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# ON GENERALIZED INTERVAL VALUED BIPOLAR FUZZY IDEALS IN ORDERED SEMIGROUPS

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**Abstract.** We introduce the notion of interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy left (resp., right, bi-) ideals in ordered semigroups. This concept is a generalized concept of interval valued bipolar fuzzy left (resp., right, bi-) ideals in ordered semigroups. We discuss some basic properties of our initialed notions. Furthermore, we characterize regular and intra-regular ordered semigroups in terms of such concepts.

**Keywords:** ordered semigroup; interval valued fuzzy left ideal; interval valued fuzzy right ideal; interval valued fuzzy bi-ideal.

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## 1. INTRODUCTION

L. A. Zadeh first introduced the notion of fuzzy sets in his definitive paper in 1965 that can be applied in many fields, including mathematics, statistics, computer science, electrical instruments, industrial operations, business, engineering, social decisions (see [25]). The concept of fuzzy semigroups has been first considered by Kuroki [14, 15, 16] and fuzzy ordered semigroups by Kehayopulu and Tsingelis [5, 6, 7]. Bhakat and Das [2] gave the concepts of $(\alpha,\beta)$-fuzzy

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subgroups by using the "belongs to" relation $(\varepsilon)$ and "quasi-coincident with" relation $(q)$ between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\varepsilon, \varepsilon \vee q)$-fuzzy subgroup. In [4] Jun and Song initiated the study of $(\alpha, \beta)$-fuzzy interior ideals of a semigroup. Shabir et al., [20] characterized regular semigroups by the properties of $(\alpha, \beta)$-fuzzy ideals, bi-ideals, and quasi-ideals. Shabir et al. [21] characterized regular and intra-regular semigroups by the properties of $(\varepsilon, \varepsilon \vee q_k)$-fuzzy left (right, quasi-, two-sided) ideals. Khan and Anis [13] characterized right weakly regular semigroups by the properties of their $(\varepsilon, \varepsilon \vee q_k)$-fuzzy ideals.

There are several kinds of extensions of fuzzy sets, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets. The notion of bipolar valued fuzzy sets is one of the extensions of fuzzy sets whose membership degree range is different from the above extensions. Lee [17] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,0]$. In a bipolar-valued fuzzy set, the membership degree 0 indicates that elements are irrelevant to the related property, the membership degrees on $(0,1]$ assign that elements somewhat satisfy the property, and the membership degree on $[-1,0)$ assign that elements somewhat satisfy the implicit counter-property [18].

N. Yaqoob et al. [24] initiated a study on bipolar $(\lambda, \delta)$-fuzzy sets in $\Gamma$-semihypergroups. They introduced the notion of bipolar $(\lambda, \delta)$-fuzzy sub-$\Gamma$-semihypergroups ($\Gamma$-hyperideals and bi-$\Gamma$-hyperideals) and discussed some fundamental results on bipolar $(\lambda, \delta)$-fuzzy sets in $\Gamma$-semihypergroups. P. Khamrot and M. Siripitukdet [12] introduced a generalization of a bipolar fuzzy subsemigroup, namely a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy subsemigroup. The notions of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy left (right, bi-) ideals are discussed. Furthermore, they also characterized regular semigroups in terms of generalized bipolar fuzzy semigroups. K. Arulmozhi et al. [1] introduced the notions of interval valued $(\widetilde{\eta}, \widetilde{\delta})$-bipolar fuzzy ideal, bi-ideal, interior ideal, $(\varepsilon, \varepsilon \vee q)$-bipolar fuzzy ideal of ordered $\Gamma$-semigroups and discussed some interesting properties.

In this paper, we introduce a generalization of interval valued bipolar fuzzy ideals in ordered semigroups. The notions of interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy left (right, bi-) ideals are discussed. Moreover, we characterize regular and intra-regular ordered semigroups using interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy left (right, bi-) ideals.

## 2. Preliminaries

A structure $(S;\circ,\leq)$ is called an *ordered semigroup* if $(S;\circ)$ is a semigroup, $(S;\leq)$ is a partially ordered set, and for any $a,b\in S$, $a\leq b$ implies $ac\leq bc$ and $ca\leq cb$ for all $c\in S$.

Throughout this paper, unless stated otherwise, $S$ always stands for an ordered semigroup.

A fuzzy subset of $S$ is a function from $S$ to the real closed interval $[0,1]$.

Now we will recall the concept of interval valued fuzzy sets. An interval number is $\widetilde{a}=[a^-,a^+]$, where $0\leq a^-\leq a^+\leq 1$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$. That is,

$$D[0,1]=\{\widetilde{a}=[a^-,a^+]:0\leq a^-\leq a^+\leq 1\}.$$

We define binary operations $\leq$, $<$, $\cap$ and $\cup$ on $D[0,1]$ as follows. Let $\widetilde{a}=[a^-,a^+]$ and $\widetilde{b}=[b^-,b^+]$ be elements of $D[0,1]$. Then

(1) $\widetilde{a}\leq\widetilde{b}$ if and only if $a^-\leq b^-$ and $a^+\leq b^+$,

(2) $\widetilde{a}=\widetilde{b}$ if and only if $a^-=b^-$ and $a^+=b^+$,

(3) $\widetilde{a}<\widetilde{b}$ if and only if $a^-\leq b^-$ and $a^+\neq b^+$,

(4) $\widetilde{a}\cap\widetilde{b}=\min\{\widetilde{a},\widetilde{b}\}=[\min\{a^-,b^-\},\min\{a^+,b^+\}]$,

(5) $\widetilde{a}\cup\widetilde{b}=\max\{\widetilde{a},\widetilde{b}\}=[\max\{a^-,b^-\},\max\{a^+,b^+\}]$.

**Definition 2.1.** *Let $S$ be a non-empty set. An interval valued bipolar fuzzy set $\widetilde{A}$ on $S$ is an object of the form*

$$\widetilde{A}:=\{(x,\widetilde{\mu}^+_{\widetilde{A}}(x),\widetilde{\mu}^-_{\widetilde{A}}(x))\mid x\in S\},$$

*where $\widetilde{\mu}^+_{\widetilde{A}}:S\to D[0,1]$ and $\widetilde{\mu}^-_{\widetilde{A}}:S\to D[-1,0]$.*

For the sake of simplicity, we shall use the symbol $\widetilde{A}=(\widetilde{\mu}^+_{\widetilde{A}},\widetilde{\mu}^-_{\widetilde{A}})$ for the bipolar fuzzy set $\widetilde{A}:=\{(x,\widetilde{\mu}^+_{\widetilde{A}}(x),\widetilde{\mu}^-_{\widetilde{A}}(x))\mid x\in S\}$.

Interval valued bipolar fuzzy subsets $\widetilde{A} = \left(\widetilde{\mu}^{+}_{\widetilde{A}}, \widetilde{\mu}^{-}_{\widetilde{A}}\right)$ and $\widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{B}}, \widetilde{\mu}^{-}_{\widetilde{B}}\right)$ of an ordered semigroup $S$, we denote $\widetilde{A} \subseteq \widetilde{B}$ if and only if

(1) $\widetilde{\mu}^{+}_{\widetilde{A}}(x) \leq \widetilde{\mu}^{+}_{\widetilde{B}}(x)$
(2) $\widetilde{\mu}^{-}_{\widetilde{A}}(x) \geq \widetilde{\mu}^{-}_{\widetilde{B}}(x)$

for all $x \in S$.

**Definition 2.2.** *Let $\widetilde{A} = \left(\widetilde{\mu}^{+}_{\widetilde{A}}, \widetilde{\mu}^{-}_{\widetilde{A}}\right)$ and $\widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{B}}, \widetilde{\mu}^{-}_{\widetilde{B}}\right)$ be two interval valued bipolar fuzzy subsets of an ordered semigroup $S$. Then for all $x \in S$, their intersection $\widetilde{A} \cap \widetilde{B}$ is defined by*

$$\widetilde{A} \cap \widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{A}\cap\widetilde{B}}(x), \widetilde{\mu}^{-}_{\widetilde{A}\cap\widetilde{B}}(x)\right).$$

*where*

$$\widetilde{\mu}^{+}_{\widetilde{A}\cap\widetilde{B}}(x) = \min\{\widetilde{\mu}^{+}_{\widetilde{A}}(x), \widetilde{\mu}^{+}_{\widetilde{B}}(x)\} \text{ and } \widetilde{\mu}^{-}_{\widetilde{A}\cap\widetilde{B}}(x) = \min\{\widetilde{\mu}^{-}_{\widetilde{A}}(x), \widetilde{\mu}^{-}_{\widetilde{B}}(x)\},$$

*for all $x \in S$. Their union $\widetilde{A} \cup \widetilde{B}$ is defined by*

$$\widetilde{A} \cup \widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{A}\cup\widetilde{B}}(x), \widetilde{\mu}^{-}_{\widetilde{A}\cup\widetilde{B}}(x)\right).$$

*where*

$$\widetilde{\mu}^{+}_{\widetilde{A}\cup\widetilde{B}}(x) = \max\{\widetilde{\mu}^{+}_{\widetilde{A}}(x), \widetilde{\mu}^{+}_{\widetilde{B}}(x)\} \text{ and } \widetilde{\mu}^{-}_{\widetilde{A}\cup\widetilde{B}}(x) = \max\{\widetilde{\mu}^{-}_{\widetilde{A}}(x), \widetilde{\mu}^{-}_{\widetilde{B}}(x)\},$$

*for all $x \in S$.*

Let $a \in S$. We denote $A_a = \{(x,y) \in S \times S \mid a \leq xy\}$.

