasinonexpansive mappings.

Encouraged by the above work, in this article, we introduce a new hybrid iterative algorithm, Picard S^* iterative algorithm with CAT(0) space setting described as:

$$
(1) \qquad \begin{cases} x_1 \in K \\ w_n = (1-\gamma_n)x_n \bigoplus \gamma_n Tx_n \\ z_n = (1-\beta_n)Tx_n \bigoplus \beta_n Tw_n \\ y_n = (1-\alpha_n)Tx_n \bigoplus \alpha_n Tz_n \\ x_{n+1} = Ty_n, \quad n \in \mathbb{N}, \end{cases}
$$

where $\{\gamma_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in (0,1). Our Picard S^* hybrid iterative algorithm gives faster convergence results than existing iterative algorithms. Then, we establish a number of existence and convergence theorems. Some useful examples are also presented to clarify facts. Finally, we present a comparison between some known iteration processes using numerical calculations. In this way, we display the efficiency of our proposed algorithm.

2. PRELIMINARIES

”For the sake of simplicity , we recall a few definitions, exceptions and conclusions.

Let (E,d) be a metric space and $x, y \in E$ with $d(x,y) = l$. A geodesic path from x to y is a isometry $c : [0,l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called a geodesic segment. A metric space E is a (uniquely) geodesic space, if every two points of E are joined by only one geodesic segment. A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic space E consists of three points x_1,x_2,x_3 of E and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\Delta(x_1,x_2,x_3)$ is the triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\overline{x_1},\overline{x_2},\overline{x_3})$ in the Euclidean space $\mathbb{R}^2$ such that

$d(x_i,x_j) = d_{\mathbb{R}^2}(\overline{x_i},\overline{x_j}), \quad \forall$ i, j = 1,2,3.

A geodesic space E is a CAT(0) space, if for each geodesic triangle $\Delta(x_1,x_2,x_3)$ in E and its comparison triangle $\overline{\Delta} := \Delta(\overline{x_1},\overline{x_2},\overline{x_3})$ in $\mathbb{R}^2$, the CAT(0) inequality

$d(x,y) = d_{\mathbb{R}^2}(\bar{x},\bar{y})$

is satisfied for all $x, y \in \Delta$ and $\bar{x},\bar{y} \in \overline{\Delta}$.

A thorough discussion of these spaces and their important role in various branches of mathematics are, for example, given in [3, 9].

One approach is due to the famous mathematician Kirk[17, 18] who established a more general result to study the fixed point results in the setting of complete CAT(0) space. Among other things, he proved that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. In this paper, we write $(1-t)x \bigoplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$d(z,x) = td(x,y)$, $d(z,y) = (1-t)d(x,y)$.

We also denote by $[x,y]$ the geodesic segment joining from x to y, i.e., $[x,y] = \{(1-t)x \bigoplus ty : t \in [0,1]\}$.

A subset of a CAT(0) space is convex if $[x,y] \subset C$ for all $x, y \in C$. For elementary facts about CAT(0) spaces, we refer the readers to [3, 4, 8, 9].

Lemma 2.1. [9] *Let E be a CAT(0) space. Then*
$d((1-s)x\bigoplus sy,z) \leq (1-s)d(x,z)+sd(y,z)$ *for all* $x,y,z\in E$ *and* $s\in[0,1]$.

Lemma 2.2. [9] *Let E be a CAT(0) space. Then*
$d((1-s)x\bigoplus sy,z)^2 \leq (1-s)d(x,z)^2+sd(y,z)^2-s(1-s)d(x,y)^2$, *for all* $x,y,z\in E$ *and* $s\in[0,1]$.

In 1976. Lim[15] introduced the concept of Δ-convergence in a general metric space. In 2008, Kirk and Panyanak[19] specialized Lim's concept to CAT(0) spaces and proved that it is similar to the weak convergence in Banach space setting. Since the notion of Δ-convergence has been widely studied.

We now give the concept of Δ-convergence and collect some of its basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space E. For $x\in E$, we set

$$r(x,\{x_n\}) = limsup_{n\to\infty} d(x,\{x_n\}).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = inf\{r(x,\{x_n\}) : x\in E\}.$$

The asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C\subset E$ is given by

$$r_C(\{x_n\}) = inf\{r(x,\{x_n\}) : x\in C\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x\in E : r(x,\{x_n\}) = r(\{x_n\})\}.$$

And the asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C\subset E$ is the set

$$A_C(\{x_n\}) = \{x\in C : r(x,\{x_n\}) = r_C(\{x_n\})\}.$$

Proposition 2.3. [13] *Let E be a complete CAT(0) space,* $\{x_n\}$ *be a bounded sequence in E and K be a closed convex subset of E. Then*

(1) there exists a unique point $u\in K$ *such that*

$$r(u,\{x_n\}) = inf_{x\in K} r(x,\{x_n\});$$

(2) $A(\{x_n\})$ *and* $A_K(\{x_n\})$ *are both singleton.*

Definition 2.4. [19, 15] Let E be a CAT(0) space. A sequence $\{x_n\}$ in E is said to Δ-converge to $p\in E$, if p is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta-\lim_{n\to\infty} x_n = p$ and call p the Δ-limit of $\{x_n\}$.