**Definition 2.3.** *Let $\widetilde{A} = \left(\widetilde{\mu}^{+}_{\widetilde{A}}, \widetilde{\mu}^{-}_{\widetilde{A}}\right)$ and $\widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{B}}, \widetilde{\mu}^{-}_{\widetilde{B}}\right)$ be two interval valued bipolar fuzzy subsets of an ordered semigroup $S$. Then for all $x \in S$, their product $\widetilde{A} \circ \widetilde{B}$ is defined by*

$$\widetilde{A} \circ \widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{A}\circ\widetilde{B}}(x), \widetilde{\mu}^{-}_{\widetilde{A}\circ\widetilde{B}}(x)\right)$$

*where*

$$\widetilde{\mu}^{+}_{\widetilde{A}\circ\widetilde{B}}(x) = \begin{cases} \bigvee_{x \leq yz} \{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(y), \widetilde{\mu}^{+}_{\widetilde{B}}(z)\} & \text{if } A_x \neq \emptyset \\ \widetilde{0} & \text{otherwise} \end{cases}$$

*and*

$$\widetilde{\mu}^{-}_{\widetilde{A}\circ\widetilde{B}}(x) = \begin{cases} \bigwedge_{x \leq yz} \{\max\{\widetilde{\mu}^{-}_{\widetilde{A}}(y), \widetilde{\mu}^{-}_{\widetilde{B}}(z)\} & \text{if } A_x \neq \emptyset \\ \widetilde{0} & \text{otherwise} \end{cases}$$

*for all* $x, y, z \in S$.

Let $A$ be a subset of $S$. The interval valued bipolar characteristic function $\widetilde{\chi}_A$ of $A$ is define by

$$\widetilde{\chi}_A = \left(\widetilde{\chi}_A^{+}(x), \widetilde{\chi}_A^{-}(x)\right)$$

for all $x \in S$, where

$$\widetilde{\chi}_{\widetilde{A}}^{+}(x) = \begin{cases} \widetilde{1} & \text{if } x \in A \\ \widetilde{0} & \text{otherwise} \end{cases}$$

and

$$\widetilde{\chi}_{\widetilde{A}}^{-}(x) = \begin{cases} -\widetilde{1} & \text{if } x \in A \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

Let $S$ be an ordered semigroup. Then, we define an interval valued bipolar fuzzy subset $\widetilde{S}$ of $S$ by

$$\widetilde{S} = \left(\widetilde{S}^{+}(x), \widetilde{S}^{-}(x)\right),$$

for all $x \in S$, where

$$\widetilde{S}^{+}(x) = \widetilde{\chi}_S^{+}(x) \text{ and } \widetilde{S}^{-}(x) = \widetilde{\chi}_S^{-}(x).$$

From now on, we let $\widetilde{\lambda}^{+}, \widetilde{\delta}^{+} \in D[0,1]$ be such that $\widetilde{0} \leq \widetilde{\lambda}^{+} < \widetilde{\delta}^{+} \leq \widetilde{1}$ and $\widetilde{\lambda}^{-}, \widetilde{\delta}^{-} \in D[-1,0]$ with $\widetilde{-1} \leq \widetilde{\delta}^{-} < \widetilde{\lambda}^{-} \leq \widetilde{0}$. Both $\widetilde{\lambda}, \widetilde{\delta}$ are arbitrary but fixed.

**Definition 2.4.** *An interval valued bipolar fuzzy subset* $\widetilde{A} = \left(\widetilde{\mu}_{\widetilde{A}}^{+}, \widetilde{\mu}_{\widetilde{A}}^{-}\right)$ *of an ordered semigroup* $S$ *is called an* interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy subsemigroup *of* $S$ *if*

$$\max\{\widetilde{\mu}_{\widetilde{A}}^{+}(xy), \widetilde{\lambda}^{+}\} \geq \min\{\widetilde{\mu}_{\widetilde{A}}^{+}(x), \widetilde{\mu}_{\widetilde{A}}^{+}(y), \widetilde{\delta}^{+}\}$$

*and*

$$\min\{\widetilde{\mu}_{\widetilde{A}}^{-}(xy), \widetilde{\lambda}^{-}\} \leq \max\{\widetilde{\mu}_{\widetilde{A}}^{-}(x), \widetilde{\mu}_{\widetilde{A}}^{-}(y), \widetilde{\delta}^{-}\},$$

*for all* $x, y \in S$.

**Example 2.5** ([1]). *Let $S = \{a_1, a_2, a_3, a_4\}$ and defined a binary operation $\circ$ on $S$ with the following Cayley table as follows:*

| $\circ$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $a_1$ | $a_1$ | $a_1$ |
| $a_2$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
| $a_3$ | $a_1$ | $a_3$ | $a_3$ | $a_3$ |
| $a_4$ | $a_1$ | $a_3$ | $a_3$ | $a_3$ |

*From the Cayley table, $S$ is a semigroup. We define a partial order relation $\leq$ on $S$ as follows:*

$$\leq = \{(a_1,a_2),(a_1,a_3),(a_1,a_4),(a_2,a_3),(a_2,a_4),(a_4,a_3)\} \cup \Delta_S,$$

*where $\Delta_S$ is an equality relation on $S$. We obtain that $S$ is an ordered semigroup. Define interval valued bipolar fuzzy subset*

$$\widetilde{A} = \left(\widetilde{\mu}^+_{\widetilde{A}}(x), \widetilde{\mu}^-_{\widetilde{A}}(x)\right),$$

*where*

$$\widetilde{\mu}^+_{\widetilde{A}}(x) = \begin{cases} [0.6,0.7] & \text{if } x = a_1 \\ [0.4,0.5] & \text{if } x = a_2 \\ [0.1,0.2] & \text{if } x = a_3 \\ [0.2,0.3] & \text{if } x = a_4, \end{cases}$$

*and*

$$\widetilde{\mu}^-_{\widetilde{A}}(x) = \begin{cases} [-0.9,-0.8] & \text{if } x = a_1 \\ [-0.7,-0.6] & \text{if } x = a_2 \\ [-0.3,-0.2] & \text{if } x = a_3 \\ [-0.6,-0.5] & \text{if } x = a_4. \end{cases}$$

*Then $\widetilde{A}$ is an interval valued bipolar $([0.5,0.6],[0.7,0.8])$-fuzzy subsemigroup of $S$.*

**Definition 2.6.** *An interval valued bipolar fuzzy subset $\widetilde{A} = \left(\widetilde{\mu}^+_{\widetilde{A}}, \widetilde{\mu}^-_{\widetilde{A}}\right)$ of an ordered semigroup $S$ is called an* interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy left ideal *of $S$ if for all $x, y \in S$:*

*(1)* $\max\{\widetilde{\mu}^+_{\widetilde{A}}(xy), \widetilde{\lambda}^+\} \geq \min\{\widetilde{\mu}^+_{\widetilde{A}}(y), \widetilde{\delta}^+\}$.

*(2) If $x \leq y$, then* $\max\{\widetilde{\mu}^+_{\widetilde{A}}(x), \widetilde{\lambda}^+\} \geq \min\{\widetilde{\mu}^+_{\widetilde{A}}(y), \widetilde{\delta}^+\}$.

*(3)* $\min\{\widetilde{\mu}^-_{\widetilde{A}}(xy), \widetilde{\lambda}^-\} \leq \max\{\widetilde{\mu}^-_{\widetilde{A}}(y), \widetilde{\delta}^-\}$.

*(4) If $x \le y$, then* $\min\{\widetilde{\mu}^-_{\widetilde{A}}(x), \widetilde{\lambda}^-\} \le \max\{\widetilde{\mu}^-_{\widetilde{A}}(y), \widetilde{\delta}^-\}$.