Lemma 2.5. *Let E be a complete CAT(0) space, K be closed convex subset of E.*

(1) If $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K. [12]

(2) Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence [19]

3. Generalized α-Nonexpansive Mappings

We present some basic properties of generalized α-nonexpansive mappings.

Proposition 3.1. *Any mapping satisfying condition(C) is a generalized α-nonexpansive mapping, but the converse is not true.*

When $\alpha = 0$, a generalized α-nonexpansive mapping reduces to a mapping satisfying condition(C). The following example shows that the reverse implication does not hold.

Example 3.1. *Let K = [0,2] be a subset of $\mathbb{R}$ endowed with the euclidean norm. Define $T : K \to K$ by:*

$$Tx = \begin{cases} 0, & \text{if } x \neq 2, \\ 1, & \text{if } x = 2. \end{cases}$$

Then for $x \in (1, \frac{4}{3}]$ and $y = 2$,

$\frac{1}{2}d(x,Tx) \leq d(x,y)$ and $d(Tx,Ty) = 1 > d(x,y)$, and T does not satisfy condition(C). Again for $x \in (1, \frac{4}{3}]$ and $y = 2$, $\frac{1}{2}d(y,Ty) \leq d(x,y)$ and $d(Tx,Ty) > d(x,y)$, hence, T does not satisfy condition(C). However, T is an α-nonexpansive with $\alpha \geq \frac{1}{4}$ and a generalized α-nonexpansive mapping with $\alpha \geq \frac{1}{6}$.

As $d(Tx,Ty) = d(0,1) = 1$ and

$$\begin{aligned} \alpha d(Tx,y) + \alpha d(Ty,x) + (1-2\alpha)d(x,y) &\leq \alpha d(0,2) + \alpha d(1,2) + (1-2\alpha)d(1,2) \\ &= 2\alpha + \alpha + (1-2\alpha).1 = 1+\alpha \end{aligned}$$

clearly, $d(Tx,Ty) = 1 \leq 1+\alpha = \alpha d(Tx,y) + \alpha d(Ty,x) + (1-2\alpha)d(x,y)$

Example 3.2. *Let $X = \{0,2,4,5\}$ be a subset of $\mathbb{R}$. Define a mertic d on X by $d(x,y) = |x-y|$. Define a mapping $T : X \to X$ by:*

T(0)= 0, T(2)= 0, T(4)= 2 and T(5)= 0.

We note that for $\alpha = \frac{1}{2}$,

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le \alpha d(Tx,y) + \alpha d(Ty,x) + (1-2\alpha)d(x,y).$$

if $x,y \neq 4,5$. *In the case x =4 and y = 5, we have*

$$\frac{1}{2}d(x,Tx) = 1 = d(x,y) \text{ and here, } d(Tx,Ty) = 2 \le \frac{1}{2}d(Tx,y) + \frac{1}{2}d(Ty,x)$$

and for y, we have $\frac{1}{2}d(y,Ty) = \frac{5}{2} > d(x,y)$

Therefore, T is generalized $\frac{1}{2}$*-nonexpansive mapping.*

However, for $x = 4, y = 5$,

$$\frac{1}{2}d(x,Tx) = 1 = d(x,y) \text{ but } d(Tx,Ty) = d(T4,T5) = 2 > d(x,y) = 1.$$

Thus, T is not a Suzuki-type generalized nonexpansive mapping.

Proposition 3.2. [1] *Let K be a nonempty subset of a CAT(0) space E and* $T : K \to K$ *a generalized* α*-nonexpansive mapping with a fixed point* $y \in K$. *Then T is a quasi-nonexpansive.*

Lemma 3.3. *Let K be a nonempty subset of a CAT(0) space E and* $T : K \to K$ *a generalized* α*-nonexpansive mapping. Then F(T) is closed. Moreover, if E is strictly convex and K is convex, then F(T) is also convex.*

Proof. Let $\{z_n\}$ be a sequence in F(T) such that $\{z_n\}$ converges to a point $z \in K$.

Since $\frac{1}{2}d(z_n,Tz_n) = 0 \le d(z_n,z)$

By definition of generalized α-nonexpansive mapping and continuity of metric on E, we have

$lim_{n\to\infty}d(z_n,Tz) = lim_{n\to\infty}d(Tz_n,Tz)$

$\le lim_{n\to\infty}[\alpha d(Tz_n,z) + \alpha d(Tz,z_n) + (1-2\alpha)d(z_n,z)]$

$= \alpha d(Tz,z_n) + (1-\alpha)d(z_n,z)]$

$= lim_{n\to\infty}d(z_n,Tz) \le \alpha lim_{n\to\infty}d(Tz,z_n) + (1-\alpha)lim_{n\to\infty}d(z_n,z)$.