**Definition 2.7.** *An interval valued bipolar fuzzy subset* $\widetilde{A} = \left(\widetilde{\mu}^+_{\widetilde{A}}, \widetilde{\mu}^-_{\widetilde{A}}\right)$ *of an ordered semigroup $S$ is called an* interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy right ideal *of $S$ if for all $x, y \in S$:*

*(1)* $\max\{\widetilde{\mu}^+_{\widetilde{A}}(xy), \widetilde{\lambda}^+\} \ge \min\{\widetilde{\mu}^+_{\widetilde{A}}(x), \widetilde{\delta}^+\}$.

*(2) If $x \le y$, then* $\max\{\widetilde{\mu}^+_{\widetilde{A}}(x), \widetilde{\lambda}^+\} \ge \min\{\widetilde{\mu}^+_{\widetilde{A}}(y), \widetilde{\delta}^+\}$.

*(3)* $\min\{\widetilde{\mu}^-_{\widetilde{A}}(xy), \widetilde{\lambda}^-\} \le \max\{\widetilde{\mu}^-_{\widetilde{A}}(x), \widetilde{\delta}^-\}$.

*(4) If $x \le y$, then* $\min\{\widetilde{\mu}^-_{\widetilde{A}}(x), \widetilde{\lambda}^-\} \le \min\{\widetilde{\mu}^-_{\widetilde{A}}(x), \widetilde{\lambda}^-\} \le \max\{\widetilde{\mu}^-_{\widetilde{A}}(y), \widetilde{\delta}^-\}$.

An interval valued bipolar fuzzy subset $\widetilde{A} = \left(\widetilde{\mu}^+_{\widetilde{A}}, \widetilde{\mu}^-_{\widetilde{A}}\right)$ of an ordered semigroup $S$ is called an *interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy ideal* of $S$ if it is both an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy left ideal and an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy right ideal of $S$.

**Definition 2.8.** *An interval valued bipolar fuzzy subset* $\widetilde{A} = \left(\widetilde{\mu}^+_{\widetilde{A}}, \widetilde{\mu}^-_{\widetilde{A}}\right)$ *of an ordered semigroup $S$ is called an* interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideal *of $S$ if for all $x, y, z \in S$:*

*(1)* $\max\{\widetilde{\mu}^+_{\widetilde{A}}(xy), \widetilde{\lambda}^+\} \ge \min\{\widetilde{\mu}^+_{\widetilde{A}}(x), \widetilde{\mu}^+_{\widetilde{A}}(y), \widetilde{\delta}^+\}$.

*(2)* $\max\{\widetilde{\mu}^+_{\widetilde{A}}(xyz), \widetilde{\lambda}^+\} \ge \min\{\widetilde{\mu}^+_{\widetilde{A}}(x), \widetilde{\mu}^+_{\widetilde{A}}(z), \widetilde{\delta}^+\}$.

*(3) If $x \le y$, then* $\max\{\widetilde{\mu}^+_{\widetilde{A}}(x), \widetilde{\lambda}^+\} \ge \min\{\widetilde{\mu}^+_{\widetilde{A}}(y), \widetilde{\delta}^+\}$.

*(4)* $\min\{\widetilde{\mu}^-_{\widetilde{A}}(xy), \widetilde{\lambda}^-\} \le \max\{\widetilde{\mu}^-_{\widetilde{A}}(x), \widetilde{\mu}^-_{\widetilde{A}}(y), \widetilde{\delta}^-\}$.

*(5)* $\min\{\widetilde{\mu}^-_{\widetilde{A}}(xyz), \widetilde{\lambda}^-\} \le \max\{\widetilde{\mu}^-_{\widetilde{A}}(x), \widetilde{\mu}^-_{\widetilde{A}}(z), \widetilde{\delta}^-\}$.

*(6) If $x \le y$, then* $\min\{\widetilde{\mu}^-_{\widetilde{A}}(x), \widetilde{\lambda}^-\} \le \max\{\widetilde{\mu}^-_{\widetilde{A}}(y), \widetilde{\delta}^-\}$.

**Example 2.9** ([1]). *Let $S = \{a_1, a_2, a_3, a_4\}$ and defined a binary operation $\circ$ on $S$ with the following Cayley table as follows:*

| $\circ$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $a_1$ | $a_1$ | $a_1$ |
| $a_2$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
| $a_3$ | $a_1$ | $a_3$ | $a_3$ | $a_3$ |
| $a_4$ | $a_1$ | $a_3$ | $a_3$ | $a_3$ |

*From the Cayley table, $S$ is a semigroup. We define a partial order relation $\le$ on $S$ as follows:*

$$\le = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4), (a_4, a_3)\} \cup \Delta_S,$$

*where $\Delta_S$ is the equality relation on S. We obtain that S is an ordered semigroup. Define interval valued bipolar fuzzy subset*

$$\widetilde{A} = \left(\widetilde{\mu}^{+}_{\widetilde{A}}(x), \widetilde{\mu}^{-}_{\widetilde{A}}(x)\right),$$

*where*

$$\widetilde{\mu}^{+}_{\widetilde{A}}(x) = \begin{cases} [0.81,0.91] & \text{if } x = a_1 \\ [0.62,0.72] & \text{if } x = a_2 \\ [0.34,0.44] & \text{if } x = a_3 \\ [0.43,0.53] & \text{if } x = a_4, \end{cases}$$

*and*

$$\widetilde{\mu}^{-}_{\widetilde{A}}(x) = \begin{cases} [-0.85,-0.75] & \text{if } x = a_1 \\ [-0.65,-0.55] & \text{if } x = a_2 \\ [-0.30,-0.20] & \text{if } x = a_3 \\ [-0.50,-0.40] & \text{if } x = a_4. \end{cases}$$

*Then $\widetilde{A}$ is an interval valued bipolar $([0.70, 0.80], [0.60, 0.90])$-fuzzy bi-ideal of S.*

**Definition 2.10.** *Let $\widetilde{A} = \left(\widetilde{\mu}^{+}_{\widetilde{A}}, \widetilde{\mu}^{-}_{\widetilde{A}}\right)$ and $\widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{B}}, \widetilde{\mu}^{-}_{\widetilde{B}}\right)$ be two interval valued bipolar fuzzy subsets of an ordered semigroup S. Then we define:*

*(1) The polar fuzzy subset $\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x), \widetilde{\mu}^{-}_{\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x)\right)$, where*

$$\widetilde{\mu}^{+}_{\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x) = \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A} \circ \widetilde{B}}(x), \widetilde{\lambda}^{+}\}, \widetilde{\delta}^{+}\}$$

*and*

$$\widetilde{\mu}^{-}_{\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x) = \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A} \circ \widetilde{B}}(x), \widetilde{\lambda}^{-}\}, \widetilde{\delta}^{-}\},$$

*for all $x \in S$.*

*(2) $\widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x), \widetilde{\mu}^{-}_{\widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x)\right)$, where*

$$\widetilde{\mu}^{+}_{\widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x) = \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A} \cap \widetilde{B}}(x), \widetilde{\lambda}^{+}\}, \widetilde{\delta}^{+}\}$$

*and*

$$\widetilde{\mu}^{-}_{\widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x) = \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A} \cap \widetilde{B}}(x), \widetilde{\lambda}^{-}\}, \widetilde{\delta}^{-}\},$$

*for all $x \in S$.*

Let $A$ be a subset of $S$. The interval valued characteristic function of $A$ is defined by

$$(\widetilde{\chi}_A)^{\lambda}_{\delta} = \left((\widetilde{\chi}^+_A)^{\lambda}_{\delta}(x), (\widetilde{\chi}^-_A)^{\lambda}_{\delta}(x)\right)$$

for all $x \in S$, where

$$\left(\widetilde{\chi}^+_A\right)^{\lambda}_{\delta}(x) = \max\{\min\{(\widetilde{\chi}^+_A)(x), \widetilde{\lambda}^+\}, \widetilde{\delta}^+\},$$

and

$$\left(\widetilde{\chi}^-_A\right)^{\lambda}_{\delta}(x) = \min\{\max\{(\widetilde{\chi}^-_A)(x), \widetilde{\lambda}^-\}, \widetilde{\delta}^-\}.$$

## 3. Regular and Intra-Regular Ordered Semigroups

In this section, we characterize regular and intra-regular ordered semigroups in terms of interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy left ideals, interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy right ideals, and interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideals of ordered semigroups.

An ordered semigroup $S$ is *regular* if, for each element $a \in S$, there exists $x \in S$ such that $a \leq axa$.