Since $(1-\alpha) > 0$, the above inequality reduces to

$lim_{n\to\infty}d(z_n,Tz) \le lim_{n\to\infty}d(z_n,z)$,

and $Tz = z$. Therefore F(T) is closed.

Next, we assume that E is strictly convex and K is convex. Fix $\lambda \in (0,1)$ and $x,y \in F(T)$ with $x \neq y$,

Put $z = \lambda x + (1-\lambda)y \in K$. (due to convexity of K)

Since $\frac{1}{2}d(x,Tx) = 0 \le d(x,z)$

Also, we have

$$d(Tx,Tz) \leq \alpha d(Tx,z) + \alpha d(Tz,x) + (1-2\alpha)d(x,z)$$
$$= \alpha d(x,z) + \alpha d(Tz,x) + (1-2\alpha)d(x,z)$$
$$= \alpha d(Tz,x) + (1-\alpha)d(x,z)$$

This implies that

$$(1-\alpha)d(Tx,Tz) \leq (1-\alpha)d(x,z)$$

Since $(1-\alpha) > 0$, we get $\quad d(Tx,Tz) \leq d(x,z)$

By a similar argument, we have $\quad d(Ty,Tz) \leq d(y,z)$

Therefore

$$d(x,y) \leq d(x,Tz) + d(Tz,y) = d(Tx,Tz) + d(Tz,Ty)$$
$$\leq d(x,z) + d(z,y) \leq d(x,y)$$

From strict convexity of E, there exists $\mu \in [0,1]$ such that $\quad Tz = \mu x + (1-\mu)y$

Now $\quad (1-\mu)d(x,y) = d(Tx,Tz)$

$$\leq d(x,z)$$
$$\leq (1-\lambda)d(x,y)$$

and $\quad \mu d(x,y) = \lambda d(x,y)$

Also $x \neq y$ implies that $d(x,y) \neq 0$

$\Rightarrow \lambda = \mu$ and hence $z = Tz$

Therefore $z \in F(T)$. □

4. Existence Results

In this section, we present some existence theorems for generalized α-nonexpansive mappings.

Theorem 4.1. *Let K be a nonempty closed convex subset of a complete CAT(0) space E. Let $T: K \to K$ be a generalized α-nonexpansive mapping. Then, $F(T) \neq \phi$ if and only if $\{T^n x\}$ is a bounded sequence for some $x \in K$.*

Proof. Suppose that $\{T^n x\}$ is a bounded sequence for some $x \in K$. Define $\{x_n\} = \{T^n x\}$ for all $n \in \mathbb{N}$. Then there exists a unique $z \in K$ such that $A(K,\{x_n\}) = \{z\}$. Now, we show that $\{d(x_n,x_{n+1})\}$ is a nonincreasing sequence.

Since $\frac{1}{2}d(x_n,Tx_n)=\frac{1}{2}d(x_n,x_{n+1})\le d(x_n,x_{n+1})$,

As T is generalized α-nonexpansive mapping, we get

$$\begin{aligned}d(x_{n+1},x_{n+2})=d(Tx_n,Tx_{n+1})&\le \alpha d(Tx_n,x_{n+1})+d(x_n,Tx_{n+1})+(1-2\alpha)d(x_n,x_{n+1})\\&\le \alpha d(x_n,x_{n+2})+(1-2\alpha)d(x_n,x_{n+1})\\&\le \alpha d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+(1-2\alpha)d(x_n,x_{n+1}).\end{aligned}$$

This implies that

$$d(x_{n+1},x_{n+2})\le d(x_n,x_{n+1}) \tag{4.1}$$

Now for all $n\in\mathbb{N}$, we claim that

$$\text{either } d(x_n,x_{n+1})\le 2d(x_n,z) \text{ or } d(x_{n+1},x_{n+2})\le 2d(x_{n+1},z)$$

Arguing by contradiction, we suppose that for some $n\in\mathbb{N}$,

$$2d(x_n,z)<d(x_n,x_{n+1}) \text{ and } 2d(x_{n+1},z)<d(x_{n+1},x_{n+2}).$$

By the triangle inequality and (4.1),

$$\begin{aligned}d(x_n,x_{n+1})&\le d(x_n,z)+d(x_{n+1},z)\\&<\frac{1}{2}d(x_n,x_{n+1})+\frac{1}{2}d(x_{n+1},x_{n+2})\\&\le\frac{1}{2}[d(x_n,x_{n+1})+d(x_n,x_{n+1})]\\&=d(x_n,x_{n+1}),\end{aligned}$$

which is a contradiction. Thus for all $n\in\mathbb{N}$,

either $\frac{1}{2}d(x_n,x_{n+1})\le d(x_n,z)$ or $\frac{1}{2}d(x_{n+1},x_{n+2})\le d(x_{n+1},z)$.

In the first case, $\frac{1}{2}d(x_n,x_{n+1})=\frac{1}{2}d(x_n,Tx_n)\le d(x_n,z)$,

Also, we have

$$d(Tx_n,Tz)\le \alpha d(Tx_n,z)+\alpha d(x_n,Tz)+(1-2\alpha)d(x_n,z).$$

This implies that

$$\limsup_{n\to\infty} d(Tx_n,Tz)\le \alpha\limsup_{n\to\infty} d(Tx_n,z)+\alpha\limsup_{n\to\infty} d(x_n,Tz)+(1-2\alpha)\limsup_{n\to\infty} d(x_n,z)$$

Thus,

$$\limsup_{n\to\infty} d(x_n,Tz)\le\limsup_{n\to\infty} d(x_n,z).$$

Consequently, $Tz \in A(K,\{x_n\})$, ensuring that $Tz = z$. Similarly, in the second case, we can deduce that $Tz = z$. Conversely, suppose that $F(T) \neq \phi$. So, there exists some $w \in F(T)$ and $T^n w = w$ for all $n \in \mathbb{N}$. Therefore, $\{T^n w\}$ is bounded. □

Theorem 4.2. *Let K be a compact subset of a complete CAT(0) space E and S a family of commutative generalized α-nonexpansive mappings on K. Then S has a common fixed point.*

Proof. It may be completed following the proof of Theorem 4.[20] □

5. Convergence Results

Lemma 5.1. [1] *Let K be a nonempty subset of a CAT(0) space E and $T : K \to K$ a generalized α-nonexpansive mapping. Then, for all $x,y \in K$:*

(1) $d(Tx,T^2x) \leq d(x,Tx)$.
(2) Either $\frac{1}{2}d(x,Tx) \leq d(x,y)$ *or* $\frac{1}{2}d(Tx,T^2x) \leq d(Tx,y)$.
(3) Either $d(Tx,Ty) \leq \alpha d(Tx,y) + \alpha d(x,Ty) + (1-2\alpha)d(x,y)$ *or*
$d(T^2x,Ty) \leq \alpha d(Tx,Ty) + \alpha d(T^2x,y) + (1-2\alpha)d(Tx,y)$.

Lemma 5.2. [1] *Let K be a nonempty subset of a CAT(0) space E and $T : K \to K$ a generalized α-nonexpansive mapping. Then, for all $x,y \in K$*

$$d(x,Ty) \leq \frac{(3+\alpha)}{(1-\alpha)}d(x,Tx) + d(x,y)$$

The demiclosed principle plays an important role in reading the asymptotic behavior for nonexpansive mappings. In [5], Xu proved the demiclosed principle for asymptotically nonexpansive mappings in uniformly convex Banach space. we are now demonstrating the demiclosed principle for generalized α-nonexpansive mappings in CAT(0) spaces that extends the result of Xu to CAT(0) spaces.

Proposition 5.3. *(Demiclosedness principle)*

Let K be a nonempty subset of a complete CAT(0) space E with the opial property and $T : K \to K$ a generalized α-nonexpansive mapping. If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty} d(Tx_n,x_n) = 0$*, then T(z) = z. i.e. (I-T) is demiclosed at 0.*

Proof. By lemma 5.2, for all $n \in \mathbb{N}$, we have

$$d(x_n, Tz) \leq \frac{(3+\alpha)}{(1-\alpha)} d(x_n, Tx_n) + d(x_n, z)$$

This implies that

$$\liminf_{n\to\infty} d(x_n, Tz) \leq \liminf_{n\to\infty} d(x_n, z)$$

By the opial property, we get T(z) = z. □

Proposition 5.4. *Let K be a nonempty closed and convex subset of a complete CAT(0) space E and $T : K \to K$ a generalized α-nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in K such that $lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - lim_{n\to\infty} x_n = w$. Then Tw = w.*

Lemma 5.5. *Let K be a nonempty closed and convex subset of a complete CAT(0) space E and $T : K \to K$ a generalized α-nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by (1). Let $z \in F(T)$, then the following assertions hold:*

(1) $max\{d(x_{n+1}, z), d(y_n, z), d(z_n, z), d(w_n, z)\} \leq d(x_n, z)$.

(2) $lim_{n\to\infty} d(x_n, z)$ exists.