**Lemma 3.1** ([23])**.** *An ordered semigroup $S$ is regular if and only if $B = (BSB]$ for any bi-ideal $B$ of $S$.*

**Lemma 3.2.** *If an interval valued bipolar fuzzy subset $\widetilde{A} = \left(\widetilde{\mu}^+_{\widetilde{A}}, \widetilde{\mu}^-_{\widetilde{A}}\right)$ of an ordered semigroup $S$ is an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideal of $S$, then $\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{S} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A} \subseteq \widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$.*

*Proof.* Let $a \in S$. If $A_a = \emptyset$, then

$$\begin{aligned}
\left(\widetilde{\mu}^+_{\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{S} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A}}\right)(a) &= \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}\circ\widetilde{S}\circ\widetilde{A}}(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&= \widetilde{\delta}^+ \\
&\leq \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&= \left(\widetilde{\mu}^+_{\widetilde{A}}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(a)
\end{aligned}$$

and

$$\left(\widetilde{\mu}^-_{\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}\right)(a) = \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}\circ\widetilde{S}\circ\widetilde{A}}(a), \widetilde{\delta}^-\}, \widetilde{\lambda}^-\}$$
$$= \widetilde{\delta}^-$$
$$\geq \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a), \widetilde{\delta}^-\}, \widetilde{\lambda}^-\}$$
$$= \left(\widetilde{\mu}^-_{\widetilde{A}}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(a).$$

Thus, $\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A} \subseteq \widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$. If $A_a \neq \emptyset$, then there exist $x,y,p,q \in S$ such that $(x,y) \in A_a$ and $(p,q) \in A_x$. That is, $a \leq xy$ and $x \leq pq$. Since $\widetilde{A}$ is an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideal of $S$, we have

$$\max\{\widetilde{\mu}^+_{\widetilde{A}}(pqy), \widetilde{\lambda}^+\} \geq \min\{\widetilde{\mu}^+_{\widetilde{A}}(p), \widetilde{\mu}^+_{\widetilde{A}}(y), \widetilde{\delta}^+\}$$

and

$$\min\{\widetilde{\mu}^-_{\widetilde{A}}(pqy), \widetilde{\lambda}^-\} \leq \max\{\widetilde{\mu}^-_{\widetilde{A}}(p), \widetilde{\mu}^-_{\widetilde{A}}(y), \widetilde{\delta}^-\}.$$

Thus,

$$\left(\widetilde{\mu}^+_{\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}\right)(a) = \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}\circ\widetilde{S}\circ\widetilde{A}}(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\}$$
$$= \max\{\min\{\bigvee_{(x,y)\in A_a}\{\min\{\widetilde{\mu}^+_{\widetilde{A}\circ\widetilde{S}}(x), \widetilde{\mu}^+_{\widetilde{A}}(y)\}\}, \widetilde{\delta}^+\}, \widetilde{\lambda}^+\}$$
$$= \max\{\min\{\bigvee_{(x,y)\in A_a}\bigvee_{(p,q)\in A_x}\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(p), \widetilde{\mu}^+_{\widetilde{S}}(q), \widetilde{\mu}^+_{\widetilde{A}}(y)\}\}, \widetilde{\delta}^+\}, \widetilde{\lambda}^+\}$$
$$= \max\{\min\{\bigvee_{(x,y)\in A_a}\bigvee_{(p,q)\in A_x}\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(p), \widetilde{\mu}^+_{\widetilde{A}}(y)\}\}, \widetilde{\delta}^+\}, \widetilde{\lambda}^+\}$$

$$= \max\{\min\{\bigvee_{(x,y)\in A_a}\bigvee_{(p,q)\in A_x}\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(p),\widetilde{\mu}^+_{\widetilde{A}}(y),\widetilde{\delta}^+\}\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}$$

$$\leq \max\{\min\{\bigvee_{(x,y)\in A_a}\bigvee_{(p,q)\in A_x}\{\max\{\widetilde{\mu}^+_{\widetilde{A}}(pqy),\widetilde{\lambda}^+\}\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}$$

$$= \max\{\max\{\bigvee_{(x,y)\in A_a}\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(pqy),\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\},\widetilde{\lambda}^+\}$$

$$\leq \max\{\min\{\max\{\widetilde{\mu}^+_{\widetilde{A}}(a),\widetilde{\lambda}^+\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}$$

$$= \max\{\max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\widetilde{\delta}^+\},\widetilde{\lambda}^+\},\widetilde{\lambda}^+\}$$

$$= \max\{\min\{\widetilde{\mu}^+(a),\widetilde{\delta}^+\},\widetilde{\lambda}^+\}$$

$$= \left(\widetilde{\mu}^+_{\widetilde{A}}\right)^{\lambda}_{\delta}(a).$$

On the other hand, we have that

$$\left(\widetilde{\mu}^-_{\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}\right)(a) = \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}\circ\widetilde{S}\circ\widetilde{A}}(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\}$$

$$= \min\{\max\{\bigwedge_{(x,y)\in A_a}\{\max\{\widetilde{\mu}^-_{\widetilde{A}\circ\widetilde{S}}(x),\widetilde{\mu}^-_{\widetilde{A}}(y)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}$$

$$= \min\{\max\{\bigwedge_{(x,y)\in A_a}\bigwedge_{(p,q)\in A_x}\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(p),\widetilde{\mu}^-_{\widetilde{S}}(q),\widetilde{\mu}^-_{\widetilde{A}}(y)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}$$

$$= \min\{\max\{\bigwedge_{(x,y)\in A_a}\bigwedge_{(p,q)\in A_x}\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(p),\widetilde{\mu}^-_{\widetilde{A}}(y)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}$$

$$= \min\{\max\{\bigwedge_{(x,y)\in A_a}\bigwedge_{(p,q)\in A_x}\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(p),\widetilde{\mu}^-_{\widetilde{A}}(y),\widetilde{\delta}^-\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}$$

$$\geq \min\{\max\{\bigwedge_{(x,y)\in A_a}\bigwedge_{(p,q)\in A_x}\{\min\{\widetilde{\mu}^-_{\widetilde{A}}(pqy),\widetilde{\lambda}^-\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}$$

$$= \min\{\min\{\bigwedge_{(x,y)\in A_a}\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(pqy),\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\},\widetilde{\lambda}^-\}$$

$$\geq \min\{\max\{\min\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\lambda}^-\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}$$

$$= \min\{\min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\},\widetilde{\lambda}^-\}$$

$$= \min\{\max\{\widetilde{\mu}^-(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\}$$

$$= \left(\widetilde{\mu}^-_{\widetilde{A}}\right)^{\lambda}_{\delta}(a).$$

Altogether, we obtain $\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{S} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A} \subseteq \widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$. □

Now, we provide a characterization of regular ordered semigroups in terms of interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideals.

**Theorem 3.3.** *Let $S$ be an ordered semigroup and $\widetilde{A} = \left(\widetilde{\mu}^{+}_{\widetilde{A}}, \widetilde{\mu}^{-}_{\widetilde{A}}\right)$ an interval valued bipolar fuzzy subset of $S$. Then the following statements are equivalent:*

*(1) $S$ is regular.*
*(2) $\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}} = \widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{S} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A}$ for any interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideal $\widetilde{A}$ of $S$.*

*Proof.* $(1) \Rightarrow (2)$. Let $\widetilde{A}$ be an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideal of $S$ and $a \in S$. Since $S$ is regular, there exists $x \in S$ such that $a \leq axa$. That is, $(ax, a) \in A_a$. Thus,

$$\begin{aligned}
\widetilde{\mu}^{+}_{\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{S} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A}}(a) &= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A} \circ \widetilde{S} \circ \widetilde{A}}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \max\{\min\{\bigvee_{a \leq yz} \{\min\{(\widetilde{\mu}^{+}_{\widetilde{A} \circ \widetilde{S}})(y), \widetilde{\mu}^{+}(z)\}\}, \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&\geq \max\{\min\{\min\{(\widetilde{\mu}^{+}_{\widetilde{A} \circ \widetilde{S}})(ax), \widetilde{\mu}^{+}(a)\}, \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \max\{\min\{(\widetilde{\mu}^{+}_{\widetilde{A} \circ \widetilde{S}})(ax), \widetilde{\mu}^{+}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \max\{\min\{\bigvee_{ax \leq pq} \{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(p), \widetilde{\mu}^{+}_{\widetilde{S}}(q)\}\}, \widetilde{\mu}^{+}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&\geq \max\{\min\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(a), \widetilde{\mu}^{+}_{\widetilde{S}}(x)\}, \widetilde{\mu}^{+}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(a), \widetilde{\mu}^{+}_{\widetilde{S}}(x), \widetilde{\mu}^{+}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(a), \widetilde{\mu}^{+}_{\widetilde{A}}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \left(\widetilde{\mu}^{+}_{\widetilde{A}}\right)^{\lambda}_{\delta}(a),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mu}^-_{\widetilde{A}\circ\frac{\widetilde{\lambda}}{\widetilde{\delta}}\widetilde{S}\circ\frac{\widetilde{\lambda}}{\widetilde{\delta}}\widetilde{A}}(a) &= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}\circ\widetilde{S}\circ\widetilde{A}}(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&= \min\{\max\{\bigwedge_{a\leq yz}\{\min\{(\widetilde{\mu}^-_{\widetilde{A}\circ\widetilde{S}})(y),\widetilde{\mu}^-(z)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&\leq \min\{\max\{\max\{(\widetilde{\mu}^-_{\widetilde{A}\circ\widetilde{S}})(ax),\widetilde{\mu}^-(a)\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&= \min\{\max\{(\widetilde{\mu}^-_{\widetilde{A}\circ\widetilde{S}})(ax),\widetilde{\mu}^-(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&= \min\{\max\{\bigwedge_{ax\leq pq}\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(p),\widetilde{\mu}^-_{\widetilde{S}}(q)\}\},\widetilde{\mu}^-(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&\leq \min\{\max\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\mu}^-_{\widetilde{S}}(x)\},\widetilde{\mu}^-(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\mu}^-_{\widetilde{S}}(x),\widetilde{\mu}^-(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\} \\
&= \left(\widetilde{\mu}^-_{\widetilde{A}}\right)^{\lambda}_{\delta}(a).
\end{aligned}$$

Hence, $\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}} \subseteq \widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}$. By Lemma 3.2, we have $\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A} \subseteq \widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$. Therefore, $\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}} = \widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}$.