Proof. By (1) and proposition (3.4)

$$\begin{aligned} d(w_n, z) &= d((1-\gamma_n)x_n \bigoplus \gamma Tx_n, z) \\ &\leq (1-\gamma_n)d(x_n, z) + \gamma_n d(Tx_n, z) \\ &\leq (1-\gamma_n)d(x_n, z) + \gamma_n d(x_n, z) \\ &= d(x_n, z). \end{aligned}$$

Now,
$$\begin{aligned} d(z_n, z) &= d((1-\beta_n)Tx_n \bigoplus \beta_n Tw_n, z) \\ &\leq (1-\beta_n)d(Tx_n, z) + \beta_n d(Tw_n, z) \\ &\leq (1-\beta_n)d(x_n, z) + \beta_n d(w_n, z) \\ &\leq (1-\beta_n)d(x_n, z) + \beta_n d(x_n, z) \\ &= d(x_n, z). \end{aligned}$$

Again,
$$\begin{aligned} d(y_n, p) &= d(T((1-\alpha_n)Tx_n \bigoplus \alpha_n Tz_n), z) \\ &\leq d((1-\alpha_n)Tx_n \bigoplus \alpha_n Tz_n, z) \\ &\leq (1-\alpha_n)d(x_n, z) + \alpha_n d(z_n, z) \\ &\leq (1-\alpha_n)d(x_n, z) + \alpha_n d(x_n, z) \\ &= d(x_n, z) \end{aligned}$$

$$d(y_n,p) \le d(x_n,z).$$

$$d(x_{n+1},z) = d(Ty_n,z) \le d(y_n,z) \le d(x_n,z)$$

Thus, the sequence $\{d(x_n,z)\}$ is nonincreasing and bounded. Hence, $lim_{n\to\infty} d(x_n,z)$ exists. □

Lemma 5.6. [21] *Suppose that E is a complete CAT(0) space and $x \in E$. $\{t_n\}$ is a sequence in [b,c] for some $b,c \in (0,1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in E such that, for some $r \ge 0$, we have*

$$\lim_{n\to\infty} supd(x_n,x) \le r, \lim_{n\to\infty} supd(y_n,x) \le r$$

and

$$\lim_{n\to\infty} supd(t_n x_n \bigoplus (1-t_n)y_n,x) \le r;$$

then

$$\lim_{n\to\infty} d(x_n,y_n) = 0$$

Theorem 5.7. *Let K be a nonempty closed and convex subset of a complete CAT(0) space E and $T: K \to K$ a generalized α-nonexpansive mapping. Let $\{x_n\}$ be a sequence with $x_1 \in K$ defined by (1). Then $F(T) \ne \phi$ iff $\{x_n\}$ is bounded and $lim_{n\to\infty} d(Tx_n,x_n) = 0$.*

Proof. Suppose $\{x_n\}$ is a bounded sequence and $lim_{n\to\infty} d(Tx_n,x_n) = 0$. Since E is complete CAT(0) space, $A(K,\{x_n\}) \ne \phi$. Let $z \in A(K,\{x_n\})$.

By definition of asymptotic radius, we have

$$r(Tz,\{x_n\}) = lim_{n\to\infty} supd(x_n,Tz).$$

Using lemma 5.2, we get

$$\begin{aligned} r(Tz,\{x_n\}) &\le \tfrac{(3+\alpha)}{(1-\alpha)} \limsup_{n\to\infty} d(Tx_n,x_n) + \limsup_{n\to\infty} d(x_n,z) \\ &= r(z,\{x_n\}) \end{aligned}$$

By the uniqueness of asymptotic center of $\{x_n\}$, we have Tz=z.

Conversely, let $F(T) \ne \phi$ and $z \in F(T)$. Then from above lemma $lim_{n\to\infty} d(x_n,z)$ exists.

Suppose $\quad lim_{n\to\infty} d(x_n,z) = r$

Also $\quad \limsup_{n\to\infty} d(y_n,z) \le r, \quad \limsup_{n\to\infty} d(z_n,z) \le r$

and $\quad \limsup_{n\to\infty} d(w_n,z) \le r$

By proposition (3.2), we get

$$\limsup_{n\to\infty} d(Tx_n,z) \le \limsup_{n\to\infty} d(x_n,z) = r,$$

$$\limsup_{n\to\infty} d(Ty_n,z) \le r,\ \limsup_{n\to\infty} d(Tz_n,z) \le r$$

and $$\limsup_{n\to\infty} d(Tw_n,z) \le r$$

Now

$$r = \lim_{n\to\infty} d(x_{n+1},z) = d(Ty_n,z)$$

$$\Rightarrow \qquad \lim_{n\to\infty} d(Ty_n,z) = r$$

As $$\lim_{n\to\infty} d(Ty_n,z) = r \le \lim_{n\to\infty} d(y_n,z)$$

This implies that $$\lim_{n\to\infty} d(y_n,z) = r$$

By (1), we have

$$r = \lim_{n\to\infty} d(y_n,z) = \lim_{n\to\infty} d(T((1-\alpha_n)Tx_n \bigoplus \alpha_n Tz_n),z)$$