(2) $\Rightarrow$ (1). Let $B$ be a bi-ideal of an ordered semigroup $S$. Then $\widetilde{\chi}_B$ is an interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy bi-ideal of $S$. Let $a\in B$. Then

$$\max\{\min\{\widetilde{\mu}^+_{\widetilde{\chi}_B\circ\widetilde{S}\circ\widetilde{\chi}_B}(a),\widetilde{\delta}^+\},\widetilde{\lambda}^+\} = \widetilde{\mu}^+_{\widetilde{\chi}_B\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{\chi}_B}(a) = \left(\widetilde{\mu}^+_{\widetilde{\chi}_B}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(a) = \widetilde{\delta}^+.$$

This implies that $A_a \neq \emptyset$, that is, there exist $b,c\in S$ such that $a\leq bc$. Thus,

$$\widetilde{\mu}^+_{\widetilde{\chi}_B\circ\widetilde{S}\circ\widetilde{\chi}_B}(a) = \bigvee_{(b,c)\in A_a}\{\min\{\widetilde{\mu}^+_{\widetilde{\chi}_B\circ\widetilde{S}}(b),\widetilde{\mu}^+_{\widetilde{\chi}_B}(c)\}\} = \widetilde{1}.$$

This yields $\widetilde{\mu}^+_{\widetilde{\chi}_B\circ\widetilde{S}}(b) = \widetilde{1}$ and $\widetilde{\mu}^+_{\widetilde{\chi}_B}(c) = \widetilde{1}$. Then $c\in B$. Since $\widetilde{\mu}^+_{\widetilde{\chi}_B\circ\widetilde{S}}(b) = \widetilde{1}$, we obtain $A_b \neq \emptyset$, and there exist $p,q\in S$ such that $b\leq pq$. Thus,

$$\bigvee_{(p,q)\in A_b}\{\min\{\widetilde{\mu}^+_{\widetilde{\chi}_B}(p),\widetilde{\mu}^+_{\widetilde{S}}(q)\}\} = \widetilde{\mu}^+_{\widetilde{\chi}_B\circ\widetilde{S}}(b) = \widetilde{1}.$$

This yields $\widetilde{\mu}^+_{\widetilde{\chi}_B}(p)=\widetilde{1}$. Then $p\in B$, and $a\leq bc\leq pqc\in BSB$. That is, $a\in(BSB]$. On the other hand, since $B$ is a bi-ideal of $S$, we have $(BSB]\subseteq B$. Therefore $B=(BSB]$. By Lemma 3.1, $S$ is regular. □

**Lemma 3.4.** *Let $\widetilde{A}=\left(\widetilde{\mu}^+_{\widetilde{A}},\widetilde{\mu}^-_{\widetilde{A}}\right)$ be an interval valued bipolar fuzzy subset of an ordered semigroup $S$. Then $\widetilde{A}$ is an interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy left ideal of $S$ if and only if $\widetilde{A}$ satisfies that*

*(1) If $x\leq y$, then*

*(a)* $\max\{\widetilde{\mu}^+_{\widetilde{A}}(x),\widetilde{\delta}^+\}\geq\min\{\widetilde{\mu}^+_{\widetilde{A}}(y),\widetilde{\lambda}^+\}$

*(b)* $\min\{\widetilde{\mu}^-_{\widetilde{A}}(x),\widetilde{\delta}^-\}\leq\max\{\widetilde{\mu}^-_{\widetilde{A}}(y),\widetilde{\lambda}^-\}$

*for all $x,y\in S$.*

*(2)* $\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}\subseteq\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$.

*Proof.* ($\Rightarrow$). Let $\widetilde{A}=\left(\widetilde{\mu}^+_{\widetilde{A}},\widetilde{\mu}^-_{\widetilde{A}}\right)$ be an interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy subset of an ordered semigroup $S$. Thus, (1) holds. Let $a\in S$. If $A_a=\emptyset$, then

$$\begin{aligned}\widetilde{\mu}^+_{\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}(a)&=\max\{\min\{\widetilde{\mu}^+_{\widetilde{S}\circ\widetilde{A}}(a),\widetilde{\lambda}^+\},\widetilde{\delta}^+\}\\&\leq\max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\widetilde{\lambda}^+\},\widetilde{\delta}^+\}\\&=\left(\widetilde{\mu}^+_{\widetilde{A}}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(a),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mu}^-_{\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}(a)&=\min\{\max\{\widetilde{\mu}^-_{\widetilde{S}\circ\widetilde{A}}(a),\widetilde{\lambda}^-\},\widetilde{\delta}^-\}\\&\geq\min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\lambda}^-\},\widetilde{\delta}^-\}\\&=\left(\widetilde{\mu}^-_{\widetilde{A}}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(a).\end{aligned}$$

Thus, $\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}\subseteq \widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$. If $A_a\neq\emptyset$, then

$$\begin{aligned}
\widetilde{\mu}^{+}_{\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}(a) &= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{S}\circ\widetilde{A}}(a),\widetilde{\lambda}^{+}\},\widetilde{\delta}^{+}\}\\
&= \max\{\min\{\bigvee_{(x,y)\in A_a}\{\min\{\widetilde{S}^{+}(x),\widetilde{\mu}^{+}(y)\}\},\widetilde{\lambda}^{+}\},\widetilde{\delta}^{+}\}\\
&= \max\{\min\{\min\{\widetilde{\mu}^{+}(y),\widetilde{\lambda}^{+}\},\widetilde{\lambda}^{+}\},\widetilde{\delta}^{+}\}\\
&\leq \max\{\min\{\max\{\widetilde{\mu}^{+}(xy),\widetilde{\delta}^{+}\},\widetilde{\lambda}^{+}\},\widetilde{\delta}^{+}\}\\
&= \max\{\min\{\widetilde{\mu}^{+}(xy),\widetilde{\lambda}^{+}\},\widetilde{\delta}^{+}\}\\
&= \max\{\min\{\min\{\widetilde{\mu}^{+}(xy),\widetilde{\lambda}^{+}\},\widetilde{\lambda}^{+}\},\widetilde{\delta}^{+}\}\\
&\leq \max\{\min\{\max\{\widetilde{\mu}^{+}(a),\widetilde{\delta}^{+}\},\widetilde{\lambda}^{+}\},\widetilde{\delta}^{+}\}\\
&= \max\{\min\{\widetilde{\mu}^{+}(a),\widetilde{\lambda}^{+}\},\widetilde{\delta}^{+}\}\\
&= \left(\widetilde{\mu}^{+}_{\widetilde{A}}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(a),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mu}^{-}_{\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}(a) &= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{S}\circ\widetilde{A}}(a),\widetilde{\lambda}^{-}\},\widetilde{\delta}^{-}\}\\
&= \min\{\max\{\bigwedge_{(x,y)\in A_a}\{\max\{\widetilde{S}^{-}(x),\widetilde{\mu}^{-}(y)\}\},\widetilde{\lambda}^{-}\},\widetilde{\delta}^{-}\}\\
&= \min\{\max\{\max\{\widetilde{\mu}^{-}(y),\widetilde{\lambda}^{-}\},\widetilde{\lambda}^{-}\},\widetilde{\delta}^{-}\}\\
&\geq \min\{\max\{\min\{\widetilde{\mu}^{-}(xy),\widetilde{\delta}^{-}\},\widetilde{\lambda}^{-}\},\widetilde{\delta}^{-}\}\\
&= \min\{\max\{\widetilde{\mu}^{-}(xy),\widetilde{\lambda}^{-}\},\widetilde{\delta}^{-}\}\\
&= \min\{\max\{\max\{\widetilde{\mu}^{-}(xy),\widetilde{\lambda}^{-}\},\widetilde{\lambda}^{-}\},\widetilde{\delta}^{-}\}\\
&\geq \min\{\max\{\min\{\widetilde{\mu}^{-}(a),\widetilde{\delta}^{-}\},\widetilde{\lambda}^{-}\},\widetilde{\delta}^{-}\}\\
&= \min\{\max\{\widetilde{\mu}^{-}(a),\widetilde{\lambda}^{-}\},\widetilde{\delta}^{-}\}\\
&= \left(\widetilde{\mu}^{-}_{\widetilde{A}}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(a).
\end{aligned}$$

Thus, $\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}\subseteq \widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$. For any two cases, we obtain that $\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}\subseteq \widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$.

($\Leftarrow$). Assume that (1) and (2) hold. Let $x, y \in S$. Then

$$\begin{aligned}
\max\{\widetilde{\mu}^{+}_{\widetilde{A}}(xy), \widetilde{\delta}^{+}\} &\geq \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(xy), \widetilde{\lambda}^{+}\}, \widetilde{\delta}^{+}\} \\
&= \left(\widetilde{\mu}^{+}_{\widetilde{A}}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(xy). \\
&\geq \widetilde{\mu}^{+}_{\widetilde{S} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A}}(xy) \\
&= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{S} \circ \widetilde{A}}(xy), \widetilde{\lambda}^{+}\}, \widetilde{\delta}^{+}\} \\
&= \max\{\min\{\bigvee_{(a,b) \in A_{xy}} \{\min\{\widetilde{\mu}^{+}_{\widetilde{S}}(a), \widetilde{\mu}^{+}_{\widetilde{A}}(b)\}\}, \widetilde{\lambda}^{+}\}, \widetilde{\delta}^{+}\} \\
&\geq \max\{\min\{\{\min\{\widetilde{\mu}^{+}_{\widetilde{S}}(x), \widetilde{\mu}^{+}_{\widetilde{A}}(y)\}\}, \widetilde{\lambda}^{+}\}, \widetilde{\delta}^{+}\} \\
&= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(y), \widetilde{\lambda}^{+}\}, \widetilde{\delta}^{+}\} \\
&\geq \min\{\widetilde{\mu}^{+}_{\widetilde{A}}(y), \widetilde{\lambda}^{+}\},
\end{aligned}$$

and

$$\begin{aligned}
\min\{\widetilde{\mu}^{-}_{\widetilde{A}}(xy), \widetilde{\delta}^{-}\} &\leq \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A}}(xy), \widetilde{\lambda}^{-}\}, \widetilde{\delta}^{-}\} \\
&= \left(\widetilde{\mu}^{-}_{\widetilde{A}}\right)^{\widetilde{\lambda}}_{\widetilde{\delta}}(xy). \\
&\leq \widetilde{\mu}^{-}_{\widetilde{S} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A}}(xy) \\
&= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{S} \circ \widetilde{A}}(xy), \widetilde{\lambda}^{-}\}, \widetilde{\delta}^{-}\} \\
&= \min\{\max\{\bigwedge_{(a,b) \in A_{xy}} \{\max\{\widetilde{\mu}^{-}_{\widetilde{S}}(a), \widetilde{\mu}^{-}_{\widetilde{A}}(b)\}\}, \widetilde{\lambda}^{-}\}, \widetilde{\delta}^{-}\} \\
&\leq \min\{\max\{\{\max\{\widetilde{\mu}^{-}_{\widetilde{S}}(x), \widetilde{\mu}^{-}_{\widetilde{A}}(y)\}\}, \widetilde{\lambda}^{-}\}, \widetilde{\delta}^{-}\} \\
&= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A}}(y), \widetilde{\lambda}^{-}\}, \widetilde{\delta}^{-}\} \\
&\leq \max\{\widetilde{\mu}^{-}_{\widetilde{A}}(y), \widetilde{\lambda}^{-}\}.
\end{aligned}$$

Therefore, $\widetilde{A}$ is an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy left ideal of $S$. □

Similarly to Lemma 3.4, we obtain the following two lemmas. Moreover, the proofs are also similar to that of Lemma 3.4.

**Lemma 3.5.** *Let $\widetilde{A}=\left(\widetilde{\mu}^{+}_{\widetilde{A}},\widetilde{\mu}^{-}_{\widetilde{A}}\right)$ be an interval valued bipolar fuzzy subset of an ordered semigroup S. Then $\widetilde{A}$ is an interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy right ideal of S if and only if $\widetilde{A}$ satisfies that*

*(1) If $x\leq y$, then*

*(a)* $\max\{\widetilde{\mu}^{+}_{\widetilde{A}}(x),\widetilde{\delta}^{+}\}\geq\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(y),\widetilde{\lambda}^{+}\}$

*(b)* $\min\{\widetilde{\mu}^{-}_{\widetilde{A}}(x),\widetilde{\delta}^{-}\}\leq\max\{\widetilde{\mu}^{-}_{\widetilde{A}}(y),\widetilde{\lambda}^{-}\}$

*for all $x,y\in S$.*

*(2)* $\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\subseteq\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$.

**Lemma 3.6.** *Let $\widetilde{A}=\left(\widetilde{\mu}^{+}_{\widetilde{A}},\widetilde{\mu}^{-}_{\widetilde{A}}\right)$ be an interval valued bipolar fuzzy subset of an ordered semigroup S. Then $\widetilde{A}$ is an interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy ideal of S if and only if $\widetilde{A}$ satisfies that*

*(1) If $x\leq y$, then*

*(a)* $\max\{\widetilde{\mu}^{+}_{\widetilde{A}}(x),\widetilde{\delta}^{+}\}\geq\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(y),\widetilde{\lambda}^{+}\}$

*(b)* $\min\{\widetilde{\mu}^{-}_{\widetilde{A}}(x),\widetilde{\delta}^{-}\}\leq\max\{\widetilde{\mu}^{-}_{\widetilde{A}}(y),\widetilde{\lambda}^{-}\}$

*for all $x,y\in S$.*

*(2)* $\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\subseteq\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$ *and* $\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}\subseteq\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}$.

Let $\widetilde{A}=\left(\widetilde{\mu}^{+}_{\widetilde{A}},\widetilde{\mu}^{-}_{\widetilde{A}}\right)$ be an interval valued bipolar fuzzy subset of an ordered semigroup $S$. Then we define the interval valued bipolar fuzzy subset $\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}=\left(\widetilde{\mu}^{+}_{\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x),\widetilde{\mu}^{-}_{\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x)\right)$ of an ordered semigroup $S$ as follows:

$$\widetilde{\mu}^{+}_{\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x)=\max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(x),\widetilde{\delta}^{+}\},\widetilde{\lambda}^{+}\},$$

and

$$\widetilde{\mu}^{-}_{\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x)=\min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A}}(x),\widetilde{\delta}^{-}\},\widetilde{\lambda}^{-}\},$$

for all $x\in S$.

**Lemma 3.7.** *Let $\widetilde{A}=\left(\widetilde{\mu}^{+}_{\widetilde{A}},\widetilde{\mu}^{-}_{\widetilde{A}}\right)$, $\widetilde{B}=\left(\widetilde{\mu}^{+}_{\widetilde{B}},\widetilde{\mu}^{-}_{\widetilde{B}}\right)$ be interval valued bipolar fuzzy subsets of an ordered semigroup S. Then $\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}\cap\widetilde{B}^{\widetilde{\lambda}}_{\widetilde{\delta}}=\widetilde{A}\cap^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}$.*

*Proof.* Let $x \in S$. Then

$$\begin{aligned}
\widetilde{\mu}^{+}_{\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}} \cap \widetilde{B}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x) &= \min\{\widetilde{\mu}^{+}_{\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x), \widetilde{\mu}^{+}_{\widetilde{B}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x)\} \\
&= \min\{\max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(x), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\}, \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{B}}(x), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\}\} \\
&= \max\{\min\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}}(x), \widetilde{\mu}^{+}_{\widetilde{B}}(x), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A} \cap \widetilde{B}}(x), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= \widetilde{\mu}^{+}_{\widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mu}^{-}_{\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}} \cap \widetilde{B}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x) &= \min\{\widetilde{\mu}^{-}_{\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x), \widetilde{\mu}^{-}_{\widetilde{B}^{\widetilde{\lambda}}_{\widetilde{\delta}}}(x)\} \\
&= \min\{\min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A}}(x), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\}, \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{B}}(x), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\}\} \\
&= \min\{\max\{\min\{\widetilde{\mu}^{-}_{\widetilde{A}}(x), \widetilde{\mu}^{-}_{\widetilde{B}}(x), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A} \cap \widetilde{B}}(x), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&= \widetilde{\mu}^{-}_{\widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}}(x).
\end{aligned}$$

Therefore, $\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}} \cap \widetilde{B}^{\widetilde{\lambda}}_{\widetilde{\delta}} = \widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}$. □

Now, we provide a characterization of regular ordered semigroups in terms of interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideals and interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy ideals.