In view of Lemma (5.6), we get

$$\lim_{n\to\infty} d(Tx_n,Tz_n) = 0$$

Now, $$r = d(Ty_n,z) \le d(Ty_n,Tx_n) + d(Tx_n,z)$$
$$\le d(Ty_n,Tx_n) + r$$

So, we get

$$\lim_{n\to\infty} d(Ty_n,Tx_n).$$

Again, $$r = d(Ty_n,z) \le d(Ty_n,Tx_n) + d(Tx_n,Tz_n) + d(Tz_n,z)$$
$$\le 0 + 0 + d(Tz_n,z)$$

hence, $$r \le \liminf_{n\to\infty} d(Tz_n,z)$$

So, we have

$$r = \lim_{n\to\infty} d(Tz_n,z) = \lim_{n\to\infty} d((1-\beta_n)Tx_n \bigoplus \beta_n Tw_n,z)$$

Again in view of Lemma 5.6, we have

$$\lim_{n\to\infty} d(Tx_n,Tw_n) = 0$$

Also, we observe that,

$$d(Ty_n,z) \le d(Ty_n,Tx_n) + d(Tx_n,Tw_n) + d(Tw_n,z)$$

Letting $n\to\infty$, we get

$$r \le \liminf_{n\to\infty} d(Tw_n,z)$$

$$\Rightarrow \qquad \lim_{n\to\infty} d(Tw_n,z) = r$$

and $r = \lim_{n\to\infty} d(Tw_n,z) \le \liminf_{n\to\infty} d(w_n,z)$

This implies that

$$\lim_{n\to\infty} d(w_n, z) = r$$

Also, by (1)

$$r = \lim_{n\to\infty} d(w_n, z) = d((1-\gamma_n)x_n \bigoplus \gamma T x_n, z)$$

Now, again in view of Lemma 5.6, we have the required result i.e.

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0.$$

□

Theorem 5.8. *Let K be a nonempty closed and convex subset of a complete CAT(0) space E and $T : K \to K$ a generalized α-nonexpansive mapping with $F(T) \neq \phi$. Let $\{x_n\}$ be a sequence with $x_1 \in K$ defined by (1). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$*.*

Proof. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} d(u_n, F(T)) = 0$.

Let $\{u_{n_j}\}$ be a subsequence of sequence $\{u_n\}$ such that $d(u_{n_j}, p_j) \leq \frac{1}{2^j}$ for all $j \geq 1$, where $\{p_j\}$ is a sequence in F(T). By Lemma 5.5, we have

$$d(u_{n_{j+1}}, p_j) \leq d(u_{n_j}, p_j) \leq \frac{1}{2^j}.$$

By triangle inequality, we conclude that

$$\begin{aligned} d(p_{j+1}, p_j) &\leq d(p_{j+1}, u_{n_{j+1}}) + d(u_{n_{j+1}}, p_j) \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}} \end{aligned}$$

A standard argument shows that $\{p_j\}$ is a cauchy sequence in F(T). By Lemma 3.5, F(T) is closed, so $\{p_j\}$ converges to some $p \in F(T)$.

Now, by triangle inequality, we have

$$d(u_{n_j}, p) \leq d(u_{n_j}, p_j) + d(p_j, p).$$

Letting $j \to \infty$ implies that p_{n_j} converges strongly to p. By Lemma 5.5, $lim_{n\to\infty} d(x_n, p)$ exists so the sequence $\{x_n\}$ converges strongly to p. □

6. Numerical Results

Example 6.1. *Let the set $K = [0,\infty]$ be equipped with the euclidean norm and $T : K \to K$ be defined as:*

$$Tx = \begin{cases} \frac{x}{2}, \; if\, x > 2, \\ 0, \; if\, x = [0,2]. \end{cases}$$

Then,

(1) T does not satisfy condition(C);

(2) T is a generalized α-nonexpansive mapping.

Proof. (1) For $x = \frac{5}{2}$ and $y = \frac{9}{2}$, we have
$\frac{1}{2}d(x,Tx) = \frac{1}{2}d(\frac{5}{2},0) = \frac{5}{4} < |x-y| = 2$ but $d(Tx,Ty) = |Tx-Ty| = \frac{9}{4} > |x-y| = 2$. Hence, T does not satisfy condition(C).

(2) With $\alpha = \frac{1}{2}$, we consider the following different modes.

Case(I). Let $x > 2$ and $0 \leq y \leq 2$, we have

$$\begin{aligned} \frac{1}{2}d(Tx,y) + \frac{1}{2}d(x,Ty) &= \frac{1}{2}|Tx-y| + \frac{1}{2}|x-Ty| \\ &= \frac{1}{2}|\frac{x}{2}-y| + \frac{1}{2}|x| \\ &\geq \frac{1}{2}|x| \\ &= |Tx-Ty| = d(Tx,Ty). \end{aligned}$$

Case(II). Let $x > 2$ and $y > 2$, we have

$$\begin{aligned} \frac{1}{2}d(Tx,y) + \frac{1}{2}d(x,Ty) &= \frac{1}{2}|Tx-y| + \frac{1}{2}|x-Ty| \\ &= \frac{1}{2}|\frac{x}{2}-y| + \frac{1}{2}|x-\frac{y}{2}| \\ &\geq \frac{1}{2}|\frac{x}{2}-\frac{y}{2}+x-y| \\ &\geq \frac{1}{2}|\frac{3}{2}x-\frac{3}{2}y| \\ &\geq \frac{1}{2}|x-y| \\ &= |Tx-Ty| = d(Tx,Ty). \end{aligned}$$

Case(III). Let $0 \leq x \leq 2$ and $0 \leq y \leq 2$, then it is obvious that

$$\frac{1}{2}d(Tx,y) + \frac{1}{2}d(x,Ty) \geq d(Tx,Ty) = 0.$$

Hence, we conclude that T is a generalized $\frac{1}{2}$-nonexpansive mapping. □

Example 6.2. [11] *Let the set $K = [-1, 1]$ be equipped with the euclidean norm and $T : K \to K$ defined as:*

$$Tx = \begin{cases} \frac{x}{2}, \ if\, x \in [-1, 0) = A, \\ -x, \ if\, x \in [0, 1]/\{\frac{1}{2}\} = B, \\ 0, \ if\, x = \frac{1}{2}. \end{cases}$$

Then

(1) T does not satisfy condition(C);

(2) T is a generalized α-nonexpansive mapping.

Karahan and Ozdemir[6] introduced a new iteration process, namely the S^* iteration process and they also used a numerical example to demonstrate that the S^* iteration process is faster than the Picard, Mann and S-iterative processes for contractions. To show efficiency of Picard S^* iteration process for generalized α-nonexpansive mapping, we use example 6.2 with parametrs $\alpha_n = \frac{1}{\sqrt{n+1}}, \beta_n = \sqrt{\frac{n}{n+1}}, \gamma_n = \frac{1}{n+1}$ and with the initial value $x_1 = -0.6$. Set the stop parameter to $d(x_n, x^*) < 10^{-15}$ for Ishikawa, S, S^* and Picard S^* iteration processes. All iterations converge to the fixed point 0. The convergence behavior of these algorithms is shown in Figure (1). Table 1 and graphical representation given below show that our iterative process (1) converges faster than all of Ishikawa, S, and S^* processes.

TABLE 1. Sequences generated by Picard S^* , S^*, S and Ishikawa iteration processes.

n	Picard S^* iteration	S^*	S iteration	Ishikawa iteration
1	-0.6	-0.6	-0.6	-0.6
2	-0.075	-0.15	-0.225	-0.312867966
3	-0.010948999	-0.034420996	-0.085983496	-0.14826831
4	-0.001839544	-0.008861794	-0.033683761	-0.068015733
5	-0.000332469	-0.00248377	-0.013473505	-0.030796599
6	-6.28692E-05	-0.000740652	-0.005481433	-0.013870635
7	-1.2265E-05	-0.000231478	-0.002261191	-0.006236351
8	-2.44846E-06	-7.50626E-05	-0.000943641	-0.002803945
9	-4.9758E-07	-2.50784E-05	-0.000397681	-0.001261854
10	-1.02577E-07	-8.58845E-06	-0.000169014	-0.000568665
11	-2.13971E-08	-3.00332E-06	-7.23601E-05	-0.000256695
12	-4.5079E-09	-1.06924E-06	-3.11803E-05	-0.000116075
13	-9.57821E-10	-3.86652E-07	-1.3513E-05	-5.25817E-05
14	-2.05022E-10	-1.41749E-07	-5.88646E-06	-2.38617E-05
15	-4.41704E-11	-5.26031E-08	-2.57614E-06	-1.08474E-05
16	-9.5711E-12	-1.9735E-08	-1.13218E-06	-4.93948E-06
17	-2.08463E-12	-7.47711E-09	-4.99489E-07	-2.25292E-06
18	-4.56154E-13	-2.8583E-09	-2.21141E-07	-1.02918E-06
19	-1.00236E-13	-1.10159E-09	-9.82256E-08	-4.70864E-07
20	-2.21109E-14	-4.27736E-10	-4.37608E-08	-2.15738E-07
21	**-4.89466E-15**	-1.67234E-10	-1.95506E-08	-9.89826E-08
22	-1.08705E-15	-6.5802E-11	-8.75721E-09	-4.54747E-08
31	.	.	.	.
32	-3.63815E-22	**-7.67289E-15**	-3.18789E-12	-2.0277E-11
33	-8.27909E-23	-3.16881E-15	-1.45733E-12	-9.42433E-12
34	.	.	.	.
38	-5.1639E-26	-3.97815E-17	-2.96561E-14	-2.06873E-13
40	-2.72209E-27	-7.03269E-18	**-6.29507E-15**	-4.51355E-14