**Theorem 3.8.** *Let $\widetilde{A} = \left(\widetilde{\mu}^{+}_{\widetilde{A}}, \widetilde{\mu}^{-}_{\widetilde{A}}\right)$, $\widetilde{B} = \left(\widetilde{\mu}^{+}_{\widetilde{B}}, \widetilde{\mu}^{-}_{\widetilde{B}}\right)$ be interval valued bipolar fuzzy subsets of an ordered semigroup $S$ . Then the following statements are equivalent:*

*(1) $S$ is regular.*
*(2) $\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A} = \widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B}$ for any interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideal $\widetilde{A}$ and interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy ideal $\widetilde{B}$ of $S$.*

*Proof.* $(1) \Rightarrow (2)$. Let $\widetilde{A}, \widetilde{B}$ be an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy bi-ideal and an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy ideal of $S$, respectively. Then, by Lemma 3.2, we have

$$\widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A} \subseteq \widetilde{A} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{S} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A} \subseteq \widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}.$$

Since $\widetilde{B}$ is an interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy ideal of $S$, by Lemma 3.6, we obtain

$$\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}\subseteq\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\subseteq\widetilde{B}^{\widetilde{\lambda}}_{\widetilde{\delta}}.$$

Thus $\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}\subseteq\widetilde{A}^{\widetilde{\lambda}}_{\widetilde{\delta}}\cap\widetilde{B}^{\widetilde{\lambda}}_{\widetilde{\delta}}=\widetilde{A}\cap^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}$. On the other hand, let $a\in S$. Since $S$ is regular, there exists $x\in S$ such that $a\leq axa\leq axaxa$, so $(a,xaxa)\in A_a$ Since $\widetilde{B}$ is an interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy ideal of $S$, we have

$$\begin{aligned}
\max\{\widetilde{\mu}^+_{\widetilde{B}}(xax),\widetilde{\lambda}^+\} &\geq \min\{\widetilde{\mu}^+_{\widetilde{B}}(ax),\widetilde{\delta}^+\}\\
&\geq \min\{\max\{\widetilde{\mu}^+_{\widetilde{B}}(ax),\widetilde{\lambda}^+\},\widetilde{\delta}^+\}\\
&\geq \min\{\min\{\widetilde{\mu}^+_{\widetilde{B}}(a),\widetilde{\delta}^+\},\widetilde{\delta}^+\}\\
&= \min\{\widetilde{\mu}^+_{\widetilde{B}}(a),\widetilde{\delta}^+\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\widetilde{\mu}^+_{\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}(a) &= \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}\circ\widetilde{B}\circ\widetilde{A}}(a),\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&= \max\{\min\{\bigvee_{(y,z)\in A_x}\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(y),\widetilde{\mu}^+_{\widetilde{B}\circ\widetilde{A}}(z)\}\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&\geq \max\{\min\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\widetilde{\mu}^+_{\widetilde{B}\circ\widetilde{A}}(xaxa)\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&= \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\widetilde{\mu}^+_{\widetilde{B}\circ\widetilde{A}}(xaxa),\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&= \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\bigvee_{(p,q)\in A_{xaxa}}\{\min\{\widetilde{\mu}^+_{\widetilde{B}}(p),\widetilde{\mu}^+_{\widetilde{A}}(q)\}\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&\geq \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\min\{\widetilde{\mu}^+_{\widetilde{B}}(xax),\widetilde{\mu}^+_{\widetilde{A}}(a)\}\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&\geq \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\min\{\max\{\widetilde{\mu}^+_{\widetilde{B}}(xax),\widetilde{\lambda}^+\},\widetilde{\mu}^+_{\widetilde{A}}(a)\}\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&\geq \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\min\{\min\{\widetilde{\mu}^+_{\widetilde{B}}(a),\widetilde{\delta}^+\},\widetilde{\mu}^+_{\widetilde{A}}(a)\}\},\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&= \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a),\widetilde{\mu}^+_{\widetilde{B}}(a),\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&= \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}\cap\widetilde{B}}(a),\widetilde{\delta}^+\},\widetilde{\lambda}^+\}\\
&= \widetilde{\mu}^+_{\widetilde{A}\cap^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}}(a),
\end{aligned}$$

and

$$\begin{aligned}\min\{\widetilde{\mu}^-_{\widetilde{B}}(xax),\widetilde{\lambda}^-\} &\leq \max\{\widetilde{\mu}^-_{\widetilde{B}}(ax),\widetilde{\delta}^-\}\\ &\leq \max\{\min\{\widetilde{\mu}^-_{\widetilde{B}}(ax),\widetilde{\lambda}^-\},\widetilde{\delta}^-\}\\ &\leq \max\{\max\{\widetilde{\mu}^-_{\widetilde{B}}(a),\widetilde{\delta}^-\},\widetilde{\delta}^-\}\\ &= \max\{\widetilde{\mu}^-_{\widetilde{B}}(a),\widetilde{\delta}^-\}.\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{\mu}^-_{\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}}(a) &= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}\circ\widetilde{B}\circ\widetilde{A}}(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &= \min\{\max\{\bigwedge_{(y,z)\in A_x}\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(y),\widetilde{\mu}^-_{\widetilde{B}\circ\widetilde{A}}(z)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &\leq \min\{\max\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\mu}^-_{\widetilde{B}\circ\widetilde{A}}(xaxa)\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\mu}^-_{\widetilde{B}\circ\widetilde{A}}(xaxa),\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\bigwedge_{(p,q)\in A_{xaxa}}\{\min\{\widetilde{\mu}^-_{\widetilde{B}}(p),\widetilde{\mu}^-_{\widetilde{A}}(q)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &\leq \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\min\{\widetilde{\mu}^-_{\widetilde{B}}(xax),\widetilde{\mu}^-_{\widetilde{A}}(a)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &\leq \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\min\{\max\{\widetilde{\mu}^-_{\widetilde{B}}(xax),\widetilde{\lambda}^-\},\widetilde{\mu}^-_{\widetilde{A}}(a)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &\leq \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\min\{\min\{\widetilde{\mu}^-_{\widetilde{B}}(a),\widetilde{\delta}^-\},\widetilde{\mu}^-_{\widetilde{A}}(a)\}\},\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}}(a),\widetilde{\mu}^-_{\widetilde{B}}(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &= \min\{\max\{\widetilde{\mu}^-_{\widetilde{A}\cap\widetilde{B}}(a),\widetilde{\delta}^-\},\widetilde{\lambda}^-\}\\ &= \widetilde{\mu}^-_{\widetilde{A}\cap^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}}(a).\end{aligned}$$

This means that $\widetilde{A}\cap^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}\subseteq\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}$. Therefore $\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}=\widetilde{A}\cap^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B}$.

$(2)\Rightarrow(1)$. Since $\widetilde{S}$ itself is an interval valued bipolar $(\widetilde{\lambda},\widetilde{\delta})$-fuzzy ideal of $S$, by hypothesis, we obtain

$$\widetilde{A}^{\widetilde{\delta}}_{\widetilde{\lambda}}=\widetilde{A}^{\widetilde{\delta}}_{\widetilde{\lambda}}\cap\widetilde{S}^{\widetilde{\delta}}_{\widetilde{\lambda}}=\widetilde{A}\cap^{\widetilde{\delta}}_{\widetilde{\lambda}}\widetilde{S}=\widetilde{A}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{S}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}.$$

By Theorem 3.3, $S$ is regular. $\square$

An ordered semigroup $S$ is *intra-regular* if, for each element $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$.

**Lemma 3.9** ([23]). *For an ordered semigroup $S$ the following conditions are equivalent.*

*(1) $S$ is intra-regular.*

*(2) $R \cap L \subseteq (LR]$ for every right ideal $R$ and every left ideal $L$ of $S$.*

Now, we characterize intra-regular ordered semigroups in terms of interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy left ideals and interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy right ideals.

**Theorem 3.10.** *Let $\widetilde{A} = \left(\widetilde{\mu}^+_{\widetilde{A}}, \widetilde{\mu}^-_{\widetilde{A}}\right)$, $\widetilde{B} = \left(\widetilde{\mu}^+_{\widetilde{B}}, \widetilde{\mu}^-_{\widetilde{B}}\right)$ be interval valued bipolar fuzzy subsets of an ordered semigroup $S$. Then the following statements are equivalent:*

*(1) $S$ is intra-regular.*

*(2) $\widetilde{A} \cap^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{B} \subseteq \widetilde{B} \circ^{\widetilde{\lambda}}_{\widetilde{\delta}} \widetilde{A}$ for every interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy right ideal $\widetilde{A}$ and every interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy left ideal $\widetilde{B}$ of $S$.*

*Proof.* $(1) \Rightarrow (2)$. Assume that $S$ is an intra-regular ordered semigroup, $\widetilde{A}$ and $\widetilde{B}$ is an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy right ideal and an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy left ideal of $S$, respectively. Let $a \in S$. Then, there exist $x, y \in S$ such that $a \leq xa^2y$, that is $(xa, ay) \in A_a$. Thus,

$$
\begin{aligned}
\widetilde{\mu}^+_{\widetilde{B} \circ^{\widetilde{\delta}}_{\widetilde{\lambda}} \widetilde{A}}(a) &= \max\{\min\{\widetilde{\mu}^+_{\widetilde{B} \circ \widetilde{A}}(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&= \max\{\min\{\bigvee_{(w,z) \in A_a} \{\min\{\widetilde{\mu}^+_{\widetilde{B}}(w), \widetilde{\mu}^+_{\widetilde{A}}(z)\}\}, \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&\geq \max\{\min\{\min\{\widetilde{\mu}^+_{\widetilde{B}}(xa), \widetilde{\mu}^+_{\widetilde{A}}(ay)\}, \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&= \max\{\min\{\widetilde{\mu}^+_{\widetilde{B}}(xa), \widetilde{\mu}^+_{\widetilde{A}}(ay), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&\geq \max\{\min\{\max\{\widetilde{\mu}^+_{\widetilde{B}}(xa), \widetilde{\lambda}^+\}, \max\{\widetilde{\mu}^+_{\widetilde{A}}(ay), \widetilde{\lambda}^+\}, \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&\geq \max\{\min\{\min\{\widetilde{\mu}^+_{\widetilde{B}}(a), \widetilde{\delta}^+\}, \min\{\widetilde{\mu}^+_{\widetilde{A}}(a), \widetilde{\delta}^+\}, \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&= \max\{\min\{\widetilde{\mu}^+_{\widetilde{B}}(a), \widetilde{\mu}^+_{\widetilde{A}}(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} \\
&= \max\{\min\{\widetilde{\mu}^+_{\widetilde{A}}(a), \widetilde{\mu}^+_{\widetilde{B}}(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\}
\end{aligned}
$$

$$= \max\{\min\{\widetilde{\mu}^{+}_{\widetilde{A}\cap\widetilde{B}}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\}$$
$$= \widetilde{\mu}^{+}_{\widetilde{A}\cap^{\widetilde{\delta}}_{\widetilde{\lambda}}\widetilde{B}}(a),$$

and

$$\begin{aligned}
\widetilde{\mu}^{-}_{\widetilde{B}\circ^{\widetilde{\delta}}_{\widetilde{\lambda}}\widetilde{A}}(a) &= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{B}\circ\widetilde{A}}(a), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&= \min\{\max\{\bigwedge_{(w,z)\in A_a}\{\max\{\widetilde{\mu}^{-}_{\widetilde{B}}(w), \widetilde{\mu}^{-}_{\widetilde{A}}(z)\}\}, \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&\leq \min\{\max\{\max\{\widetilde{\mu}^{-}_{\widetilde{B}}(xa), \widetilde{\mu}^{-}_{\widetilde{A}}(ay)\}, \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{B}}(xa), \widetilde{\mu}^{-}_{\widetilde{A}}(ay), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&\leq \min\{\max\{\min\{\widetilde{\mu}^{-}_{\widetilde{B}}(xa), \widetilde{\lambda}^{-}\}, \min\{\widetilde{\mu}^{-}_{\widetilde{A}}(ay), \widetilde{\lambda}^{-}\}, \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&\leq \min\{\max\{\max\{\widetilde{\mu}^{-}_{\widetilde{B}}(a), \widetilde{\delta}^{-}\}, \min\{\widetilde{\mu}^{-}_{\widetilde{A}}(a), \widetilde{\delta}^{-}\}, \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{B}}(a), \widetilde{\mu}^{-}_{\widetilde{A}}(a), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A}}(a), \widetilde{\mu}^{-}_{\widetilde{B}}(a), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&= \min\{\max\{\widetilde{\mu}^{-}_{\widetilde{A}\cap\widetilde{B}}(a), \widetilde{\delta}^{-}\}, \widetilde{\lambda}^{-}\} \\
&= \widetilde{\mu}^{-}_{\widetilde{A}\cap^{\widetilde{\delta}}_{\widetilde{\lambda}}\widetilde{B}}(a).
\end{aligned}$$

This means that $\widetilde{A}\cap^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{B} \subseteq \widetilde{B}\circ^{\widetilde{\lambda}}_{\widetilde{\delta}}\widetilde{A}$.

$(2) \Rightarrow (1)$. Assume that (2) holds. Let $R$ and $L$ is a right ideal and a left ideal of $S$, respectively. Then $\widetilde{\chi}_R$ and $\widetilde{\chi}_L$ is an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy right ideal and an interval valued bipolar $(\widetilde{\lambda}, \widetilde{\delta})$-fuzzy left ideal of $S$, respectively. Let $a \in R\cap L$. Then, by hypothesis,

$$\begin{aligned}
\widetilde{\delta}^{+} &= \max\{\min\{\widetilde{\chi}^{+}_{R\cap L}(a), \widetilde{\delta}^{+}\}, \widetilde{\lambda}^{+}\} \\
&= (\widetilde{\chi}^{+}_{R\cap L})^{\widetilde{\delta}}_{\widetilde{\lambda}}(a) \\
&= ((\widetilde{\chi}^{+}_{R})^{\widetilde{\delta}}_{\widetilde{\lambda}} \cap (\widetilde{\chi}^{+}_{L})^{\widetilde{\delta}}_{\widetilde{\lambda}})(a) \\
&= (\widetilde{\chi}^{+}_{R} \cap^{\widetilde{\delta}}_{\widetilde{\lambda}} \widetilde{\chi}^{+}_{L})(a) \\
&\leq (\widetilde{\chi}^{+}_{L} \circ^{\widetilde{\delta}}_{\widetilde{\lambda}} \widetilde{\chi}^{+}_{R})(a)
\end{aligned}$$

$$= \max\{\min\{(\widetilde{\chi}_L^+ \circ \widetilde{\chi}_R^+)(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\}$$

$$= \max\{\min\{\widetilde{\chi}_{(LR]}^+(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\}$$

$$\leq \widetilde{\delta}^+.$$

This means that $\max\{\min\{\widetilde{\chi}_{(LR]}^+(a), \widetilde{\delta}^+\}, \widetilde{\lambda}^+\} = \widetilde{\delta}^+$ and then $\widetilde{\chi}_{(LR]}^+(a) = \widetilde{1}$. Thus, $a \in (LR]$ and

$$\begin{aligned}
\widetilde{\delta}^- &= \min\{\max\{\widetilde{\chi}_{R\cap L}^-(a), \widetilde{\delta}^-\}, \widetilde{\lambda}^-\} \\
&= (\widetilde{\chi}_{R\cap L}^-)_{\widetilde{\lambda}}^{\widetilde{\delta}}(a) \\
&= ((\widetilde{\chi}_R^-)_{\widetilde{\lambda}}^{\widetilde{\delta}} \cap (\widetilde{\chi}_L^-)_{\widetilde{\lambda}}^{\widetilde{\delta}})(a) \\
&= (\widetilde{\chi}_R^- \cap_{\widetilde{\lambda}}^{\widetilde{\delta}} \widetilde{\chi}_L^-)(a) \\
&\geq (\widetilde{\chi}_L^- \circ_{\widetilde{\lambda}}^{\widetilde{\delta}} \widetilde{\chi}_R^-)(a) \\
&= \min\{\max\{(\widetilde{\chi}_L^- \circ \widetilde{\chi}_R^-)(a), \widetilde{\delta}^-\}, \widetilde{\lambda}^-\} \\
&= \min\{\max\{\widetilde{\chi}_{(LR]}^-(a), \widetilde{\delta}^-\}, \widetilde{\lambda}^-\} \\
&\geq \widetilde{\delta}^-.
\end{aligned}$$

This means that $\min\{\max\{\widetilde{\chi}_{(LR]}^-(a), \widetilde{\delta}^-\}, \widetilde{\lambda}^-\} = \widetilde{\delta}^-$ and then $\widetilde{\chi}_{(LR]}^-(a) = -\widetilde{1}$. Thus, $a \in (LR]$. Therefore $R \cap L \subseteq (LR]$, by Lemma 3.9, $S$ is intra-regular. □

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## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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