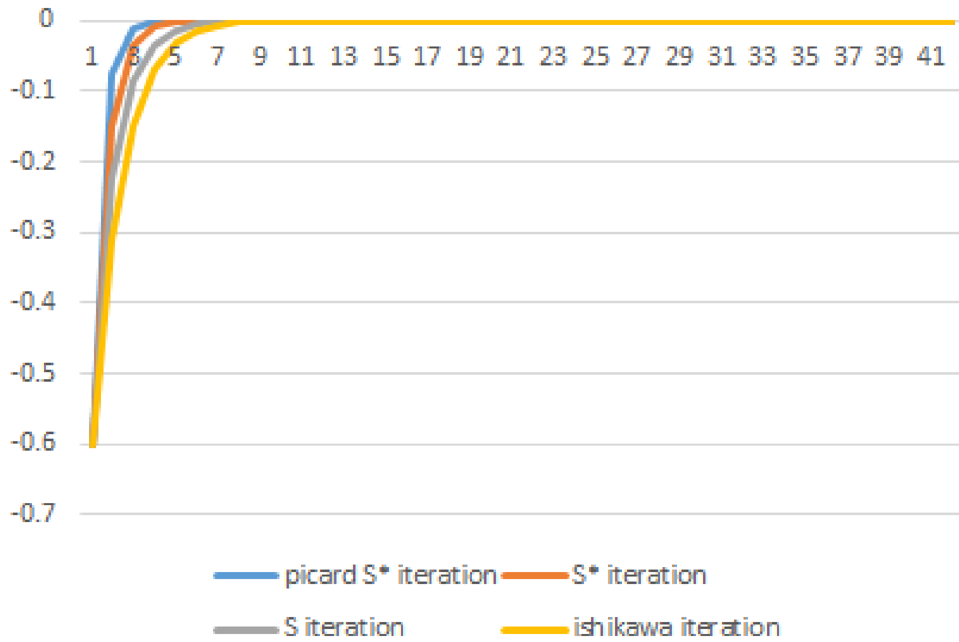

FIGURE 1. Convergence of Picard S^* , S and Ishikawa iterations to the fixed point 0.

ACKNOWLEDGEMENTS

The authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript. The first author is thankful to University Grants Commission for financial assistance.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A. Panwar, P. Lamba, Approximating fixed points of generalized α-nonexpansive mappings in CAT(0) spaces, Adv. Appl. Math. Sci. 19(9) (2020), 907-915.

[2] A. Panwar, Reena, Approximation of fixed points of a multivalued ρ-quasi-nonexpansive mapping for newly defined hybrid iterative scheme, J. Interdiscip. Math. 22(4) (2019), 593-607.

[3] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry, Graduate Studies in Mathematics, American Mathematical Society, Providence, 2001.

[4] F. Bruhat, J. Tits, Groupes Réductifs Sur Un Corps Local, Publications Mathématiques de l'Institut des Hautes Études Scientifiques. 41 (1972), 5–251.

[5] H.K. Xu, Existence and convergence for fixed points of mappings of asymptotically nonexpansive type, Nonlinear Anal., Theory Meth. Appl. 16(12) (1991), 1139–1146.

[6] I. Karahan, M.A. Ozdemir, General iterative method for approximation of fixed points and their applications, Adv. Fixed. Point. Theory. 3(3) (2013), 510-526.

[7] K. Aoyama, F. Kohsaka, Fixed point theorem for α-nonexpansive mappings in Banach spaces, Nonlinear Anal., Theory Meth. Appl. 74 (2011), 4387–4391.

[8] K.S. Brown, Buildings, Springer, New York, 1989.

[9] M.R. Bridson, A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, 1999.

[10] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically non-expansive mappings, J. Nonlinear Convex Anal. 8(1) (2007), 61–79.

[11] R. Pant, R. Shukla, Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces, Numer. Funct. Anal. Optim. 38 (2017), 248–266.

[12] S. Dhompongsa, W.A. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, J. Nonlinear Convex Anal. 8(1) (2007), 35–45.

[13] S. Dhompongsa, W.A. Kirk, B. Sims, Fixed points of uniformly Lipschitzian mappings, Nonlinear Anal., Theory Meth. Appl. 65(4) (2006), 762–772.

[14] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974), 147-150.

[15] T.C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc. 60 (1976), 179–182.

[16] T. Suzuki, Fixed points theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008), 1088-1095.

[17] W.A. Kirk, Geodesic geometry and fixed point theory. Colecc. Abierta. In Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Univ. Sevilla Secr. Publ., Seville; 64 (2003), 195–225.

[18] W.A. Kirk, Geodesic geometry and fixed point theory II, In International Conference on Fixed point Theory and Apllications, Yokohama Publishers, Yokohama, Japan, (2004), 113-142.

[19] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal., Theory Meth. Appl. 68(12) (2008), 3689–3696.

[20] W.G. Dotson, Fixed points of quasi-nonexpansive mappings, J. Aust. Math. Soc. 13(2) (1972), 167-170.

[21] W. Laowang, B. Panyanak, Approximating fixed points of nonexpansive nonself mappings in CAT(0) spaces, Fixed Point Theory Appl. 2010 (2010), 367274.

[22] W.R. Mann, Mean value methods in Iteration, Proc. Amer. Math. Soc. 4 (1953), 506-510.